Reasoning about typicality with low complexity Description Logics: the logic $\mathcal{EL}^{+^\bot}\mathbf{T}$

Laura Giordano[1], Valentina Gliozzi[2], Nicola Olivetti[3], and Gian Luca Pozzato[2]

[1] Dip. di Informatica - Università del Piemonte Orientale - laura@mfn.unipmn.it
[2] Dip. di Informatica - Università di Torino - {gliozzi,pozzato}@di.unito.it
[3] LSIS-UMR CNRS 6168 Univ. "P. Cézanne" - nicola.olivetti@univ-cezanne.fr

Abstract. We present an extension of the low complexity Description Logic $\mathcal{EL}^{+^\bot}$ for reasoning about prototypical properties and inheritance with exceptions. We add to $\mathcal{EL}^{+^\bot}$ a typicality operator $\mathbf{T}$, which is intended to select the "most normal" instances of a concept. In the resulting logic, called $\mathcal{EL}^{+^\bot}\mathbf{T}$, the knowledge base may contain subsumption relations of the form "$\mathbf{T}(C)$ is subsumed by P", expressing that typical C-members have the property P. We show that the problem of entailment in $\mathcal{EL}^{+^\bot}\mathbf{T}$ is in co-NP by proving a small model result.

1 Introduction

In Description Logics (DLs), the need of representing prototypical properties and of reasoning about defeasible inheritance of such properties naturally arises. The traditional approach to address this problem is to handle defeasible inheritance by integrating some kind of nonmonotonic reasoning mechanism. This has led to the study of nonmonotonic extensions of DLs [3, 4, 6–8, 15]. However, finding a suitable nonmonotonic extension for inheritance with exceptions is far from obvious.

In this work we continue our investigation started in [9], where we have proposed the logic $\mathcal{ALC} + \mathbf{T}$ for defeasible reasoning in the description logic $\mathcal{ALC}$. $\mathcal{ALC} + \mathbf{T}$ is obtained by adding a typicality operator $\mathbf{T}$ to $\mathcal{ALC}$. The intended meaning of the operator $\mathbf{T}$ is that, for any concept C, $\mathbf{T}(C)$ singles out the instances of C that are considered as "typical" or "normal". Thus assertions as "typical football players love football" are represented by $\mathbf{T}(\mathit{FootballPlayer}) \sqsubseteq \mathit{FootballLover}$. The semantics of the typicality operator $\mathbf{T}$ turns out to be strongly related to the semantics of nonmonotonic entailment in KLM logic $\mathbf{P}$ [14].

In our setting, we assume that the TBox element of a KB comprises, in addition to the standard concept inclusions, a set of inclusions of the type $\mathbf{T}(C) \sqsubseteq D$ where D is a concept not mentioning $\mathbf{T}$. For instance, a KB may contain: $\mathbf{T}(\mathit{Dog}) \sqsubseteq \mathit{Affectionate}$; $\mathbf{T}(\mathit{Dog}) \sqsubseteq \mathit{CarriedByTrain}$; $\mathbf{T}(\mathit{Dog} \sqcap \mathit{PitBull}) \sqsubseteq \mathit{NotCarriedByTrain}$; $\mathit{CarriedByTrain} \sqcap \mathit{NotCarriedByTrain} \sqsubseteq \bot$, corresponding to the assertions: typically dogs are affectionate, normally dogs can be transported by train, whereas typically a dog belonging to the race of pitbull cannot (since pitbulls are considered as reactive dogs); the fourth inclusion represents the disjointness of the two concepts *CarriedByTrain* and *NotCarriedByTrain*. Notice that, in standard DLs, replacing the second and the third inclusion with $Dog \sqsubseteq CarriedByTrain$ and $Dog \sqcap PitBull \sqsubseteq NotCarriedByTrain$, respectively, we would simply get that there are not pitbull dogs, thus the KB would collapse. This collapse is avoided as we do not assume that $\mathbf{T}$ is monotonic, that is to say $C \sqsubseteq D$ does not imply $\mathbf{T}(C) \sqsubseteq \mathbf{T}(D)$.

By the properties of $\mathbf{T}$, some inclusions are entailed by the above KB, as for instance $\mathbf{T}(Dog \sqcap CarriedByTrain) \sqsubseteq Affectionate$. In our setting we can also use the $\mathbf{T}$ operator to state that some domain elements are typical instances of a given concept. For instance, an ABox may contain either $\mathbf{T}(Dog)(fido)$ or $\mathbf{T}(Dog \sqcap PitBull)(fido)$. In the two cases, the expected conclusions are entailed: $CarriedByTrain(fido)$ and $NotCarriedByTrain(fido)$, respectively.

