Nested Sequent Calculi for Conditional Logics

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Abstract. Nested sequent calculi are a useful generalization of ordinary sequent calculi, where sequents are allowed to occur within sequents. Nested sequent calculi have been profitably employed in the area of (multi)-modal logic to obtain analytic and modular proof systems for these logics. In this work, we extend the realm of nested sequents by providing nested sequent calculi for the basic conditional logic CK and some of its significant extensions. The calculi are internal (a sequent can be directly translated into a formula), cut-free and analytic. Moreover, they can be used to design (sometimes optimal) decision procedures for the respective logics, and to obtain complexity upper bounds. Our calculi are an argument in favour of nested sequent calculi for modal logics and alike, showing their versatility and power.

1 Introduction

The recent history of the conditional logics starts with the work by Lewis [16, 17], who proposed them in order to formalize a kind of hypothetical reasoning (if A were the case then B), that cannot be captured by classical logic with material implication. One original motivation was to formalize *counterfactual sentences*, i.e. conditionals of the form "if A were the case then B would be the case", where A is false. Conditional logics have found an interest in several fields of artificial intelligence and knowledge representation. They have been used to reason about prototypical properties [10] and to model belief change [14, 12]. Moreover, conditional logics can provide an axiomatic foundation of nonmonotonic reasoning [5, 15], here a conditional $A \Rightarrow B$ is read as "in normal circumstances if A then B". Recently, a kind of (multi)-conditional logics [3, 4] have been used to formalize epistemic change in a multi-agent setting.

Semantically, all conditional logics enjoy a possible world semantics, with the intuition that a conditional $A \Rightarrow B$ is true in a world x, if B is true in the set of worlds where A is true and that are most similar/closest/"as normal as" x. Since there are different ways of formalizing "the set of worlds similar/closest/..." to a given world, there are expectedly rather different semantics for conditional logics, from the most general selection function semantics to the stronger sphere semantics.

From the point of view of proof-theory and automated deduction, conditional logics do not have however a state of the art comparable with, say, the one of modal logics, where there are well-established alternative calculi, whose proof-theoretical and computational properties are well-understood. This is partially due to the mentioned lack of a unifying semantics; as a matter of fact the most general semantics, the *selection function* one, is of little help for proof-theory, and the preferential/sphere semantics only captures a subset of (actually rather strong) systems. Similarly to modal logics and other extensions/alternative to classical logics two types of calculi have been studied: *external* calculi which make use of labels and relations on them to import the semantics into the syntax, and *internal* calculi which stay within the language, so that a "configuration" (sequent, tableaux node...) can be directly interpreted as a formula of the language. Just to mention some work, to first stream belongs [2] proposing a calculus for (unnested) cumulative logic **C** (see below). More recently, [18] presents modular labeled calculi (of optimal complexity) for CK and some of its extensions, basing on the selection function semantics, and [13] presents modular labeled calculi for preferential logic PCL and its extensions. The latter calculi take advantage of a sort of hybrid modal translation. To the second stream belong the calculi by Gent [9] and by de Swart [21] for Lewis' logic VC and neighbours. These calculi manipulate set of formulas and provide a decision procedure, although they comprise an infinite set of rules. Very recently, some internal calculi for CK and some extensions (with any combination of MP, ID, CEM) have been proposed by Pattinson and Schröder [19]. The calculi are obtained by a general method for closing a set of rules (corresponding to Hilbert axioms) with respect to the cut rule. These calculi have optimal complexity; notice that some of the rules do not have a fixed number of premises. These calculi have been extended to preferential conditional logics [20], i.e. including cumulativity (CM) and or-axiom (CA), although the resulting systems are fairly complicated.

In this paper we begin to investigate *nested sequents* calculi for conditional logics. Nested sequents are a natural generalization of ordinary sequents where sequents are allowed to occur within sequents. However a nested sequent always corresponds to a formula of the language, so that we can think of the rules as operating "inside a formula", combining subformulas rather than just combining outer occurrences of formulas as in ordinary sequents[3]. Limiting to modal logics, nested calculi have been provided, among others, for modal logics by Brünnler [7, 6] and by Fitting [8].

In this paper we treat the basic normal conditional logic CK (its role is the same as K in modal logic) and its extensions with ID and CEM. We also consider the *flat* fragment (i.e., without nested conditionals) of CK+CSO+ID, coinciding with the logic of cumulativity **C** introduced in [15]. The calculi are rather natural, all rules have a fixed number of premises. The completeness is established by cut-elimination, whose peculiarity is that it must take into account the substitution of equivalent antecedents of conditionals (a condition corresponding to normality). The calculi can be used to obtain a decision procedure for the respective logics by imposing some restrictions preventing redundant applications of rules. In all cases, we get a PSPACE upper bound, a bound that for CK+ID and CK+CSO+ID is optimal (but not for CK+CEM that is known to be coNP). For flat CK+CSO+ID = cumulative logic **C** we also get a PSPACE bound, we are not aware of better upper bound for this logic (although we may suspect that it is not optimal). We can see the present work as a further argument in favor of nested sequents as a useful tool to provide natural, yet computationally adequate, calculi for modal extensions of classical logics.

Technical details and proofs can be found in the accompanying report [1].

2 Conditional Logics

A propositional conditional language $\mathcal{L}$ contains: - a set of propositional variables ATM; - the symbol of *false* $\bot$; - a set of connectives $\top$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\Rightarrow$. We define

[3] In this sense, they are a special kind of "deep inference" calculi by Guglielmi and colleagues.

formulas of $\mathcal{L}$ as follows: - $\bot$ and the propositional variables of ATM are *atomic formulas*; - if A and B are formulas, then $\neg A$ and $A \otimes B$ are *complex formulas*, where $\otimes \in \{\wedge, \vee, \rightarrow, \Rightarrow\}$. We adopt the *selection function semantics*, that we briefly recall here. We consider a non-empty set of possible worlds $\mathcal{W}$. Intuitively, the selection function f selects, for a world w and a formula A, the set of worlds of $\mathcal{W}$ which are *closer* to w given the information A. A conditional formula $A \Rightarrow B$ holds in a world w if the formula B holds in *all the worlds selected by f for w and A*.

