

Prototypical reasoning with low complexity

Description Logics: preliminary results

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Abstract. We present an extension $\mathcal{EL}^{++}\mathbf{T}$ of the description logic \mathcal{EL}^{++} for reasoning about prototypical properties and inheritance with exceptions. $\mathcal{EL}^{++}\mathbf{T}$ is obtained by adding to \mathcal{EL}^{++} a typicality operator \mathbf{T} , which is intended to select the “typical” instances of a concept. In $\mathcal{EL}^{++}\mathbf{T}$ knowledge bases may contain inclusions of the form “ $\mathbf{T}(C)$ is subsumed by P ”, expressing that typical C -members have the property P . We show that the problem of entailment in $\mathcal{EL}^{++}\mathbf{T}$ is in co-NP.

1 Introduction

In Description Logics (DLs) the need of representing prototypical properties and of reasoning about defeasible inheritance of such properties naturally arises. The traditional approach is to handle defeasible inheritance by integrating some kind of nonmonotonic reasoning mechanism. This has led to study nonmonotonic extensions of DLs [2–6, 12]. However, finding a suitable nonmonotonic extension for inheritance with exceptions is far from being obvious.

In this work we introduce a defeasible extension of the description logic \mathcal{EL}^{++} called $\mathcal{EL}^{++}\mathbf{T}$, continuing the investigation started in [7], where we extended the logic \mathcal{ALC} with a typicality operator \mathbf{T} . The intended meaning of the operator \mathbf{T} is that, for any concept C , $\mathbf{T}(C)$ singles out the instances of C that are considered as “typical” or “normal”. Thus assertions as “typical football players love football” are represented by $\mathbf{T}(\text{FootballPlayer}) \sqsubseteq \text{FootballLover}$. The semantics of the typicality operator \mathbf{T} turns out to be strongly related to the semantics of nonmonotonic entailment in KLM logic \mathbf{P} [11].

In our setting, we assume that the TBox element of a KB comprises, in addition to the standard concept inclusions, a set of inclusions of the type $\mathbf{T}(C) \sqsubseteq D$ where D is a concept not mentioning \mathbf{T} . For instance, a KB may contain: $\mathbf{T}(\text{Dog}) \sqsubseteq \text{Affectionate}$; $\mathbf{T}(\text{Dog}) \sqsubseteq \text{CarriedByTrain}$; $\mathbf{T}(\text{Dog} \sqcap \text{PitBull}) \sqsubseteq \text{NotCarriedByTrain}$; $\text{CarriedByTrain} \sqcap \text{NotCarriedByTrain} \sqsubseteq \perp$, corresponding to the assertions: typically dogs are affectionate, normally dogs can be transported by train, whereas typically a dog belonging to the race of pitbull cannot (since pitbulls are considered as reactive dogs); the fourth inclusion represents the disjointness of the two concepts *CarriedByTrain* and *NotCarriedByTrain*. Notice that, in standard DLs, replacing the second and the third inclusion with $\text{Dog} \sqsubseteq \text{CarriedByTrain}$ and $\text{Dog} \sqcap \text{PitBull} \sqsubseteq \text{NotCarriedByTrain}$, respectively, we would simply get that there are not pitbull dogs, thus the KB would collapse.

This collapse is avoided as we do not assume that \mathbf{T} is monotonic, that is to say $C \sqsubseteq D$ does not imply $\mathbf{T}(C) \sqsubseteq \mathbf{T}(D)$.

By the properties of \mathbf{T} , some inclusions are entailed by the above KB, as for instance $\mathbf{T}(Dog \sqcap CarriedByTrain) \sqsubseteq Affectionate$. In our setting we can also use the \mathbf{T} operator to state that some domain elements are typical instances of a given concept. For instance, an ABox may contain either $\mathbf{T}(Dog)(fido)$ or $\mathbf{T}(Dog \sqcap PitBull)(fido)$. In the two cases, the expected conclusions are entailed: $CarriedByTrain(fido)$ and $NotCarriedByTrain(fido)$, respectively.