In this work, we extend our approach based on the typicality operator to *low complexity* Description Logics, focusing on the logic $\mathcal{EL}^{+^\bot}$ of the well known $\mathcal{EL}$ family. The logics of the $\mathcal{EL}$ family allow for conjunction ($\sqcap$) and existential restriction ($\exists R.C$). Despite their relatively low expressivity, a renewed interest has recently emerged for these logics. Indeed, theoretical results have shown that $\mathcal{EL}$ has better algorithmic properties than its counterpart $\mathcal{FL}_0$, which allows for conjunction and value restriction ($\forall R.C$). Also, it has turned out that the logics of the $\mathcal{EL}$ family are relevant for several applications, in particular in the bio-medical domain; for instance, medical terminologies, such as the GALEN Medical Knowledge Base, the Systemized Nomenclature of Medicine, and the Gene Ontology, can be formalized in small extensions of $\mathcal{EL}$.

We present a *small model* result for the logic $\mathcal{EL}^{+^\bot}\mathbf{T}$. More precisely, we show that, given an $\mathcal{EL}^{+^\bot}\mathbf{T}$ knowledge base, KB, if KB is satisfible, then there is a *small* model satisfying KB, whose size is polynomial in the size of KB. The construction of the model exploits the facts that (1) it is possible to reuse the same domain element (instance of a concept C) to fulfill existential formulas $\exists r.C$ w.r.t. domain elements; (2) we can restrict our attention to a specific class of models which are called multi-linear and include a polynomial number of chains of elements of polynomial length. The construction of the model allows us to conclude that the problem of deciding entailment in $\mathcal{EL}^{+^\bot}\mathbf{T}$ is in co-NP. For reasoning about *irrelevance*, we deal with the inheritance of defeasible properties by introducing default rules. Given the complexity of entailment in $\mathcal{EL}^{+^\bot}\mathbf{T}$, credulous reasoning in the resulting default theory is in Σ^p_2. Preliminary results about this work have been presented in [12].

2 The Logic $\mathcal{EL}^{+^\bot}\mathbf{T}$

We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individuals $\mathcal{O}$. The language $\mathcal{L}$ of the logic $\mathcal{EL}^{+^\bot}\mathbf{T}$ is defined by distinguishing *concepts* and *extended concepts* as follows: (Concepts) $A \in \mathcal{C}$, $\top$, and $\bot$ are *concepts* of $\mathcal{L}$; if $C, D \in \mathcal{L}$ and $r \in \mathcal{R}$, then $C \sqcap D$ and $\exists r.C$ are *concepts* of $\mathcal{L}$. (Extended concepts) if C is a concept, then C and $\mathbf{T}(C)$ are extended concepts of $\mathcal{L}$. A knowledge base is a pair (TBox,ABox). TBox contains (i) a finite set of GCIs $C \sqsubseteq D$, where C is an extended concept (either C' or $\mathbf{T}(C')$), and D is a concept, and (ii) a finite set of role inclusions (RIs) $r_1 \circ r_2 \circ \cdots \circ r_n \sqsubseteq r$. ABox contains expressions of the form $C(a)$ and $r(a, b)$ where C is an extended concept, $r \in \mathcal{R}$, and $a, b \in \mathcal{O}$. In order to provide a semantics to the operator $\mathbf{T}$, we extend the definition of a model used in "standard" terminological logic $\mathcal{EL}^{+^\perp}$:

Definition 1 (Semantics of T). *A model $\mathcal{M}$ is any structure $\langle \Delta, <, I \rangle$, where Δ is the domain; $<$ is an irreflexive and transitive relation over Δ, and satisfies the following* Smoothness Condition*: for all $S \subseteq \Delta$, for all $a \in S$, either $a \in Min_<(S)$ or $\exists b \in Min_<(S)$ such that $b < a$, where $Min_<(S) = \{a : a \in S$ and $\nexists b \in S$ s.t. $b < a\}$. I is the extension function that maps each extended concept C to $C^I \subseteq \Delta$, and each role r to a $r^I \subseteq \Delta^I \times \Delta^I$. For concepts of $\mathcal{EL}^{+^\perp}$, C^I is defined in the usual way. For the* $\mathbf{T}$ *operator: $(\mathbf{T}(C))^I = Min_<(C^I)$. A model satisfying a KB (TBox,ABox) is defined as usual. Moreover, we assume the unique name assumption.*