Definition 1 (Selection function semantics). *A model is a triple $\mathcal{M} = \langle \mathcal{W}, f, [\ ] \rangle$ where: - $\mathcal{W}$ is a non empty set of* worlds*; - f is the* selection function $f : \mathcal{W} \times 2^{\mathcal{W}} \longrightarrow 2^{\mathcal{W}}$*; - $[\ ]$ is the* evaluation function, *which assigns to an atom $P \in ATM$ the set of worlds where P is true, and is extended to boolean formulas as usual, whereas for conditional formulas $[A \Rightarrow B] = \{w \in \mathcal{W} \mid f(w, [A]) \subseteq [B]\}$.*

We have defined f taking $[A]$ rather than A (i.e. $f(w, [A])$ rather than $f(w, A)$) as an argument; this is equivalent to define f on formulas, i.e. $f(w, A)$ but imposing that if $[A] = [A^{'}]$ in the model, then $f(w, A) = f(w, A^{'})$. This condition is called *normality*.

The semantics above characterizes the *basic conditional system*, called CK [17]. An axiomatization of CK is given by ($\vdash$ denotes provability in the axiom system):

- any axiomatization of the classical propositional calculus;
- If $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$ (Modus Ponens)
- If $\vdash A \leftrightarrow B$ then $\vdash (A \Rightarrow C) \leftrightarrow (B \Rightarrow C)$ (RCEA)
- If $\vdash (A_1 \wedge \cdots \wedge A_n) \rightarrow B$ then $\vdash (C \Rightarrow A_1 \wedge \cdots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)$ (RCK)

Other conditional systems are obtained by assuming further properties on the selection function; we consider the following standard extensions of the basic system CK:

System	Axiom	Model condition
ID	$A \Rightarrow A$	$f(w, [A]) \subseteq [A]$
CEM	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$	$\mid f(w, [A]) \mid \leq 1$
CSO	$(A \Rightarrow B) \wedge (B \Rightarrow A) \rightarrow ((A \Rightarrow C) \rightarrow (B \Rightarrow C))$	$f(w, [A]) \subseteq [B]$ and $f(w, [B]) \subseteq [A]$ implies $f(w, [A]) = f(w, [B])$

The above axiomatization is complete with respect to the semantics [17].

3 Nested Sequent Calculi $\mathcal{N}S$ for Conditional Logics

In this section we present nested sequent calculi $\mathcal{N}S$, where S is an abbreviation for CK+X, and X=$\{$CEM, ID, CEM+ID$\}$. As usual, completeness is an easy consequence of the admissibility of cut. We are also able to turn $\mathcal{N}S$ into a terminating calculus, which gives us a decision procedure for the respective conditional logics.

Definition 2. *A nested sequent Γ is defined inductively as follows: - A finite multiset of formulas is a nested sequent. - If A is a formula and Γ is a nested sequent, then $[A : \Gamma]$ is a nested sequent. - A finite multiset of nested sequents is a nested sequent.*

A nested sequent can be displayed as

$$A_1, \ldots, A_m, [B_1 : \Gamma_1], \ldots, [B_n : \Gamma_n],$$

where $n, m \geq 0$, $A_1, \ldots, A_m, B_1, \ldots, B_n$ are formulas and $\Gamma_1, \ldots, \Gamma_n$ are nested sequents. The depth $d(\Gamma)$ of a nested sequent Γ is defined as follows: - if $\Gamma = A_1 \ldots, A_n$, then $d(\Gamma) = 0$; - if $\Gamma = [A : \Delta]$, then $d(\Gamma) = 1 + d(\Delta)$ - if $\Gamma = \Gamma_1, \ldots, \Gamma_n$, then $d(\Gamma) = max(\Gamma_i)$. A nested sequent can be directly interpreted as a formula, just replace "," by $\vee$ and ":" by $\Rightarrow$. More explicitly, the interpretation of a nested sequent $A_1, \ldots, A_m, [B_1 : \Gamma_1], \ldots, [B_n : \Gamma_n]$ is inductively defined by the formula $\mathcal{F}(\Gamma) = A_1 \vee \ldots \vee A_m \vee (B_1 \Rightarrow \mathcal{F}(\Gamma_1)) \vee \ldots \vee (B_n \Rightarrow \mathcal{F}(\Gamma_n))$. For example, the nested sequent $A, B, [A : C, [B : E, F]], [A : D]$ denotes the formula $A \vee B \vee (A \Rightarrow C \vee (B \Rightarrow (E \vee F))) \vee (A \Rightarrow D)$.

The specificity of nested sequent calculi is to allow inferences that apply within formulas. In order to introduce the rules of the calculus, we need the notion of context. Intuitively a context denotes a "hole", a *unique* empty position, within a sequent that can be filled by a formula/sequent. We use the symbol $(\ )$ to denote the empty context. A context is defined inductively as follows:

Definition 3. *If Δ is a nested sequent, $\Gamma(\ ) = \Delta, (\ )$ is a context with depth $d(\Gamma(\ )) = 0$; if Δ is a nested sequent and $\Sigma(\ )$ is a context, $\Gamma(\ ) = \Delta, [A : \Sigma(\ )]$ is a context with depth $d(\Gamma(\ )) = 1 + d(\Sigma(\ ))$.*

Finally we define the result of filling "the hole" of a context by a sequent:

Definition 4. *Let $\Gamma(\ )$ be a context and Δ be a sequent, then the sequent obtained by filling the context by Δ, denoted by $\Gamma(\Delta)$ is defined as follows: - if $\Gamma(\ ) = \Lambda, (\ )$, then $\Gamma(\Delta) = \Lambda, \Delta$; - if $\Gamma(\ ) = \Lambda, [A : \Sigma(\ )]$, then $\Gamma(\Delta) = \Lambda, [A : \Sigma(\Delta)]$.*