In this work, we present some preliminary results on *low complexity* Description Logics extended with the typicality operator \mathbf{T} . In particular we focus on the logic $\mathcal{EL}^{+\perp}$ of the well known \mathcal{EL} family. The logics of the \mathcal{EL} family allow for conjunction (\sqcap) and existential restriction ($\exists R.C$). Despite their relatively low expressivity, a renewed interest has recently emerged for these logics. Indeed, theoretical results have shown that \mathcal{EL} has better algorithmic properties than its counterpart \mathcal{FL}_0 , which allows for conjunction and value restriction ($\forall R.C$). Also, it has turned out that the logics of the \mathcal{EL} family are relevant for several applications, in particular in the bio-medical domain; for instance, medical terminologies, such as GALEN, SNOMED, and the Gene Ontology used in bioinformatics, can be formalized in small extensions of \mathcal{EL} .

We present some results about the complexity of $\mathcal{EL}^{+\perp}\mathbf{T}$. We show that, given an $\mathcal{EL}^{+\perp}\mathbf{T}$ KB, if it is satisfiable, then there is a *small* model whose size is polynomial in the size of KB. The construction of the model exploits the facts that (1) it is possible to reuse the same domain element (instance of a concept C) to fulfill existential formulas $\exists r.C$ w.r.t. domain elements; (2) we can restrict our attention to a class of models in which the preference relation $<$ is multi-linear and polynomial, that is it determines a set of disjoint chains of elements of polynomial length. The construction of the model allows us to conclude that the problem of deciding entailment in $\mathcal{EL}^{+\perp}\mathbf{T}$ is in CO-NP.

Technical details and proofs can be found in the accompanying report [10].

2 The Logic $\mathcal{EL}^{+\perp}\mathbf{T}$

We consider an alphabet of concept names \mathcal{C} , of role names \mathcal{R} , and of individuals \mathcal{O} . The language \mathcal{L} of the logic $\mathcal{EL}^{+\perp}\mathbf{T}$ is defined by distinguishing *concepts* and *extended concepts* as follows: (Concepts) $A \in \mathcal{C}$, \top , and \perp are *concepts* of \mathcal{L} ; if $C, D \in \mathcal{L}$ and $r \in \mathcal{R}$, then $C \sqcap D$ and $\exists r.C$ are *concepts* of \mathcal{L} . (Extended concepts) if C is a concept, then C and $\mathbf{T}(C)$ are extended concepts of $\mathcal{EL}^{+\perp}\mathbf{T}$. A knowledge base is a pair (TBox, ABox). TBox contains (i) a finite set of GCIs $C \sqsubseteq D$, where C is an extended concept (either C' or $\mathbf{T}(C')$), and D is a concept, and (ii) a finite set of role inclusions (RIs) $r_1 \circ r_2 \circ \dots \circ r_n \sqsubseteq r$. ABox contains expressions of the form $C(a)$ and $r(a, b)$ where C is an extended concept, $r \in \mathcal{R}$, and $a, b \in \mathcal{O}$.

In order to provide a semantics to the operator \mathbf{T} , we extend the definition of a model used in “standard” terminological logic $\mathcal{EL}^{+\perp}$:

Definition 1 (Semantics of \mathbf{T}). A model \mathcal{M} is any structure $\langle \Delta, <, I \rangle$, where Δ is the domain; $<$ is an irreflexive and transitive relation over Δ , and satisfies the following Smoothness Condition: for all $S \subseteq \Delta$, for all $a \in S$, either $a \in \text{Min}_{<}(S)$ or $\exists b \in \text{Min}_{<}(S)$ such that $b < a$, where $\text{Min}_{<}(S) = \{a : a \in S \text{ and } \nexists b \in S \text{ s.t. } b < a\}$. I is the extension function that maps each extended concept C to $C^I \subseteq \Delta$, and each role r to a $r^I \subseteq \Delta^I \times \Delta^I$. For concepts of \mathcal{ALC} , C^I is defined in the usual way. For the \mathbf{T} operator: $(\mathbf{T}(C))^I = \text{Min}_{<}(C^I)$. A model satisfying a KB $(TBox, ABox)$ is defined as usual. Moreover, we assume the unique name assumption.