Notice that the meaning of $\mathbf{T}$ can be split into two parts: for any a of the domain Δ, $a \in (\mathbf{T}(C))^I$ just in case (i) $a \in C^I$, and (ii) there is no $b \in C^I$ such that $b < a$. In order to isolate the second part of the meaning of $\mathbf{T}$, we introduce a new modality $\Box$. The basic idea is simply to interpret the preference relation $<$ as an accessibility relation. By the Smoothness Condition, it turns out that $\Box$ has the properties as in Gödel-Löb modal logic of provability G. The interpretation of $\Box$ in $\mathcal{M}$ is as follows: $(\Box C)^I = \{a \in \Delta \mid$ for every $b \in \Delta$, if $b < a$ then $b \in C^I\}$. We have that a is a typical instance of C ($a \in (\mathbf{T}(C))^I$) iff $a \in C^I$ and, for all $b < a$, $b \notin C^I$, namely we have that $a \in (\mathbf{T}(C))^I$ iff $a \in (C \sqcap \Box\neg C)^I$. From now on, we consider $\mathbf{T}(C)$ as an abbreviation for $C \sqcap \Box\neg C$. The Smoothness Condition ensures that typical elements of C^I exist whenever $C^I \neq \emptyset$, by preventing infinitely descending chains of elements.

3 Constructing small models for $\mathcal{EL}^{+^\perp}\mathbf{T}$

We can show that, given a model $\mathcal{M} = \langle \Delta, <, I \rangle$ of a KB, we can build a *small* model of KB whose size is polynomial in the size of the KB. As we will see, this will provide a complexity upper bound for the logic $\mathcal{EL}^{+^\perp}\mathbf{T}$.

First of all, we must introduce an appropriate normal form[1] for KBs, in particular for TBoxes. Given a KB=(TBox,ABox), we say that it is normal if:

- all the inclusion relations in TBox have one of the following forms: $C_1 \sqsubseteq D$; $C_1 \sqcap C_2 \sqsubseteq D$; $C_1 \sqsubseteq \exists r.C_2$; $\exists r.C_1 \sqsubseteq D$; $\mathbf{T}(C_1) \sqsubseteq C_2$; $\mathbf{T}(C_1 \sqcap C_2) \sqsubseteq D$; $\mathbf{T}(C_1) \sqsubseteq \exists r.C_2$; $\mathbf{T}(\exists r.C_1) \sqsubseteq D$, where $C_1, C_2 \in \mathcal{C} \cup \{\top\}$ and $D \in \mathcal{C} \cup \{\bot\}$;
- all role inclusions in TBox are of the form $r \sqsubseteq s$ or $r_1 \circ r_2 \sqsubseteq s$.

[1] The normal form presented in this paper is not *minimal*. We could further reduce the normal form without reducing the expressivity of the language.

By extending the results presented in [1], we can show that any KB can be turned into a normalized KB' that is a *conservative* extension of KB, that is to say every model satisfying KB' is also a model of KB, whereas every model of KB can be extended to a model of KB' by appropriately choosing the interpretations of the additional concept and role names introduced by the normalization procedure. Furthermore, it can be shown that the size of KB' is linear in the size of KB, and that the normalization procedure can be done in linear time. Without loss of generality, from now on we only refer to normalized KBs. Starting from a normalized KB, we can now prove the following theorem, whose proof lasts until the end of the section.

Theorem 1 (Small model theorem). *Let KB=(TBox,ABox) be an* $\mathcal{EL}^{+^{\perp}}\mathbf{T}$ *knowledge base. For all models* $\mathcal{M} = \langle \Delta, <, I \rangle$ *of KB and all* $x \in \Delta$*, there exists a model* $\mathcal{N} = \langle \Delta^{\circ}, <^{\circ}, I^{\circ} \rangle$ *of KB such that (i)* $x \in \Delta^{\circ}$*, (ii) for all* $\mathcal{EL}^{+^{\perp}}\mathbf{T}$ *concepts* C*,* $x \in C^{I}$ *iff* $x \in C^{I^{\circ}}$*, and (iii)* $\mid \Delta^{\circ} \mid$ *is polynomial in the size of KB.*

We sketch the proof through the following four steps. In the first step, in order to reduce the size of the model, we cut a portion of it that includes x. In particular, we keep only those domain elements needed to retain the values of formulas in x. Intuitively, we reuse the same domain element to make true existential formulas in different domain elements. Moreover, we need to add new elements to the domain of the constructed model, in order to keep the same evaluation of existential formulas as in the initial model. As it is not guaranteed that the model obtained has a polynomial size in the size of the KB, we have to refine this construction. Our goal is to obtain a multi-linear model (Definition 2 below), that can be further transformed into a model of polynomial size. The second and the third steps are devoted to build a multi-linear model. In the fourth step we show that we can further reduce the size of the model by shortening the length of the linear descending chains to a polynomial size.