The calculi $\mathcal{N}S$ are shown in Figure 1. As usual, we say that a nested sequent Γ is *derivable* in $\mathcal{N}S$ if it admits a *derivation*. A derivation is a tree whose nodes are nested sequents. A branch is a sequence of nodes $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ Each node Γ_i is obtained from its immediate successor Γ_{i-1} by applying *backward* a rule of $\mathcal{N}S$, having Γ_{i-1} as the conclusion and Γ_i as one of its premises. A branch is closed if one of its nodes is an instance of axioms (AX) and $(AX_\top)$, otherwise it is open. We say that a tree is closed if all its branches are closed. A nested sequent Γ has a derivation in $\mathcal{N}S$ if there is a closed tree having Γ as a root. As an example, Figure 2 shows a derivation of (an instance of) the axiom ID.

$$\Gamma(P, \neg P)\ \underset{P \in ATM}{(AX)} \qquad \Gamma(\top)\ (AX_\top) \qquad \frac{\Gamma(A) \qquad \Gamma(B)}{\Gamma(A \wedge B)}(\wedge^+) \qquad \frac{\Gamma(\neg A, \neg B)}{\Gamma(\neg(A \wedge B))}(\wedge^-)$$

$$\frac{\Gamma(A, B)}{\Gamma(A \vee B)}(\vee^+) \qquad \frac{\Gamma(\neg A) \qquad \Gamma(\neg B)}{\Gamma(\neg(A \vee B))}(\vee^-) \qquad \frac{\Gamma(\neg A, B)}{\Gamma(A \rightarrow B)}(\rightarrow^+) \qquad \frac{\Gamma(A) \qquad \Gamma(\neg B)}{\Gamma(\neg(A \rightarrow B))}(\rightarrow^-)$$

$$\frac{\Gamma(A)}{\Gamma(\neg\neg A)}(\neg) \qquad \frac{\Gamma([A : B])}{\Gamma(A \Rightarrow B)}(\Rightarrow^+) \qquad \frac{\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \qquad A, \neg A' \qquad A', \neg A}{\Gamma(\neg(A \Rightarrow B), [A' : \Delta])}(\Rightarrow^-)$$

$$\frac{\Gamma([A : \Delta, \neg A])}{\Gamma([A : \Delta])}(ID) \qquad \frac{\Gamma([A : \Delta, \Sigma], [B : \Sigma]) \qquad A, \neg B \qquad B, \neg A}{\Gamma([A : \Delta], [B : \Sigma])}(CEM)$$

Fig. 1. The nested sequent calculi $\mathcal{N}S$.

The following lemma shows that axioms can be generalized to any formula F:

Lemma 1. *Given any formula F, the sequent $\Gamma(F, \neg F)$ is derivable in $\mathcal{N}S$.*

$$\dfrac{\dfrac{\dfrac{}{[P : P, \neg P]}\,(AX)}{[P : P]}\,(ID)}{P \Rightarrow P}\,(\Rightarrow^{+})$$

Fig. 2. A derivation of the axiom ID.

The easy proof is by induction on the complexity of F.

In [19] the authors propose optimal sequent calculi for CK and its extensions by any combination of ID, MP and CEM. It is not difficult to see that the rules CK_g, $CKID_g$, $CKCEM_g$, $CKCEMID_g$ of their calculi are derivable in our calculi.

3.1 Basic structural properties of $\mathcal{N}S$

First of all, we show that weakening and contraction are height-preserving admissible in the calculi $\mathcal{N}S$. Furthermore, we show that all the rules of the calculi, with the exceptions of $(\Rightarrow^-)$ and (CEM), are height-preserving invertible. As usual, we define the height of a derivation as the height of the tree corresponding to the derivation itself.

Lemma 2 (Admissibility of weakening). *Weakening is height-preserving admissible in $\mathcal{N}S$: if $\Gamma(\Delta)$ (resp. $\Gamma([A : \Delta])$) is derivable in $\mathcal{N}S$ with a derivation of height h, then also $\Gamma(\Delta, F)$ (resp. $\Gamma([A : \Delta, F])$) is derivable in $\mathcal{N}S$ with a proof of height $h' \leq h$, where F is either a formula of a nested sequent $[B : \Sigma]$.*

The easy proof is by induction on the height of the derivation of $\Gamma(\Delta)$.

Lemma 3 (Invertibility). *All the rules of $\mathcal{N}S$, with the exceptions of $(\Rightarrow^-)$ and (CEM), are height-preserving invertible: if Γ has a derivation of height h and it is an instance of a conclusion of a rule $(\mathbf{R})$, then also Γ_i, $i = 1, 2$, are derivable in $\mathcal{N}S$ with derivations of heights $h_i \leq h$, where Γ_i are instances of the premises of $(\mathbf{R})$.*

Proof. Let us first consider the rule (ID). In this case, we can immediately conclude because the premise $\Gamma([A : \Delta, \neg A])$ is obtained by weakening, which is height-preserving admissible (Lemma 2), from $\Gamma([A : \Delta])$. For the other rules, we proceed by induction on the height of the derivation of Γ. We only show the most interesting case of $(\Rightarrow^+)$. For the base case, consider $(*)$ $\Gamma(A \Rightarrow B, \Delta)$ where either (i) $P \in \Delta$ and $\neg P \in \Delta$, i.e. $(*)$ is an instance of (AX), or (ii) $\top \in \Delta$, i.e. $(*)$ is an instance of $(AX_\top)$; we immediately conclude that also $\Gamma([A : B], \Delta)$ is an instance of either (AX) in case (i) or $(AX_\top)$ in case (ii). For the inductive step, we consider each rule ending (looking forward) the derivation of $\Gamma(A \Rightarrow B)$. If the derivation is ended by an application of $(\Rightarrow^+)$ to $\Gamma([A : B])$, we are done. Otherwise, we apply the inductive hypothesis to the premise(s) and then we conclude by applying the same rule. $\square$

It can be observed that a "weak" version of invertibility also holds for the rules $(\Rightarrow^-)$ and (CEM). Roughly speaking, if $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$, which is an instance of the conclusion of $(\Rightarrow^-)$, is derivable, then also the sequent $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$, namely the left-most premise in the rule $(\Rightarrow^-)$, is derivable too. Similarly for (CEM).