Notice that the meaning of \mathbf{T} can be split into two parts: for any a of the domain Δ , $a \in (\mathbf{T}(C))^I$ just in case (i) $a \in C^I$, and (ii) there is no $b \in C^I$ such that $b < a$. In order to isolate the second part of the meaning of \mathbf{T} , we introduce a new modality \square . The basic idea is simply to interpret the preference relation $<$ as an accessibility relation. By the Smoothness Condition, it turns out that \square has the properties as in Gödel-Löb modal logic of provability \mathbf{G} . The interpretation of \square in \mathcal{M} is as follows: $(\square C)^I = \{a \in \Delta \mid \text{for every } b \in \Delta, \text{ if } b < a \text{ then } b \in C^I\}$. We have that a is a typical instance of C ($a \in (\mathbf{T}(C))^I$) iff $a \in C^I$ and, for all $b < a$, $b \notin C^I$, namely we have that $a \in (\mathbf{T}(C))^I$ iff $a \in (C \square \square \neg C)^I$. From now on, we consider $\mathbf{T}(C)$ as an abbreviation for $C \square \square \neg C$. The Smoothness Condition ensures that typical elements of C^I exist whenever $C^I \neq \emptyset$, by preventing infinitely descending chains of elements.

3 Complexity of $\mathcal{EL}^{+\perp} \mathbf{T}$

In order to give a complexity upper bound for the logic $\mathcal{EL}^{+\perp} \mathbf{T}$, we have shown that, given a model $\mathcal{M} = \langle \Delta, <, I \rangle$ of a KB, we can build a *small* model of KB whose size is polynomial in the size of the KB.

Theorem 1 (Small model theorem). Let $KB = (TBox, ABox)$ be an $\mathcal{EL}^{+\perp} \mathbf{T}$ knowledge base. For all models $\mathcal{M} = \langle \Delta, <, I \rangle$ of KB and all $x \in \Delta$, there exists a model $\mathcal{N} = \langle \Delta^\circ, <^\circ, I^\circ \rangle$ of KB such that (i) $x \in \Delta^\circ$, (ii) for all $\mathcal{EL}^{+\perp} \mathbf{T}$ concepts C , $x \in C^I$ iff $x \in C^{I^\circ}$, and (iii) $|\Delta^\circ|$ is polynomial in the size of KB.

Due to space limitations, here we only give a sketch of the proof, whose details can be found in [10]. First of all, in order to reduce the size of the model, we cut a portion of it that includes x . We build a model \mathcal{M}' by means of the following construction. For each atomic concept $C \in \mathcal{C}$ and for each role $r \in \mathcal{R}$ we let $S(C)$ and $R(r)$ be the mappings computed by the algorithm defined in [1] to compute subsumption by means of completion rules. As usual, for a given individual a in the ABox, we write a^I to denote the element of Δ corresponding to the extension of a in \mathcal{M} . We make use of three sets of elements: Δ_0 will be part of the domain of the model being constructed, and it contains a portion of the domain Δ of the initial model. All elements introduced in the domain must be processed in order to satisfy the existential formulas. *Unres* is used to keep track of not yet processed elements. Finally, Δ_1 is a set of elements that

will belong to the domain of the constructed model. Each element w_C of Δ_1 is created for one atomic concept C , and is used to satisfy all existential formulas $\exists r.C$ throughout the model. In the following by w_C we mean the world of Δ_1 which is added for the atomic concept C . We provide an algorithmic description of the construction of model \mathcal{M}' from the given model \mathcal{M} . Observe that \mathcal{M} can be an infinite model.

