Belief Revision and the Ramsey Test: A Solution

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Abstract. We offer a solution to the *triviality result* by Gärdenfors. His result claims that there are no non-trivial belief revision systems which satisfy the AGM postulates and are compatible with the Ramsey Test. We show that this result does no longer apply if we restrict some of the AGM postulates to deal only with the non-conditional part of epistemic states.

1 Introduction

In [1] Alchourrón, Gärdenfors and Makinson have proposed a set of rationality postulates, called AGM, that govern *belief revision*, a particular form of belief change. The process of revision modifies the original epistemic state by inserting a new piece of information represented by a formula. The new epistemic state resulting from revision has to be consistent if so is the incoming formula, and it has to differ as little as possible from the original epistemic state. The latter requirement is the so-called Minimal Change Principle.

More precisely, the AGM postulates characterize the revision of *belief sets*, i.e. deductively closed sets of *propositional* formulas: if K is a belief set, and A is a propositional formula, the belief revision operators characterized by AGM postulates return a new belief set $K * A$ in which A has been consistently added. The belief sets considered by the AGM postulates are therefore rather simple knowledge structures in which only factual knowledge of the world can be represented. On the contrary, all the knowledge, such as plausibility judgements, expectations, "revision strategies" that influence the result of a belief revision, are left outside the belief sets.

In [6] Gärdenfors has proposed to enrich the notion of belief set. To this purpose, he has proposed to extend the propositional language $\mathcal{L}$ with the conditional operator $>$, and to consider belief sets as deductively closed sets of formulas of $\mathcal{L}_>$ rather than of $\mathcal{L}$. These enriched belief sets contain therefore two kinds of formulas: propositional and conditional formulas. According to Gärdenfors, the meaning of conditional formulas can be expressed in terms of belief revision by means of the so-called Ramsey Test (RT). The Ramsey Test suggests an acceptability criterion for conditional formulas, which informally claims that the

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conditional sentence “if A, then B” is accepted in the belief set K if and only if B belongs to the revision of K by A. Ramsey Test has been expressed by Gärdenfors as follows:

(RT) $A > B \in K$ iff $B \in K * A$,

where $*$ represents a belief revision operator.

The importance of the Ramsey Test is twofold. By the Ramsey Test, conditionals and belief revision become two sides of the same coin. On the one hand, the Ramsey Test may provide a semantics of conditionals (i.e. conditional sentences) in terms of epistemic dynamics. This is actually the formalization, in the context of belief revision theory, of the seminal ideas by the philosopher F.P. Ramsey [16] about the meaning of conditional sentences.

On the other hand, the Ramsey Test postulates a correspondence between conditional formulas and belief revision which might be used to represent the “revision strategies” within the belief sets. By means of the Ramsey Test, one could define explicitly a revision operator by a set of conditional formulas. In this way, conditional formulas could provide a way of axiomatizing revision.

Unfortunately, the Ramsey Test is incompatible with the AGM postulates for belief revision, and in particular with the Minimal Change Principle. More precisely, the Ramsey Test entails the Monotonicity Principle according to which given two belief sets K and K', if $K \subseteq K'$, then $K * A \subseteq K' * A$. The Monotonicity Principle, combined with the rationality postulates for belief revision leads to a well known *triviality result* by Gärdenfors [6] according to which there are no significant belief revision systems compatible with the Ramsey Test.

Gärdenfors’ negative result has stimulated a wide debate and literature aiming to reconcile the two sides of the coin: belief revision and conditional logic. The proposals can be divided in a few categories. Some authors maintain that the Ramsey Test links conditionals and *belief update*, which is another kind of belief change operation, different from belief revision. [10]. Other authors suggest to avoid the triviality result by considering a weaker version of the Ramsey Test [17,13]. Others, finally, propose to exclude conditional formulas from belief sets, ascribing them a different epistemic status [12].

In this paper we show that the correspondence postulated by the Ramsey Test can be safely assumed (avoiding the triviality result) if we weaken some of the revision postulates so that they only apply to the propositional (i.e.non-conditional) part of an epistemic state.