(STEP 1) We build a model $\mathcal{M}'$ by means of the following construction. For each atomic concept $C \in \mathcal{C}$ and for each role $r \in \mathcal{R}$ we let $S(C)$ and $R(r)$ be the mappings computed by the algorithm defined in [2] to compute subsumption by means of completion rules, whose meaning is that $D \in S(C)$ implies $C \sqsubseteq D$ and $(C, D) \in R(r)$ implies $C \sqsubseteq \exists r.D$. As usual, for a given individual a in the ABox, we write a^{I} to denote the element of Δ corresponding to the extension of a in $\mathcal{M}$. We make use of three sets of elements: Δ_0 will be part of the domain of the model being constructed, and it contains a portion of the domain Δ of the initial model. All elements introduced in the domain must be processed in order to satisfy the existential formulas. *Unres* is used to keep track of not yet processed elements. Finally, Δ_1 is a set of new elements that will belong to the domain of the constructed model. Each element w_C of Δ_1 is created for a corresponding atomic concept C and is used to satisfy any existential formula $\exists r.C$ throughout the model. In the following, by w_C we mean the element of Δ_1 which is added for the atomic concept C. We provide an algorithmic description of the construction of model $\mathcal{M}'$ from the given model $\mathcal{M}$. Observe that $\mathcal{M}$ can be an infinite model.

$\Delta_0 := \{x\} \cup \{a^I \in \Delta \mid a$ occurs in the ABox $\}$
Unres$:=\{x\} \cup \{a^I \in \Delta \mid a$ occurs in the ABox $\}$
$\Delta_1 := \emptyset$
while *Unres* $\neq \emptyset$ **do**
  extract one y from *Unres*
  for each $\exists r.C$ occurring in KB s.t. $y \in (\exists r.C)^I$ **do**
    if $\nexists w_C \in \Delta_1$ **then**
      choose $w \in \Delta$ s.t. $(y, w) \in r^I$ and $w \in C^I$
      $\Delta_0 := \Delta_0 \cup \{w\}$
      Unres$:=$*Unres* $\cup \{w\}$
      create a new element w_C associated with C
      $\Delta_1 := \Delta_1 \cup \{w_C\}$
      add $w <' w_C$
      add (y, w_C) to $r^{I'}$
    else
      add (y, w_C) to $r^{I'}$
  for each $y_i \in \Delta$ such that $y_i < y$ **do**
    $\Delta_0 := \Delta_0 \cup \{y_i\}$
    Unres$:=$*Unres* $\cup \{y_i\}$
for each $w_C, w_D \in \Delta_1$ with $C \neq D$ **do**
  if $(C, D) \in R(r)$ **then** add (w_C, w_D) to $r^{I'}$

The model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ is defined as follows:

- $\Delta' = \Delta_0 \cup \Delta_1$
- we extend $<'$ computed by the algorithm by adding $u <' v$ if $u < v$, for each $u, v \in \Delta'$;
- the extension function I' is defined as follows:
  - for all atomic concepts $C \in \mathcal{C}$, for all elements in Δ', we define:
    - for each $u \in \Delta_0$, we let $u \in C^{I'}$ if $u \in C^I$;
    - for each $w_D \in \Delta_1$, we let $w_D \in C^{I'}$ if $C \in S(D)$.
  - for all roles r, we extend $r^{I'}$ constructed by the algorithm by means of the following role closure rules:
    - for all inclusions $r \sqsubseteq s \in$ TBox, if $(u, v) \in r^{I'}$ then add (u, v) to $s^{I'}$;
    - for all inclusions $r_1 \circ r_2 \sqsubseteq s \in$ TBox, if $(u, v) \in r_1^{I'}$ and $(v, w) \in r_2^{I'}$ then add (u, w) to $s^{I'}$.
  - I' is extended so that it assigns a^I to each individual a in the ABox.

It is easy to see that relation $<'$ is irreflexive, transitive and satisfies the Smoothness Condition. Moreover, by induction on the structure of C, it can be proved that:

Lemma 1. *Given any* $\mathcal{EL}^{+^{\perp}}\mathbf{T}$ *concept* C *occurring in KB, for all* $y \in \Delta_0$, $y \in C^I$ *iff* $y \in C^{I'}$.