Since the rules are invertible, it follows that *contraction* is admissible, that is to say:

Lemma 4 (Admissibility of contraction). *Contraction is height-preserving admissible in $\mathcal{N}S$: if $\Gamma(F, F)$ has a derivation of height h, then also $\Gamma(F)$ has a derivation of height $h' \leq h$, where F is either a formula or a nested sequent $[A : \Sigma]$.*

3.2 Soundness of the calculi $\mathcal{N}S$

To improve readability, we slightly abuse the notation identifying a sequent Γ with its interpreting formula $\mathcal{F}(\Gamma)$, thus we shall write $A \Rightarrow \Delta$, $\Gamma \wedge \Delta$, etc. instead of $A \Rightarrow \mathcal{F}(\Gamma), \mathcal{F}(\Gamma) \wedge \mathcal{F}(\Delta)$. First of all we prove that nested inference in sound (similarly to Brünnler [7], Lemma 2.8).

Lemma 5. *Let $\Gamma(\ )$ be any context. If the formula $A_1 \wedge \ldots \wedge A_n \rightarrow B$, with $n \geq 0$, is (CK+X)-valid, then also $\Gamma(A_1) \wedge \ldots \wedge \Gamma(A_n) \rightarrow \Gamma(B)$ is (CK+X) valid.*

Proof. By induction on the depth of a context $\Gamma(\ )$. Let $d(\Gamma(\ )) = 0$, then $\Gamma = \Lambda, (\ )$. Since $A_1 \wedge \ldots \wedge A_n \rightarrow B$ is valid, by propositional reasoning, we have that also $(\Lambda \vee A_1) \wedge \ldots (\Lambda \vee A_n) \rightarrow (\Lambda \vee B)$ is valid, that is $\Gamma(A_1) \wedge \ldots \wedge \Gamma(A_n) \rightarrow \Gamma(B)$ is valid. Let $d(\Gamma(\ )) > 0$, then $\Gamma(\ ) = \Delta, [C : \Sigma(\ )]$. By inductive hypothesis, we have that $\Sigma(A_1) \wedge \ldots \wedge \Sigma(A_n) \rightarrow \Sigma(B)$ is valid. By (RCK), we obtain that also $(C \Rightarrow \Sigma(A_1)) \wedge \ldots \wedge (C \Rightarrow \Sigma(A_n)) \rightarrow (C \Rightarrow \Sigma(B))$ is valid. Then, we get that $(\Lambda \vee (C \Rightarrow \Sigma(A_1))) \wedge \ldots \wedge (\Lambda \vee (C \Rightarrow \Sigma(A_n))) \rightarrow (\Lambda \vee (C \Rightarrow \Sigma(B)))$ is also valid, that is $\Gamma(A_1) \wedge \ldots \wedge \Gamma(A_n) \rightarrow \Gamma(B)$ is valid. □

Theorem 1. *If Γ is derivable in $\mathcal{N}S$, then Γ is valid.*

Proof. By induction on the height of the derivation of Γ. If Γ is an axiom, that is $\Gamma = \Gamma(P, \neg P)$, then trivially $P \vee \neg P$ is valid; by Lemma 5 (case $n = 0$), we get $\Gamma(P, \neg P)$ is valid. Similarly for $\Gamma(\top)$. Otherwise Γ is obtained by a rule (**R**):
- (**R**) is a propositional rule, say $\Gamma_1, \Gamma_2/\Delta$, we first prove that $\Gamma_1 \wedge \Gamma_2 \rightarrow \Delta$ is valid. All rules are easy, since for the empty context they are nothing else than trivial propositional tautologies. We can then use Lemma 5 to propagate them to any context. For instance, let the rule (**R**) be ($\neg\vee$). Then $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$ and, by the previous lemma, we get that $\Gamma(\neg A) \wedge \Gamma(\neg B) \rightarrow \Gamma(\neg(A \vee B))$. Thus if Γ is derived by (**R**) from Γ_1, Γ_2, we use the inductive hypothesis that Γ_1 and Γ_2 are valid and the above fact to conclude.
- (**R**) is ($\Rightarrow^+$): trivial by inductive hypothesis.
- (**R**) is ($\Rightarrow^-$) then $\Gamma = \Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ is derived from (i) $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$, (ii) $\neg A, A'$, (iii) $\neg A', A$. By inductive hypothesis we have that $A \leftrightarrow A'$ is valid. We show that also $(*)$ $[\neg(A \Rightarrow B) \vee (A' \Rightarrow (\Delta \vee \neg B))] \rightarrow [\neg(A \Rightarrow B) \vee (A' \Rightarrow \Delta)]$ is valid, then we apply Lemma 5 and the inductive hypothesis to conclude. To prove (*), by (RCK) we have that the following is valid: $[(A' \Rightarrow B) \wedge (A' \Rightarrow (\Delta \vee \neg B))] \rightarrow (A' \Rightarrow \Delta)$. Since $A \leftrightarrow A'$ is valid, by (RCEA) we get that $(A \Rightarrow B) \rightarrow (A' \Rightarrow B)$ is valid, so that also $(A \Rightarrow B) \rightarrow ((A' \Rightarrow (\Delta \vee \neg B)) \rightarrow (A' \Rightarrow \Delta))$ is valid, then we conclude by propositional reasoning.
- (**R**) is (ID), then $\Gamma = \Gamma([A : \Delta])$ is derived from $\Gamma([A : \Delta, \neg A])$. We show that $(A \Rightarrow (\Delta \vee \neg A)) \rightarrow (A \Rightarrow \Delta)$ is valid in CK+ID, then we conclude by Lemma 5 and by the inductive hypothesis. The mentioned formula is derivable: by (RCK) we obtain $(A \Rightarrow A) \rightarrow ((A \Rightarrow (\Delta \vee \neg A)) \rightarrow (A \Rightarrow \Delta))$ so that we conclude by (ID).