Moreover, our weakened reformulation of the AGM postulates has an intuitive support. Learning new information may change our expectations and plausibility judgements about the world. In short, even if consistent, new information may change our revision strategies. This means that we cannot assume the Minimal Change Principle on the revision strategies themselves. Since by the Ramsey Test the revision strategies are represented by conditional formulas, we cannot assume the Minimal Change Principle on conditional formulas.

The following examples show that even in the case of consistent revision conditional sentences might be naturally added or removed.

Example 1. A woman has been murdered last night. Mary and John, her neighbours, are the main suspects. To solve up his doubts about who is the murder, the detective decides to look for the gun, believing that if the gun is found in John's room, then John is the culprit, and if it is found in Mary's room, Mary is the culprit. His knowledge base can be formalized as follows:

$$K = (gun\_John > John), (gun\_Mary > Mary)$$

Suppose now that the gun is found in John's room, but with Mary's fingerprints. The detective would conclude that Mary is the culprit and therefore abandon his conditional belief $(gun\_John > John)$. This contrasts with the Minimal Change Principle (and more precisely with AGM postulate (K*4)), according to which the detective's knowledge base after learning the information $gun\_John \wedge fingerprint\_Mary$ has to differ as little as possible from his original knowledge base, and, in particular, it has to contain all the conditional sentences previously believed.

The next example comes from Darwiche and Pearl [2]

Example 2. The same murder than in example 1. The detective in charge of the inquiry believes that only one out of John and Mary is the culprit, and therefore that if Mary has killed, then John has not and, viceversa, if John has killed, then Mary has not. His knowledge can be represented as follows:

$$K = \{(Mary > \neg John), (John > \neg Mary)\}$$

Suppose that he learns that John has killed the woman. He would conclude that John, and not Mary, has killed. However, his new knowledge base cannot be represented by:

$$K * John = \{John, \neg Mary, (Mary > \neg John), (John > \neg Mary)\}$$

like the Minimal Change Principle would impose.

Indeed, suppose that later the detective discovers that also Mary is involved in the murder. Would he conclude that John is no longer involved? No, he would rather conclude that they both have participated in the murder. Therefore, the right representation of the detective's knowledge base after learning that John is the culprit is the following:

$$K * John = \{John, \neg Mary, (Mary > John \wedge Mary), (John > \neg Mary).\}$$

in which conditionals have changed, in spite of the fact that the revision has been done with a consistent information. This clearly violates the Minimal Change Principle, and more precisely AGM postulates (K*3) and (K*4).

The plan of the paper is the following. In section 2 we present our restricted formulation of AGM postulates; in section 3 we show that there is a non-trivial belief revision system satisfying our postulates and the Ramsey Test; in section 4 we consider some related works.

2 Belief Revision

Let the conditional language $\mathcal{L}_>$ be the extension of the propositional language $\mathcal{L}$ by the conditional operator $>$. Let S be a set of formulas of $\mathcal{L}_>$, we define $Cn_{PC}(S) = \{A \in \mathcal{L}_> \text{ s.t. } S \models_{PC} A\}$; the relation $S \models_{PC} A$ means that A is a propositional consequence of S, where conditional formulas are treated as atoms.

An *epistemic state* is any set of formulas of $\mathcal{L}_>$ which is deductively closed with respect to Cn_{PC}. A *belief set* is a deductively closed set of formulas of $\mathcal{L}$. We introduce a belief function $[\,]$ that associates to each epistemic state K its corresponding belief set $[K] = K \cap \mathcal{L}$. A revision operator $*$ is any function that takes an epistemic state and a formula in $\mathcal{L}$ as input, and gives an epistemic state as output. The *expansion* of an epistemic state K by a formula A is the set $K + A = Cn_{PC}(K \cup \{A\})$.