By making use of the above lemma, we can show that:

Lemma 2. *$\mathcal{M}'$ is a model of KB.*

$\mathcal{M}'$ is not guaranteed to have polynomial size in the KB because in line 18 we add an element y_i for each $y_i < y$, then the size of Δ_0 may be arbitrarily large. For this reason, we refine our construction in order to build a multi-linear model, that we will be able to further refine in order to obtain a model of polynomial size. In STEP 2, we replicate some domain elements, namely those belonging to more than one descending chain of $<$.

(STEP 2) We build a model $\mathcal{M}'' = \langle \Delta'', <'', I'' \rangle$ from $\mathcal{M}' = \langle \Delta', <', I' \rangle$. We define $\Delta'_s = \{x\} \cup \{u \in \Delta' \mid \nexists v \in \Delta'$ such that $u <' v\}$ to be the set containing x and all the domain elements in Δ' that are not preferred to any other element according to $<'$. Let $k = \mid \Delta'_s \mid$. For each $y_j \in \Delta'_s$ we define $\Delta_{y_j} = \{z \in \Delta' \mid z <' y_j\} \cup \{y_j\}$ the set of all domain elements preferred to y_j. As two sets Δ_{y_j} and Δ_{y_i} with $y_j \neq y_i$, are not guaranteed to be disjoint, we rename the elements of each set Δ_{y_j}, as follows. We consider for each $j = 1, \dots, k$ a renaming function (i.e. a bijection) f_j whose domain is Δ_{y_j} that makes a copy $\Delta_{f_j(y_j)}$ of Δ_{y_j} which is (i) disjoint from any Δ_{y_l}, and (ii) disjoint from any other $\Delta_{f_l(y_l)}$ with $l \neq j$. Moreover, for $y_j = x$, we let $x = f_j(x)$, that is, x is not renamed in Δ_x.

We define a model $\mathcal{M}'' = \langle \Delta'', <'', I'' \rangle$ as follows: $\Delta'' = \Delta_{f_1(y_1)} \cup \Delta_{f_2(y_2)} \dots \cup \Delta_{f_k(y_k)}$. The relation $<''$ is defined as follows: $u <'' v$ iff $u, v \in \Delta_{f_j(y_j)}$ so that $u = f_j(z)$ and $v = f_j(w)$, where $z, w \in \Delta_{y_j}$ and $z <' w$. Observe that elements in different components Δ_{y_j} are incomparable w.r.t. $<''$. Finally, given any atomic concept $C \in \mathcal{C}$, for $u \in \Delta_{f_j(y_j)}$ with $u = f_j(w)$, we let $u \in C^{I''}$ iff $w \in C^{I'}$. Moreover, given any role $r \in \mathcal{R}$, for $u \in \Delta_{f_j(y_j)}$ with $u = f_j(w)$ and $v \in \Delta_{f_i(y_i)}$ with $v = f_i(z)$, we let $(u, v) \in r^{I''}$ iff $(w, z) \in r^{I'}$. It can proved that:

Lemma 3. *$\mathcal{M}''$ is a model of KB and, given any concept C of $\mathcal{EL}^{+^{\bot}}\mathbf{T}$, $x \in C^{I''}$ iff $x \in C^{I'}$.*

(STEP 3) First of all, we introduce the notion of *multi-linear* model of a KB. Intuitively, a model $\mathcal{M} = \langle \Delta, <, I \rangle$ is multi-linear if the relation $<$ forms a set of chains of domain elements, that is:

Definition 2 (Multi-linear model). *Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, we say that it is multi-linear if the following properties hold for every $u, v, z \in \Delta$:*

(i) if $u < z$ and $v < z$ and $u \neq v$, then $u < v$ or $v < u$;
(ii) if $z < u$ and $z < v$ and $u \neq v$, then $u < v$ or $v < u$.