- (**R**) is (CEM), thus $\Gamma = \Gamma([A : \Delta], [A', \Sigma])$ and it is derived from (i) $\Gamma([A : \Delta, \Sigma], [A' : \Sigma])$, (ii) $\neg A, A'$, (iii) $\neg A', A$. By inductive hypothesis $A \leftrightarrow A'$ is valid. We first show that $(**)$ $(A \Rightarrow (\Delta \vee \Sigma)) \rightarrow ((A \Rightarrow \Delta) \vee (A' \Rightarrow \Sigma))$. Then we conclude as before by Lemma 5 and inductive hypothesis. To prove (**), we notice that the following is derivable by (RCK): $(A \Rightarrow (\Delta \vee \Sigma)) \rightarrow [(A \Rightarrow \neg\Delta) \rightarrow (A \Rightarrow \Sigma)]$. By (CEM), the following is valid: $(A \Rightarrow \Delta) \vee (A \Rightarrow \neg\Delta)$. Thus we get that $(A \Rightarrow (\Delta \vee \Sigma)) \rightarrow [(A \Rightarrow \Delta) \vee (A \Rightarrow \Sigma)]$ is valid. Since $A \leftrightarrow A'$ is valid, (by RCEA) we have that also $(A \Rightarrow \Sigma) \rightarrow (A' \Rightarrow \Sigma)$ is valid, obtaining (**). □

3.3 Completeness of the calculi $\mathcal{N}S$

Completeness is an easy consequence of the admissibility of the following rule *cut*:

$$\frac{\Gamma(F) \qquad \Gamma(\neg F)}{\Gamma(\emptyset)}\,(cut)$$

where F is a formula. The standard proof of admissibility of cut proceeds by a double induction over the complexity of F and the sum of the heights of the derivations of the two premises of (cut), in the sense that we replace one cut by one or several cuts on formulas of smaller complexity, or on sequents derived by shorter derivations. However, in $\mathcal{N}S$ the standard proof does not work in the following case, in which the cut formula F is a conditional formula $A \Rightarrow B$:

$$\dfrac{\dfrac{(1)\ \Gamma([A : B], [A' : \Delta])}{(3)\ \Gamma(A \Rightarrow B, [A' : \Delta])}\,(\Rightarrow^+) \qquad \dfrac{(2)\ \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \qquad A, \neg A' \qquad A', \neg A}{\Gamma(\neg(A \Rightarrow B), [A' : \Delta])}\,(\Rightarrow^+)}{\Gamma([A' : \Delta])}\,(cut)$$

Indeed, even if we apply the inductive hypothesis on the heights of the derivations of the premises to cut (2) and (3), obtaining (modulo weakening, which is admissible by Lemma 2) a derivation of $(2')$ $\Gamma([A' : \Delta, \neg B], [A' : \Delta])$, we cannot apply the inductive hypothesis on the complexity of the cut formula to $(2')$ and $(1')$ $\Gamma([A : \Delta, B], [A' : \Delta])$ (obtained from (1) again by weakening). Such an application would be needed in order to obtain a proof of $\Gamma([A' : \Delta], [A' : \Delta])$ and then to conclude $\Gamma([A' : \Delta])$ since contraction is admissible (Lemma 4).

In order to prove the admissibility of cut for $\mathcal{N}S$, we proceed as follows. First, we show that if $A, \neg A'$ and $A', \neg A$ are derivable, then if $\Gamma([A : \Delta])$ is derivable, then $\Gamma([A' : \Delta])$, obtained by replacing $[A : \Delta]$ with $[A' : \Delta]$, is also derivable. We prove that cut is admissible by "splitting" the notion of cut in two propositions:

Theorem 2. *In $\mathcal{N}S$, the following propositions hold:* (A) *If $\Gamma(F)$ and $\Gamma(\neg F)$ are derivable, so is $\Gamma(\emptyset)$, i.e. (cut) is admissible in $\mathcal{N}S$;* (B) *if (I) $\Gamma([A : \Delta])$, (II) $A, \neg A'$ and (III) $A', \neg A$ are derivable, then $\Gamma([A' : \Delta])$ is derivable.*

Proof. The proof of both is by mutual induction. To make the structure of the induction clear call: $Cut(c, h)$ the property (A) for any Γ and any formula F of complexity c and such that the sum of the heights of derivation of the premises is h. Similarly call $Sub(c)$ the assertion that (B) holds for any Γ and any formula A of complexity c. Then we show that following facts:

(i) $\forall h\ Cut(0, h)$
(ii) $\forall c\ Cut(c, 0)$
(iii) $\forall c' < c\ Sub(c) \rightarrow (\forall c' < c\ \forall h'\ Cut(c', h') \wedge \forall h' < h\ Cut(c, h') \rightarrow Cut(c, h))$
(iv) $\forall h\ Cut(c, h) \rightarrow Sub(c)$

This will prove that $\forall c\ \forall h Cut(c, h)$ and $\forall c\ \ Sub(c)$, that is (A) and (B) hold. The proof of (iv) (that is that $Sub(c)$ holds) in itself is by induction on the height h of the derivation of the premise (I) of (B). To save space, we only present the most interesting cases.

Inductive step for (A): we distinguish the following two cases:

• (case 1) the last step of *one* of the two premises is obtained by a rule (**R**) in which F is *not* the principal formula. This case is standard, we can permute (**R**) over the cut, i.e. we cut the premise(s) of (**R**) and then we apply (**R**) to the result of cut.

