Transient analysis of continuous time Markov chains

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Characterisation of the transient behaviour

- aims to calculate $H(t)$ where
$$[H(t)]_{i,j} = Pr\{X(t) = j | X(0) = i\}$$
- $H(t)$ must be such that it satisfies
$$\frac{dH(t)}{dt} = H(t)Q$$
where Q is the infinitesimal generator of the CTMC
- initial condition: $H(0) = I$ where I is an identity matrix, i.e.,
$$Pr\{X(0) = j | X(0) = i\} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Matrix exponential function

- define the exponential of a matrix as
$$\exp(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$
- as a consequence we have
$$\frac{d\exp(Qt)}{dt} = \frac{d\sum_{k=0}^{\infty}\frac{(Qt)^k}{k!}}{dt} = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d(Qt)^k}{dt} =$$
$$\sum_{k=1}^{\infty}\frac{Q(Qt)^{k-1}}{(k-1)!} = Q\exp(Qt)$$
- it is easy to check that $\exp(Qt)$ and Q are commutative, i.e.,
$$Q\exp(Qt) = \exp(Qt)Q$$

Transient behaviour

- $H(t) = \exp(Qt)$ is the solution of
$$\frac{dH(t)}{dt} = H(t)Q, \quad H(0) = I$$
- and in theory $H(t)$ can be calculated directly from
$$\exp(Qt) = \sum_{k=0}^{\infty}\frac{(Qt)^k}{k!}$$
- but with numerical trouble as $(Qt)^k$ can diverge
- for example:
$$Q = \begin{vmatrix} -2 & 2 \\ 3 & -3 \end{vmatrix} \qquad Q^{10} = \begin{vmatrix} 3906250 & -3906250 \\ -5859375 & 5859375 \end{vmatrix}$$

Randomization

- choose a q such that
$$q > \max_i |Q_{i,i}|$$
- introduce the matrix
$$\hat{Q} = \frac{Q}{q} + I$$
- so we have $Q = q(\hat{Q} - I)$
- each entry of $\hat{Q}$ is between 0 and 1
- $\hat{Q}$ is matrix with row sums equal to 1

Randomization

- $H(t)$ can be calculated as
$$H(t) = \exp(Qt) = \exp(q(\hat{Q} - I)t)$$
- for two matrices, A and B, that commute
$$\exp(A + B) = \exp(A)\exp(B)$$
- from which
$$H(t) = \exp(\hat{Q}qt)\exp(-Iqt) = \exp(\hat{Q}qt)\exp(-qt) =$$
$$\sum_{k=0}^{\infty}\frac{(\hat{Q}qt)^k}{k!}\exp(-qt) = \sum_{k=0}^{\infty}\hat{Q}^k\frac{(qt)^k}{k!}\exp(-qt)$$

Randomization

- we derived the form
$$H(t) = \sum_{k=0}^{\infty}\hat{Q}^k\frac{(qt)^k}{k!}\exp(-qt)$$
- why is it advantageous?
- $\hat{Q}$ describes a discrete time Markov chain
- $\hat{Q}^k$ has row sums equal to 1 for any k
- the quantities
$$\frac{(qt)^k}{k!}\exp(-qt)$$
are the Poisson probabilities

Randomization

- easy error control based on the Poisson probabilities
- approximation of $H(t)$ with precision ϵ is
$$\sum_{k=0}^{M(\epsilon)}\hat{Q}^k\frac{(qt)^k}{k!}\exp(-qt)$$
where $M(\epsilon)$ is such that
$$1 - \sum_{k=0}^{M(\epsilon)-1}\frac{(qt)^k}{k!}\exp(-qt) > \epsilon, \quad 1 - \sum_{k=0}^{M(\epsilon)}\frac{(qt)^k}{k!}\exp(-qt) < \epsilon$$
- calculate the Poisson probabilities by recursion as
$$\frac{(qt)^k}{k!}\exp(-qt) = \frac{(qt)^{k-1}}{(k-1)!}\exp(-qt)\frac{qt}{k}$$

Randomization

- randomization has a nice stochastic interpretation
- the transient behaviour of a CTMC can be “simulated” by a DTMC
- number of transition in $[0, t]$ is distributed according to the Poisson distribution