We now define a multi-linear model $\mathcal{M}^* = \langle \Delta'', <^*, I'' \rangle$ as follows: we let $<^*$ be any total order on each $\Delta_{f_j(y_j)}$ which respects $<''$; the elements in different components remain incomparable. More precisely $<^*$ satisfies:

- if $u <'' v$ then $u <^* v$
- for each $u, v \in \Delta_{f_j(y_j)}$, with $u \neq v$, $u <^* v$ or $v <^* u$
- for each $u \in \Delta_{f_i(y_i)}, v \in \Delta_{f_j(y_j)}$, with $i \neq j$, $u \not<^* v$ and $v \not<^* u$

Again, we can prove that:

Lemma 4. *$\mathcal{M}^*$ is a model of KB and, given any concept C of $\mathcal{EL}^{+^\perp}\mathbf{T}$, $x \in C^{I^*}$ iff $x \in C^{I''}$.*

(STEP 4) We conclude the proof by constructing a model $\mathcal{N} = \langle \Delta^\circ, <^\circ, I^\circ \rangle$ whose domain has polynomial size in the size of KB. Let the size of the initial KB be n. We know that $\mathcal{M}^*$ contains a polynomial number of linear chains of domain elements related by $<^*$, each one starting from a domain elements in Δ_1 (built by the algorithm in STEP 1) or from one domain element in $\{x, a_1^I, \ldots, a_k^I\}$, where $a_1^I, \ldots, a_k^I$ are the domain elements corresponding to the individuals in the ABox. We know that there are $O(n)$ chains, as Δ_1 contains one domain element for each atomic concept in $\mathcal{EL}^{+^\perp}$ and the domain elements $a_1^I, \ldots, a_k^I$ are $O(n)$. However, we have no bound on the length of the chains.

We want to show that the linear chains in the model can be reduced to finite chains of polynomial length in the size of the KB. To this purpose, given $\mathcal{M}^*$, we build a new multi-linear model $\mathcal{N} = \langle \Delta^\circ, <^\circ, I^\circ \rangle$ whose descending chains have polynomial length.

Let us consider a chain $w_0, w_1, w_2, \ldots$ in the multilinear model $\mathcal{M}^*$. Observe that, given two elements w_i and w_j in the chain such that $w_i < w_j$, the set of negated box formulas $\neg\Box\neg C$ of which w_i is an instance is a subset of the set of negated box formulas of which w_j is an instance. We can thus shrink each chain by retaining only those elements w_i and w_j such that $w_i < w_j$ implies that there exists a formula $\neg\Box\neg C$ such that w_j is an instance of $\neg\Box\neg C$, while w_i is not. As there is only a finite polynomial number of such box formulas $\neg\Box\neg C$, each chain will contain only a polynomial number of elements.

The resulting model $\mathcal{N} = \langle \Delta^\circ, <^\circ, I^\circ \rangle$ is defined as follows: Δ° is the set of all the domain elements in Δ^* which have not been removed during the chain transformation process; the relation $<^\circ$ is defined so that, for all $x, y \in \Delta^\circ$, $x <^\circ y$ if and only if $x <^* y$; the interpretation of atomic concepts in the domain elements is left unchanged. It can be shown that $\mathcal{N}$ is a multi-linear model of the KB and that the valuation in x is the same in $\mathcal{N}$ and in $\mathcal{M}^*$. Since, the number of chains in $\mathcal{N}$ is polynomial in the size of the KB and each chain has polynomial length, the resulting model $\mathcal{N}$ has polynomial size. This concludes the proof of the Small model theorem (Theorem 1).

Given the small model theorem above, we can conclude that, when evaluating the entailment, we can restrict our consideration to small models, namely, to polynomial multi-linear models of the KB. As usual, we write KB $\models \alpha$ to say that a query α holds in all the models of the KB. A query α is either a formula of the form $C(a)$ or a subsumption relation $C \sqsubseteq D$. We write KB $\models_s \alpha$ to say that α holds in all polynomial multi-linear models of the KB.

Theorem 2. *KB $\models \alpha$ if and only if KB $\models_s \alpha$.*

We can prove an upper bound on the complexity of entailment in $\mathcal{EL}^{+^\perp}\mathbf{T}$.

Theorem 3 (Complexity entailment in $\mathcal{EL}^{+^\perp}\mathbf{T}$). *The problem of deciding whether KB $\models \alpha$ is in* CO-NP.

Proof. Let us consider the complementary problem of deciding whether KB $\not\models \alpha$. This problem can be solved by a nondeterministic polynomial time algorithm which guesses a model $\mathcal{N}$ of polynomial size and a domain element x of the model, and then checks in polynomial time that $\mathcal{N}$ is a model of the KB and that x falsifies α.