• (case 2) F is the principal formula in the last step of *both* derivations of the premises of the cut inference. There are seven subcases: F is introduced a) by $(\wedge^-)$ - $(\wedge^+)$, b) by $(\vee^-)$ - $(\vee^+)$, c) by $(\rightarrow^-)$ - $(\rightarrow^+)$, d) by $(\Rightarrow^-)$ - $(\Rightarrow^+)$, e) by $(\Rightarrow^-)$ - (ID), f) by $(\Rightarrow^-)$ - (CEM), g) by (CEM) - (ID). We only show the most interesting case d), where the derivation is as follows:

$$\dfrac{\dfrac{(1)\ \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \quad A, \neg A' \quad A', \neg A}{\Gamma(\neg(A \Rightarrow B), [A' : \Delta])}(\Rightarrow^-) \qquad \dfrac{(2)\ \Gamma([A : B], [A' : \Delta])}{(3)\ \Gamma(A \Rightarrow B, [A' : \Delta])}(\Rightarrow^+)}{\Gamma([A' : \Delta])}(cut)$$

First of all, since we have proofs for $A, \neg A'$ and for $A', \neg A$ and $cp(A) < cp(A \Rightarrow B)$, we can apply the inductive hypothesis for (B) to (2), obtaining a proof of $(2')\ \Gamma([A' : B], [A' : \Delta])$. By Lemma 2, from (3) we obtain a proof of at most the same height of $(3')\ \Gamma(A \Rightarrow B, [A' : \Delta, \neg B])$. We can then conclude as follows: we first apply the inductive hypothesis on the height for (A) to cut (1) and $(3')$, obtaining a derivation of $(4)\ \Gamma([A' : \Delta, \neg B])$. By Lemma 2, we have also a derivation of $(4')\ \Gamma([A' : \Delta, \neg B], [A' : \Delta])$. Again by Lemma 2, from $(2')$ we obtain a derivation of $(2'')\ \Gamma([A' : \Delta, B], [A' : \Delta])$. We then apply the inductive hypothesis on the complexity of the cut formula to cut $(2'')$ and $(4')$, obtaining a proof of $\Gamma([A' : \Delta], [A' : \Delta])$, from which we conclude since contraction is admissible (Lemma 4).

Inductive step for (B) (that is statement (iv) of the induction): we have to consider all possible rules ending (looking forward) the derivation of $\Gamma([A : \Delta])$. We only show the most interesting case, when $(\Rightarrow^-)$ is applied by using $[A : \Delta]$ as principal formula. The derivation ends as follows:

$$\dfrac{(1)\ \Gamma(\neg(C \Rightarrow D), [A : \Delta, \neg D]) \qquad (2)\ C, \neg A \qquad (3)\ A, \neg C}{\Gamma(\neg(C \Rightarrow D), [A : \Delta])}(\Rightarrow^-)$$

We can apply the inductive hypothesis to (1) to obtain a derivation of $(1')\ \Gamma(\neg(C \Rightarrow D), [A' : \Delta, \neg D])$. Since weakening is admissible (Lemma 2), from (II) we obtain a derivation of $(II')\ C, A, \neg A'$, from (III) we obtain a derivation of $(III')\ A', \neg A, \neg C$. Again by weakening, from (2) and (3) we obtain derivations of $(2')\ C, \neg A, \neg A'$ and $(3')\ A', A, \neg C$, respectively. We apply the inductive hypothesis of (A) that is that cut holds for the formula A (of a given complexity c) and conclude as follows:

$$\dfrac{(1')\ \Gamma(\neg(C\Rightarrow D),[A':\Delta,\neg D]) \qquad \dfrac{(II')\ C,A,\neg A' \quad (2')\ C,\neg A,\neg A'}{C,\neg A'}\,(cut) \qquad \dfrac{(III')\ A',\neg A,\neg C \quad (3')\ A',A,\neg C}{A',\neg C}\,(cut)}{\Gamma(\neg(C\Rightarrow D),[A':\Delta])}\,(\Rightarrow^-)$$

□

Theorem 3 (Completeness of $\mathcal{N}S$). *If Γ is valid, then it is derivable in $\mathcal{N}S$.*

Proof. We prove that the axioms are derivable and that the set of derivable formulas is closed under (Modus Ponens), (RCEA), and (RCK). A derivation of an instance of ID has been shown in Figure 2. Here is a derivation of an instance of CEM:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A:B,\neg B]}\,(AX) \quad \dfrac{}{A,\neg A}\,(AX) \quad \dfrac{}{\neg A,A}\,(AX)}{[A:B],[A:\neg B]}\,(CEM)}{[A:B],A\Rightarrow\neg B}\,(\Rightarrow^+)}{A\Rightarrow B,A\Rightarrow\neg B}\,(\Rightarrow^+)}{(A\Rightarrow B)\vee(A\Rightarrow\neg B)}\,(\vee^+)$$

For (Modus Ponens), the proof is standard and is omitted to save space. For (RCEA), we have to show that if $A \leftrightarrow B$ is derivable, then also $(A \Rightarrow C) \leftrightarrow (B \Rightarrow C)$ is derivable. As usual, $A \leftrightarrow B$ is an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. Since $A \leftrightarrow B$ is derivable, and since $(\wedge^+)$ and $(\rightarrow^+)$ are invertible (Lemma 3), we have a derivation for $A \rightarrow B$, then for $(1)\ \neg A, B$, and for $B \rightarrow A$, then for $(2)\ A, \neg B$. We derive $(A \Rightarrow C) \rightarrow (B \Rightarrow C)$ (the other half is symmetric) as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A\Rightarrow C),[B:C,\neg C]}\,(AX) \quad (1)\ \neg A,B \quad (2)\ A,\neg B}{\neg(A\Rightarrow C),[B:C]}\,(\Rightarrow^-)}{\neg(A\Rightarrow C),B\Rightarrow C}\,(\Rightarrow^+)}{(A\Rightarrow C)\rightarrow(B\Rightarrow C)}\,(\rightarrow^+)$$