1. $\Delta_0 := \{x\} \cup \{a^I \in \Delta \mid a \text{ occurs in the ABox}\}$
2. $Unres := \{x\} \cup \{a^I \in \Delta \mid a \text{ occurs in the ABox}\}$
3. $\Delta_1 := \emptyset$
4. **while** $Unres \neq \emptyset$ **do**
5. extract one y from $Unres$
6. **for each** $\exists r.C$ occurring in KB s.t. $y \in (\exists r.C)^I$ **do**
7. **if** $\nexists w_C \in \Delta_1$ **then**
8. choose $w \in \Delta$ s.t. $(y, w) \in r^I$ and $w \in C^I$
9. $\Delta_0 := \Delta_0 \cup \{w\}$
10. $Unres := Unres \cup \{w\}$
11. create a new element w_C associated with C
12. $\Delta_1 := \Delta_1 \cup \{w_C\}$
13. add $w <' w_C$
14. add (y, w_C) to $r^{I'}$
15. **else**
16. add (y, w_C) to $r^{I'}$
17. **for each** $y_i \in \Delta$ such that $y_i < y$ **do**
18. $\Delta_0 := \Delta_0 \cup \{y_i\}$
19. $Unres := Unres \cup \{y_i\}$
20. **for each** $w_C, w_D \in \Delta_1$ with $C \neq D$ **do**
21. **if** $(C, D) \in R(r)$ **then** add (w_C, w_D) to $r^{I'}$

The model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ is defined as follows:

- $\Delta' = \Delta_0 \cup \Delta_1$
- we extend $<'$ computed by the algorithm by adding $u <' v$ if $u < v$, for each $u, v \in \Delta'$;
- the extension function I' is defined as follows:
 - for all atomic concepts $C \in \mathcal{C}$, for all worlds in Δ' , we define: for each $u \in \Delta_0$, we let $u \in C^{I'}$ if $u \in C^I$; for each $w_D \in \Delta_1$, we let $w_D \in C^{I'}$ if $C \in S(D)$.
 - for all roles r , we extend $r^{I'}$ constructed by the algorithm by means of the following role closure rules: for all inclusions $r \sqsubseteq s \in \text{TBox}$, if $(u, v) \in r^{I'}$ then add (u, v) to $s^{I'}$; for all inclusions $r_1 \circ r_2 \sqsubseteq s \in \text{TBox}$, if $(u, v) \in r_1^{I'}$ and $(v, w) \in r_2^{I'}$ then add (u, w) to $s^{I'}$.
 - I' is extended so that it assigns a^I to each individual a in the ABox.

\mathcal{M}' is not guaranteed to have polynomial size in the KB because in line 18 we add an element y_i for each $y_i < y$, then the size of Δ_0 may be arbitrarily large. For this reason, we refine our construction in order to build a *multi-linear* model, that we will be able to further refine in order to obtain a model of polynomial size. Intuitively, a model is multi-linear if the relation $<$ forms a set of chains

of domain elements, that is, for every u, v, z of the domain, we have that: (i) if $u < z$ and $v < z$ and $u \neq v$, then $u < v$ or $v < u$; (ii) if $z < u$ and $z < v$ and $u \neq v$, then $u < v$ or $v < u$. The proof is ended by constructing a model $\mathcal{N} = \langle \Delta^\circ, <^\circ, I^\circ \rangle$ whose domain has polynomial size in the size of KB. The multi-linear model contains a polynomial number of linear chains of domain elements that can be reduced to finite chains of polynomial length. The idea is as follows. Let us consider a chain w_0, w_1, w_2, \dots in the multi-linear model. We can observe that, given w_i and w_j in the chain such that $w_i < w_j$, the set of negated box formulas $\neg \Box \neg C$ of which w_i is an instance is a subset of the set of negated box formulas of which w_j is an instance. We build a new chain containing a subset of the elements of the given chain by removing from the chain some domain elements so that the number of negated box formulas strictly decreases along the chain. We keep on removing domain elements from the chain until, for each element w of the chain, there is at least a box formula $\Box \neg C$ of which w is an instance, while the domain element preceding w in the chain is not an instance of $\Box \neg C$. As there is only a finite polynomial number of such box formulas, we can only retain a finite polynomial number of worlds in the chain. The same transformation is applied to all the polynomial chains in the multi-linear model.