Definition 1. *A* belief revision system *is a triple* $\langle \mathbf{K}, *, [\,] \rangle$*, where* $\mathbf{K}$ *is a set of epistemic states closed under the revision operator* $*$*, and* $[\,]$ *is the belief function. The operator* $*$ *satisfies the following postulates:*

- $(B*1)$ $K*A$ *is an epistemic state;*
- $(B*2)$ $A \in K*A$*;*
- $(B*3)$ $[K*A] \subseteq [K+A]$*;*
- $(B*4)$ *if* $\neg A \notin [K]$*, then* $[K+A] \subseteq [K*A]$*;*
- $(B*5)$ $K*A \vdash_{PC} \bot$ *only if* $\vdash_{PC} \neg A$*;*
- $(B*6)$ *if* $A \equiv B$*, then* $K*A = K*B$*;*
- $(B*7)$ $[K*(A \wedge B)] \subseteq [(K*A)+B]$*;*
- $(B*8)$ *if* $\neg B \notin [K*A]$*, then* $[(K*A)+B] \subseteq [K*(A \wedge B)]$
- $(B*\top)$ *for any* K *consistent,* $K*\top = K$*.*

Postulates (B*1), (B*2), (B*5), (B*6) are AGM postulates (K*1), (K*2), (K*5), (K*6). Postulates (B*3), (B*4), (B*7), (B*8) are the restriction of AGM postulates (K*3), (K*4), (K*7), (K*8) to belief sets. Postulates (K*3),(K*4) represent the Minimal Change Principle. The restriction of (K*3), (K*4) means that the Minimal Change Principle is restricted to the non-conditional part of epistemic states. As we have argued in the introduction, the Minimal Change Principle cannot be assumed for conditional formulas. A similar remark applies to (K*7) and (K*8) that are a generalization of (K*3),(K*4)[1]. Postulate $(B*\top)$ expresses a rather unquestionable property of revision. It comes for free from the original AGM postulates (K*3),(K*4); we have introduced it as we can no longer derive it from our corresponding (B*3), (B*4).

The restriction we have put on AGM postulates have an impact on the closure properties of belief revision systems. AGM postulates entail that belief revision systems are closed with respect to expansion, i.e. if $\mathbf{K}$ is the set of all the epistemic states of a belief revision system, then for any epistemic state K, if $K \in \mathbf{K}$, then also $K + A \in \mathbf{K}$. This property follows immediately from $(K*3), (K*4)$

[1] Given $(B*\top)$, (K*3),(K*4) derive respectively from (K*7),(K*8) by taking $A = \top$.

together with the closure with respect to the revision operator. In contrast, this property cannot be derived from our modified postulates. In our setting, the closure with respect to revision does not imply the closure with respect to expansion. As we have argued in the introduction, the revision of an epistemic state with a formula A, even when A is consistent with the state, may affect the *conditional formulas* holding in that state in a way that is not reflected by the simple expansion operation.

As we will see in the next section, these restrictions are sufficient to avoid the *triviality result* by Gärdenfors.

3 Non-triviality

In this section, we show that there is a non-trivial belief revision system satisfying postulates $(B*1)-(B*\top)$ and the Ramsey Test.

Definition 2. *A belief revision system* $\langle \mathbf{K}, *, [\,] \rangle$ *is non-trivial if there are three formulas* A*,* B*,* C *in* $\mathcal{L}$*, which are pairwise disjoint (i.e. such that* $\vdash_{PC} \neg(A \wedge B)$*,* $\vdash_{PC} \neg(B \wedge C)$*,* $\vdash_{PC} \neg(A \wedge C)$*), and an epistemic state* $K \in \mathbf{K}$ *such that* $\neg A \notin K$*,* $\neg B \notin K$*, and* $\neg C \notin K$*.*

To fit our result in its proper place, we recall Gärdenfors's triviality result.

Theorem 1 ([4], page 85). *There is no non-trivial belief revision system which satisfies AGM postulates (K*1)-(K*8) and (RT).*

Actually, Gärdenfors result is stronger, as he has shown that the AGM postulates (K*1),(K*2),(K*4), (K*5) alone, together with (RT) imply the triviality of the belief revision system. In contrast, we show that our reformulated postulates $(B*1)-(B*\top)$ are compatible with the Ramsey Test, in the sense that they do not entail the triviality of the belief revision system.

We proceed as follows. We consider a Spohn system as defined in [19,2] and we show how to build a belief revision system satisfying postulates $(B*1)-(B*\top)$ and (RT). Since we have proved in [9] that there is one non-trivial Spohn system, we will be able to conclude that there is one non-trivial belief revision system.