For (RCK), suppose that we have a derivation in $\mathcal{N}S$ of $(A_1 \wedge \ldots \wedge A_n) \rightarrow B$. Since $(\rightarrow^+)$ is invertible (Lemma 3), we have also a derivation of $B, \neg(A_1 \wedge \ldots \wedge A_n)$. Since $(\wedge^-)$ is also invertible, then we have a derivation of $B, \neg A_1, \ldots, \neg A_n$ and, by weakening (Lemma 2), of $(1)\ \neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2, \ldots, \neg A_n]$, from which we conclude as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)\ \neg(C\Rightarrow A_1),\ldots,\neg(C\Rightarrow A_n),[C:B,\neg A_1,\neg A_2,\ldots,\neg A_n]}{\cdots}}{\neg(C\Rightarrow A_1),\ldots,\neg(C\Rightarrow A_n),[C:B,\neg A_1,\neg A_2]}\,(\Rightarrow^-) \quad \dfrac{}{C,\neg C}\,(AX) \quad \dfrac{}{\neg C,C}\,(AX)}{\neg(C\Rightarrow A_1),\ldots,\neg(C\Rightarrow A_n),[C:B,\neg A_1]}\,(\Rightarrow^-) \quad \dfrac{}{C,\neg C}\,(AX) \quad \dfrac{}{\neg C,C}\,(AX)}{\neg(C\Rightarrow A_1),\ldots,\neg(C\Rightarrow A_n),[C:B]}\,(\Rightarrow^-)}{\neg(C\Rightarrow A_1\wedge\ldots\wedge C\Rightarrow A_n),[C:B]}\,(\wedge^-)}{\neg(C\Rightarrow A_1\wedge\ldots\wedge C\Rightarrow A_n),C\Rightarrow B}\,(\Rightarrow^+)}{(C\Rightarrow A_1\wedge\ldots\wedge C\Rightarrow A_n)\rightarrow(C\Rightarrow B)}\,(\rightarrow^+)$$

□

3.4 Termination and complexity of $\mathcal{N}S$

The rules ($\Rightarrow^-$), (CEM), and (ID) may be applied infinitely often. In order to obtain a terminating calculus, we have to put some restrictions on the application of these rules. Let us first consider the systems without CEM. We put the following restrictions:

- apply ($\Rightarrow^-$) to $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ only if $\neg B \notin \Delta$;
- apply (ID) to $\Gamma([A : \Delta])$ only if $\neg A \notin \Delta$.

These restrictions impose that ($\Rightarrow^-$) is applied only once to each formula $\neg(A \Rightarrow B)$ with a context $[A' : \Delta]$ in each branch, and that (ID) is applied only once to each context $[A : \Delta]$ in each branch.

Theorem 4. *The calculi $\mathcal{N}S$ with the termination restrictions is sound and complete for their respective logics.*

Proof. We show that it is useless to apply the rules $(\Rightarrow^-)$ and (ID) without the restrictions. We only present the case of $(\Rightarrow^-)$. Suppose it is applied twice on $\Gamma([A : \Delta], [B : \Sigma])$ in a branch. Since $(\Rightarrow^-)$ is "weakly" invertible, we can assume, without loss of generality, that the two applications of $(\Rightarrow^-)$ are consecutive, starting from $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B, \neg B])$. By Lemma 4 (contraction), we have a derivation of $\Gamma([A : \Delta, \Sigma], [B : \Sigma])$, and we can conclude with a (single) application of $(\Rightarrow^-)$. □

The above restrictions ensure a terminating proof search for the nested sequents for CK and CK+ID, in particular:

Theorem 5. *The calculi $\mathcal{N}CK$ and $\mathcal{N}CK + ID$ with the termination restrictions give a* PSPACE *decision procedure for their respective logics.*

For the systems allowing CEM, we need a more sophisticated machinery[4], that allows us also to conclude that:

Theorem 6. *The calculi $\mathcal{N}CK{+}CEM$ and $\mathcal{N}CK{+}CEM{+}ID$ with the termination restrictions give a* PSPACE *decision procedure for their respective logics.*

It is worth noticing that our calculi match the PSPACE lower-bound of the logics CK and CK+ID, and are thus optimal with respect to these logics. On the contrary the calculi for CK+CEM(+ID) are not optimal, since validity in these logics is known to be decidable in CONP. In future work we shall try to devise an optimal decision procedure by adopting a suitable strategy.

4 A calculus for the flat fragment of CK+CSO+ID

In this section we show another application of nested sequents to give an analytic calculus for the flat fragment, i.e. without nested conditionals $\Rightarrow$, of CK+CSO+ID. This logic is well-known and it corresponds to logic **C**, the logic of *cumulativity*, the weakest system in the family of KLM logics [15]. Formulas are restricted to boolean combinations of propositional formulas and conditionals $A \Rightarrow B$ where A and B are propositionals. A sequent has then the form:

$$A_1, \ldots, A_m, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$$

[4] The termination of the calculi with (CEM) can be found in [1].

where B_i and Δ_i are propositional. The logic has also an alternative semantics in terms of *weak preferential models*. The rules of $\mathcal{N}CK + CSO + ID$ are those ones of $\mathcal{N}CK + ID$ (restricted to the flat fragment) where the rule $(\Rightarrow^-)$ is replaced by the rule (CSO):

$$\dfrac{\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D] \qquad \Gamma, \neg(C \Rightarrow D), [A : C] \qquad \Gamma, \neg(C \Rightarrow D), [C : A]}{\Gamma, \neg(C \Rightarrow D), [A : \Delta]}\ (CSO)$$

A derivation of an instance of CSO is easy and left to the reader. More interestingly, in Figure 3 we give an example of derivation of the cumulative axiom $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \wedge B) \Rightarrow C$.