Given Theorem 1 above, when evaluating the entailment, we can restrict our consideration to small models, namely, to polynomial multi-linear models of the KB. We write $\text{KB} \models \alpha$ to say that a query α holds in all the models of the KB. A query α is either a formula of the form $C(a)$ or a subsumption relation $C \sqsubseteq D$. We write $\text{KB} \models_s \alpha$ to say that α holds in all polynomial multi-linear models of the KB. It holds that $\text{KB} \models \alpha$ if and only if $\text{KB} \models_s \alpha$. As a consequence, we can give an upper bound on the complexity of $\mathcal{EL}^{+\perp} \mathbf{T}$:

Theorem 2. *In $\mathcal{EL}^{+\perp} \mathbf{T}$, the problem of deciding whether $\text{KB} \models \alpha$ is in co-NP. The problems of satisfiability of a KB and of concept satisfiability are in NP. The problems of subsumption and of instance checking are in co-NP.*

4 Conclusions and future issues

We have presented the description logic $\mathcal{EL}^{+\perp} \mathbf{T}$, that is $\mathcal{EL}^{+\perp}$ extended by a typicality operator \mathbf{T} intended to select the “most normal” instances of a concept. Whereas for $\mathcal{ALC} + \mathbf{T}$ deciding satisfiability (subsumption) is EXPTIME complete (see [9]), we have shown here that for $\mathcal{EL}^{+\perp} \mathbf{T}$ the complexity is significantly smaller, namely it reduces to NP for satisfiability (and co-NP for subsumption). This result is obtained by a “small” model property (of a particular kind: multi-linear) that fails for the whole $\mathcal{ALC} + \mathbf{T}$ as well as for \mathcal{ALC} . We believe that this bound is also a lower bound, but we have not proved it so far. Although validity/satisfiability for KLM logic \mathbf{P} is known to be (co)NP hard, in $\mathcal{EL}^{+\perp} \mathbf{T}$, we can only directly encode nonmonotonic assertions $A \sim B$ where A is a conjunction of atoms and B is either an atom or \perp . As far as we know, the complexity of this fragment of \mathbf{P} is unknown. Thus a lower bound for $\mathcal{EL}^{+\perp} \mathbf{T}$ cannot be obtained from known results about KLM logic \mathbf{P} .

The logic $\mathcal{EL}^{++}\mathbf{T}$ in itself is not sufficient for prototypical reasoning and inheritance with exceptions, in particular we need a stronger (nonmonotonic) mechanism to cope with the problem known as *irrelevance*. Concerning the example of the Introduction, we would like to conclude that typical red dogs are affectionate, since the color of a dog is irrelevant with respect to the property of being affectionate. However, as the property of being red is not a property of typical dogs (or of all dogs), in $\mathcal{EL}^{++}\mathbf{T}$ we are not able to conclude $\mathbf{T}(Dog \sqcap Red) \sqsubseteq Affectionate$. One possibility is to consider a stronger (nonmonotonic) entailment relation $\mathcal{EL}^{++}\mathbf{T}_{min}$ determined by restricting the entailment of $\mathcal{EL}^{++}\mathbf{T}$ to “minimal models”, as defined in [8] for $\mathcal{ALC} + \mathbf{T}$. Intuitively minimal models are those that maximise “typical instances” of a concept. As shown in [8], for $\mathcal{ALC} + \mathbf{T}_{min}$, minimal entailment can be decided in $\text{CO-NEXP}^{\text{NP}}$. We believe that for $\mathcal{EL}^{++}\mathbf{T}_{min}$ we can obtain a smaller complexity upper bound on the base of the results presented here.

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