A Spohn system is a structure $\langle \mathbf{R}, *_s \rangle$, where

- $\mathbf{R} = \{k : W \to Ord\}$ is a set of functions from the set W of all classical interpretations to the set of ordinals. The elements k of $\mathbf{R}$ are called *rankings*, and the elements w of W are called *worlds*. It is assumed that for all rankings $k \in \mathbf{R}$, there is a world w such that $k(w) = 0$.
- $k(A) = \min\{k(w) : w \models A\}$.
- The operator $*_s$ of type: $\mathbf{R} \times \mathcal{L} \longrightarrow \mathbf{R}$ is defined as follows [2]:

$$k *_s A(w) = \begin{cases} k(w) - k(A) \text{ if } w \models A \\ k(w) + 1 \text{ otherwise} \end{cases}$$

[2] This is the simplified version of Spohn's function proposed in [2]

Given a Spohn system , let $Bel : \mathbf{R} \rightarrow P(\mathcal{L})$, be defined as follows

$$Bel(k) = \{A \in \mathcal{L} \mid \forall w \in W \ (k(w) = 0 \rightarrow w \models A)\}.$$

Starting from a Spohn system $\langle \mathbf{R}, *_s \rangle$, we can define a belief revision system like the one described in section 2, in which epistemic states are deductively closed sets of formulas in $\mathcal{L}_>$, and the revision operator satisfies postulates $(B * 1) - (B * \top)$ as follows.

Definition 3 (Construction).

- *Let $\mathcal{L}*_>$ be the subset of $\mathcal{L}_>$ defined as follows:*
  - *if $A \in \mathcal{L}$, then $A \in \mathcal{L}*_>$;*
  - *if $A \in \mathcal{L}$ and $B \in \mathcal{L}*_>$, then $A > B \in \mathcal{L}*_>$.*
- *First, we define the set $S(k)$ by stipulating:*
  - *let $A \in \mathcal{L}$. If $A \in Bel(k)$, then $A \in S(k)$;*
  - *let $A > B \in \mathcal{L}*_>$ If $B \in S(k *_s A)$, then $A > B \in S(k)$;*
  - *no other formula is in $S(k)$.*
- *Then, we let $K = Cn_{PC}(S(k))$ and*
  $\mathbf{K} = \{K \mid K = Cn_{PC}(S(k))$ *and* $k \in \mathbf{R}\}$.
  We define an operator $*$ on $\mathbf{K}$ as follows
  $K * A = Cn_{PC}(S(k *_s A))$
  We finally consider the structure $\langle \mathbf{K}, *, [\ ] \rangle$.

We now show that, given any Spohn system $< \mathbf{R}, *_s >$ the corresponding structure built as indicated in the previous definition is a belief revision system. To this aim, we start noticing that the sets K of $\mathbf{K}$ are epistemic states, as they are closed with respect to Cn_{PC} by definition. Furthermore, we show that the revision operator satisfies postulates $(B*1) - (B*\top)$. To this purpose, we need the two following lemmas.

Lemma 1. *Given $K \in \mathbf{K}$, with $K = Cn_{PC}(S(k))$, for $k \in \mathbf{R}$, we have that if $\bot \notin Bel(k)$ then $\bot \notin K$.*

Lemma 2. *. Let $K \in \mathbf{K}$, with $K = Cn_{PC}(S(k))$, where $k \in \mathbf{R}$. For any formula $A \in \mathcal{L}*_>$, $A \in K$ if and only if $A \in S(k)$.*

Proof. $\Leftarrow$ It holds by definition.
$\Rightarrow$ By definition, in $S(k)$ there are only formulas of $\mathcal{L}*_>$. If $A \in K$ then $S(k) \vdash_{PC} A$, thus there exist formulas $D_1, \dots, D_n, E_1, \dots, E_k \in S(k)$ such that $D_1, \dots, D_n \in \mathcal{L}$, $E_1, \dots, E_k \in \mathcal{L} *_> -\mathcal{L}$, and $D_1, \dots, D_n, E_1, \dots, E_k \vdash_{PC} A$.
We distinguish two cases: (i) if $A \in \mathcal{L}$, then $D_1, \dots, D_n, E_1, \dots, E_k \vdash_{PC} A$ iff $D_1, \dots, D_n \vdash_{PC} A$, for $E_1, \dots, E_k$ do not occur in $\{D_1, \dots, D_n\}$, nor in A. Since $\{D_1, \dots, D_n\} \subseteq S(k)$ iff $\{D_1, \dots, D_n\} \subseteq Bel(k)$ and $Bel(k) = Cn_{PC}(Bel(k))$, we can conclude that $A \in Bel(k)$, and therefore $A \in S(k)$.
(ii) If $A \in \mathcal{L} *_> -\mathcal{L}$, then $D_1, \dots, D_n, E_1, \dots, E_k \vdash_{PC} A$ iff $A = E_i \in \{E_1, \dots, E_k\}$, and therefore re $A \in S(k)$.