4 Reasoning about irrelevance in $\mathcal{EL}^{+^\bot}\mathbf{T}$

The logic $\mathcal{EL}^{+^\bot}\mathbf{T}$ allows us to capture - through cautious monotonicity [14] - some form of inheritance of typical properties among concepts. For instance, from the KB in the introduction it is possible to conclude that typical dogs that are carried by train are affectionate, i.e., $\mathbf{T}(Dog \sqcap CarriedByTrain) \sqsubseteq Affectionate$. However, there are cases in which cautious monotonicity is not strong enough to derive the intended conclusions. For instance, we would like to conclude that typical red dogs are also affectionate. As the property of being red is not a property neither of all dogs, nor of typical dogs, cautious monotonicity is not applicable to conclude that typical red dogs are affectionate. This is the problem of irrelevance. As the color of a dog is irrelevant with respect to the property of being affectionate, and typical dogs are affectionate, we would like anyhow to conclude that $\mathbf{T}(Dog \sqcap Red) \sqsubseteq Affectionate$. To allow this form of inheritance among concepts, we can introduce default rules like the following:

$$\frac{\mathbf{T}(Dog) \sqsubseteq Affectionate \quad : \quad \mathbf{T}(Dog \sqcap Red) \sqsubseteq Affectionate}{\mathbf{T}(Dog \sqcap Red) \sqsubseteq Affectionate}$$

If typical dogs are affectionate, and *it is consistent* to assume that typical red dogs are affectionate, then we can conclude that typical red dogs are affectionate.

In general, if $C_1 \sqsubseteq C_2$, we expect that all the defeasible properties of C_2 are inherited by C_1, unless they are inconsistent with other properties of C_1. In the following, we enforce this requirement for all concepts C_2 minimally subsuming C_1, where C_1 and C_2 are any concepts occuring in the normalized KB. Let D be any concept occurring in the KB as the right hand side of an inclusion $T(C) \sqsubseteq D$. We introduce the following default rule:

$$\frac{\mathbf{T}(C_2) \sqsubseteq D \quad : \quad \mathbf{T}(C_1) \sqsubseteq D}{\mathbf{T}(C_1) \sqsubseteq D}$$

To allow default rules, we extend the KB to a defeasible KB including, in addition to an ABox and a TBox, a defeasible part consisting of a set of default rules. Observe that, as a difference with terminological default theories in [3], here prerequisites, justifications and consequents of default rules are inclusions, rather than concept terms. As defaults are normal, existence of an extension is guaranteed. Multiple extensions are possible due to multiple inheritance. As C_1, C_2 and D are subformulas of the initial KB, a polynomial number of default rules (namely, $O(n^3)$ rules) is needed. Given the complexity of entailment and satisfiability in $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ in the previous section, the verification that an inclusion (for instance, $\mathbf{T}(Dog \sqcap Red) \sqsubseteq Affectionate$) holds in an extension of a defeasible KB (credulous reasoning) is in Σ_2^p, while the verification that an inclusion holds in all the extensions of a defeasible KB (skeptical reasoning) is in Π_2^p.

5 Conclusions and related work

We have introduced the description logic $\mathcal{EL}^{+^{\bot}}\mathbf{T}$, obtained by extending $\mathcal{EL}^{+^{\bot}}$ with a tipicality operator $\mathbf{T}$ intended to select the "most normal" instances of a concept. Whereas for $\mathcal{ALC} + \mathbf{T}$ deciding satisfiability (subsumption) is EXPTIME complete (see [11]), we have shown here that for $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ the complexity is significantly smaller, namely it reduces to NP for satisfiability (and CO-NP for subsumption). This result is obtained by a "small" model property (of a particular kind: multi-linear) that fails for the whole $\mathcal{ALC} + \mathbf{T}$ as well as for $\mathcal{ALC}$. We believe that this bound is also a lower bound, but we have not proved it so far. Although validity/satisfiability for KLM logic $\mathbf{P}$ is known to be (co)NP hard, in $\mathcal{EL}^{+^{\bot}}\mathbf{T}$, we can only directly encode nonmonotonic assertions $A \mid\!\sim B$ where A is a conjunction of atoms and B is either an atom or $\bot$. As far as we know, the complexity of this fragment of $\mathbf{P}$ is unknown. Thus a lower bound for $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ cannot be obtained from known results about KLM logic $\mathbf{P}$.