$$\dfrac{\dfrac{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A \wedge B : C, \neg C] \quad \dfrac{\dfrac{\Sigma, [A \wedge B : A, \neg A, \neg B)]}{\Sigma, [A \wedge B : A, \neg(A \wedge B)]}\ (\wedge^-)}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A \wedge B : A]}\ (ID) \quad \begin{array}{c}\Pi_1 \\ \neg(A \Rightarrow B), \neg(A \Rightarrow C), [A : A \wedge B]\end{array}}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A \wedge B : C]}\ (CSO)}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), (A \wedge B) \Rightarrow C}\ (\Rightarrow^+)$$

where Π_1 is the following derivation:

$$\dfrac{\dfrac{\Sigma, [A : A, \neg A]}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A : A]}\ (ID) \quad \dfrac{\Sigma, [A : B, \neg B] \quad \dfrac{\Sigma, [A : A, \neg A]}{\Sigma, [A : A]}\ (ID) \quad \dfrac{\Sigma, [A : A, \neg A]}{\Sigma, [A : A]}\ (ID)}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A : B]}\ (CSO)}{\neg(A \Rightarrow B), \neg(A \Rightarrow C), [A : A \wedge B]}\ (\wedge^+)$$

Fig. 3. A derivation of the cumulative axiom $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \wedge B) \Rightarrow C$. We omit the first propositional steps, and we let $\Sigma = \neg(A \Rightarrow B), \neg(A \Rightarrow C)$.

Definition 5. *A sequent Γ is* reduced *if it has the form $\Gamma = \Lambda, \Pi, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$, where Λ is a multiset of literals and Π is a multiset of* negative *conditionals.*

The following proposition is a kind of *disjunctive property* for reduced sequents.

Proposition 1. *Let $\Gamma = \Lambda, \Pi, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$ be reduced, if Γ is derivable then for some i, the sequent $\Lambda, \Pi, [B_i : \Delta_i]$ is (height-preserving) derivable.*

Proposition 2. *Let $\Gamma = \Sigma, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$ be any sequent, if Γ is derivable then it can be (height-preserving) derived from some* reduced *sequents $\Gamma_i = \Sigma_j, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$. Moreover all rules applied to derive Γ from Γ_i are either propositional rules or the rule $(\Rightarrow^+)$.*

Proposition 2 can be proved by permuting (downwards) all the applications of propositional and $(\Rightarrow^+)$ rules.

Proposition 3. *Let $\Gamma = \Sigma, [A : \Delta], [A : \Delta]$ be derivable, then $\Gamma = \Sigma, [A : \Delta]$ is (height-preserving) derivable.*

Proof. By Proposition 2, Γ is height-preserving derivable from a set of reduced sequents $\Sigma_i, [A : \Delta], [A : \Delta]$. By Proposition 1, each $\Sigma_i, [A : \Delta]$ is derivable; we then obtain $\Sigma, [A : \Delta]$ by applying the same sequence of rules. $\square$

Theorem 7. *Contraction is admissible in $\mathcal{N}CK + CSO + ID$: if $\Gamma(F, F)$ is derivable, then $\Gamma(F)$ is (height-preserving) derivable, where F is either a formula or a nested sequent $[A : \Sigma]$.*

Proof. (Sketch) If F is a formula the proof is essentially the same as the one for Lemma 4 in $\mathcal{N}S$. If $F = [A : \Sigma]$ we apply Proposition 3. □

Observe that the standard inductive proof of contraction does not work in the case $F = [A : \Delta]$, that is why we have obtained it by Proposition 3 which in turn is based on Proposition 1, a kind of disjunctive property. The same argument *does not extend* immediatly to the full language with nested conditionals.

As usual, we obtain completeness by cut-elimnation. As in case of $\mathcal{N}S$, the proof is by mutual induction together with a substitution property and is left to the reader due to space limitations.

Theorem 8. *In $\mathcal{N}CK + CSO + ID$, the following propositions hold:* (A) *If $\Gamma(F)$ and $\Gamma(\neg F)$ are derivable, then so is Γ;* (B) *if (I) $\Gamma([A : \Delta])$, (II) $\Gamma([A : A'])$, and (III) $\Gamma([A' : A])$ are derivable, then so is $\Gamma([A' : \Delta])$.*

Theorem 9. *The calculus $\mathcal{N}CK + CSO + ID$ is sound and complete for the flat fragment of* CK+CSO+ID.

Proof. For soundness just check the validity of the (CSO) rule. For completeness, one can derive all instances of CSO axioms. Moreover the rules RCK and RCEA are derivable too (by using the rules (ID) and (CSO)). For closure under modus ponens, as usual we use the previous Theorem 8. Details are left to the reader. □

Termination of this calculus can be proved similarly to Theorem 5, details will be given in a full version of the paper.

Theorem 10. *The calculus $\mathcal{N}CK + CSO + ID$ with the termination restrictions give a* PSPACE *decision procedure for the flat fragment of* CK+CSO+ID.

We do not know whether this bound is optimal. The study of the optimal complexity for CK+CSO+ID is still open. A NEXP tableau calculus for cumulative logic **C** has been proposed in [11]. In [20] the authors provide calculi for full (i.e. with nested conditionals) CK+ID+CM and CK+ID+CM+CA: these logics are related to CSO, but they do not concide with it, even for the flat fragment as CSO = CM+RT (restricted transitivity). Their calculi are internal, but rather complex as the make use of ingenious but highly combinatorial rules. They obtain a PSPACE bound in all cases.

5 Conclusions and Future Works

In this work we have provided nested sequent calculi for the basic normal conditional logic and a few extensions of it. The calculi are analytic and their completeness is established via cut-elimination. The calculi can be used to obtain a decision procedure, in some cases of optimal complexity. We have also provided a nested sequent calculus for the flat fragment of CK+CSO+ID, corresponding to the cumulative logic **C** of the

KLM framework. Even if for some of the logics considered in this paper there exist other proof systems, we think that nested calculi are particularly natural internal calculi. Obviously, it is our goal to extent them to a wider spectrum of conditional logics, in particular preferential conditional logics, which still lack "natural" and internal calculi. We also intend to study improvements of the calculi towards efficiency, based on a better control of duplication. Finally, we wish to take advantage of the calculi to study logical properties of the corresponding systems (disjunction property, interpolation) in a constructive way.

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