The logic $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ in itself is not sufficient for prototypical reasoning and inheritance with exceptions, in particular we need a stronger (nonmonotonic) mechanism to cope with the problem known as *irrelevance*. Concerning the example of the introduction, we would like to conclude that typical red dogs are affectionate, since the color of a dog is irrelevant with respect to the property of being affectionate. However, as the property of being red is not a property neither of all dogs, nor of typical dogs, in $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ we are not able to conclude $\mathbf{T}(Dog \ \sqcap \ Red) \sqsubseteq Affectionate$. One possibility is to consider a stronger (nonmonotonic) entailment relation $\mathcal{EL}^{+^{\bot}}\mathbf{T}_{min}$ determined by restricting the entailment of $\mathcal{EL}^{+^{\bot}}\mathbf{T}$ to "minimal models", as defined in [10] for $\mathcal{ALC} + \mathbf{T}$. Intuitively, minimal models are those that maximise "typical instances" of a concept. As shown in [10], for $\mathcal{ALC} + \mathbf{T}_{min}$, minimal entailment can be decided in CO-NEXP$^{\text{NP}}$. We believe that for $\mathcal{EL}^{+^{\bot}}\mathbf{T}_{min}$ we can obtain a smaller complexity upper bound on the base of the results presented here.

Several approaches to handle prototypical reasoning and inheritance with exceptions in DL have been proposed in the literature, all of them are based on the integration of DLs with some nonmonotonic reasoning mechanism: either default logic (see [3, 15, 4]), or autoepistemic logic (see [7] and [13] for some recents developmentes) containing two epistemic operators, or finally circuscription (see [6] and [5]). As already observed, the use of normal defaults proposed in the previous section is rather different from [3]: we use defaults for handling irrelevant data and not to ascribe prototypical properties to individuals. Circumscription can model inheritance with exeception by means of a set of abnormability predicates. In [5] circumscription is applied to low complexity DLs. Concerning $\mathcal{EL}^{\bot}$

the authors have shown that all reasoning tasks remain ExpTime-hard, whereas they fall in the second level of the polynomial hierarchy for a restricted type of KB called *left local*. In future work we will deal with the precise relation between our **T**-DLs with the other nonmonotonic extensions of DLs mentioned above, notably with circumscription. In this setting, a natural question is to compare $\mathcal{EL}^{+^{\perp}}\mathbf{T}_{min}$ with circumscribed $\mathcal{EL}^{\perp}$ and see whether we get the same complexity bounds or not.

Acknowledgements. The work has been partially supported by Regione Piemonte, Project ICT4Law - *ICT Converging on Law: Next Generation Services for Citizens, Enterprises, Public Administration and Policymakers*".

References

1. F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *LTCS-Report LTCS-05-01, Inst. for Theoretical Computer Science, TU Dresden, http://lat.inf.tudresden.de/research/reports.html*, 2005.
2. F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc. of IJCAI'05, Professional Book Center* pp. 364-369, 2005.
3. F. Baader and B. Hollunder. Embedding defaults into terminological knowledge representation formalisms. *J. Autom. Reasoning*, 14(1):149–180, 1995.
4. F. Baader and B. Hollunder. Priorities on defaults with prerequisites, and their application in treating specificity in terminological default logic. *J. of Automated Reasoning (JAR)*, 15(1):41–68, 1995.
5. P. Bonatti, M. Faella, and L. Sauro. Defeasible inclusions in low-complexity dls: Preliminary notes. In *Proc. of IJCAI 2009*, 2009.
6. P. A. Bonatti, C. Lutz, and F. Wolter. Description logics with circumscription. In *Proc. of KR*, pages 400–410, 2006.
7. F. M. Donini, D. Nardi, and R. Rosati. Description logics of minimal knowledge and negation as failure. *ACM Trans. Comput. Log.*, 3(2):177–225, 2002.
8. T. Eiter, T. Lukasiewicz, R. Schindlauer, and H. Tompits. Combining answer set programming with description logics for the semantic web. In *KR 2004, 141-151.*
9. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Preferential Description Logics. In LPAR 2007. LNAI 4790, pp. 257-272, 2007.
10. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Reasoning About Typicality in Preferential Description Logics. In JELIA 2008. LNAI 5293, pp. 192-205, 2008.
11. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. On Extending Description Logics for Reasoning About Typicality: a First Step. In *Technical Report 116/09, Dip. di Informatica, Univ. di Torino*, 2009.
12. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Prototypical reasoning with low complexity Description Logics: Preliminary results. In LPNMR 2009, LNAI 5753, 430-436, 2009.
13. P. Ke and U. Sattler. Next Steps for Description Logics of Minimal Knowledge and Negation as Failure. In DL2008, 2008.
14. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44(1-2):167–207, 1990.
15. U. Straccia. Default inheritance reasoning in hybrid kl-one-style logics. In *Proc. of IJCAI*, pages 676–681, 1993.