We now show that the revision operator $*$ defined above satisfies the revision postulates of section 2.

Theorem 1 *The structure defined in 3 is a belief revision system.*

Proof. In light of the previous lemmas and considerations, we have to show that the operator $*$ satisfies postulates $(B * 1) - (B * \top)$.

- $(\mathbf{B} * \mathbf{1})$ $K * A$ is an epistemic state.
  This follows from the definition of $K * A$.
- $(\mathbf{B} * \mathbf{2})$ if $A \in \mathcal{L}$, then $A \in K * A$.
  By the definition of Spohn operator, if $k *_s A(w) = 0$ then $w \models A$, since if $w \not\models A$, then it would be $k *_s A(w) = k(w) + 1 > 0$. By the definition of Bel, it follows that $A \in Bel(k *_s A)$ and, by the definition of $K * A$, we have that $A \in K * A$.
- $(\mathbf{B} * \mathbf{3})$ $[K * A] \subseteq [K + A]$.
  By definition of $*_s$, for any w such that $k(w) = 0$ and $w \models A$, $k *_s A(w) = 0$. Therefore, $\{w : k(w) = 0 \text{ and } w \models A\} \subseteq \{w : k *_s A(w) = 0\}$.
  By definition of Bel, it follows that $Bel(k *_s A) \subseteq Cn_{PC}(Bel(k) \cup \{A\})$.
  By lemma 2, we know that $[K * A] = \{B \in \mathcal{L} : B \in K * A\} = \{B \in \mathcal{L} : B \in S(k *_s A)\} = Bel(k *_s A)$.
  By the same reasoning, we also know that $[K] = Bel(k)$.
  Moreover, we can show that K, $[K + A] = [K] + A$. To see this, suppose the contrary holds. Then there would be a formula $B \in \mathcal{L}$ such that $B \in [K + A]$, but $B \notin [K] + A$. Therefore, there would be some formulas $E_1 \dots E_n \in \mathcal{L} *_> -\mathcal{L}$ such that $E_1 \dots E_n, A \vdash_{PC} B$. But since $B \in \mathcal{L}$, it would follow that $A \vdash_{PC} B$ and that therefore $B \in [K] + A = Bel(k) + A$. We can therefore conclude that $[K * A] \subseteq [K + A]$.
- $(\mathbf{B} * \mathbf{4})$ if $\neg A \notin [K]$, then $[K + A] \subseteq [K * A]$.
  If $\neg A \notin [K]$, then $\neg A \notin Bel(k)$. Therefore, there is a world w such that $k(w) = 0$ and $w \models A$. By definition of $k(A)$, this entails that $k(A) = 0$. From this fact and the definition of $*_s$, it follows that for any w, if $k *_s (w) = 0$ then $k(w) = 0$ and $w \models A$. We can conclude that $\{w : k *_s A(w) = 0\} \subseteq \{w : k(w) = 0 \text{ and } w \models A\}$. By definition of Bel, it follows that $Bel(k) + A \subseteq Bel(k *_s A)$.
  Moreover, by definition of S and by lemma 2, we know that $Bel(k *_s A) = [K * A]$, and $Bel(k) + A = [K] + A = [K + A]$. Thus, if $\neg A \notin [K]$ then $[K + A] \subseteq [K * A]$.
- $(\mathbf{B} * \mathbf{5})$ $K * A \vdash_{PC} \perp$ only if $\vdash_{PC} \neg A$.
  Since the set W of a Spohn system is the set of all classical interpretations, if $\not\vdash_{PC} \neg A$, then there exists $w : w \models A$. In particular, there exists w such that $w \models A$ and $k(w) = k(A)$. By definition of $*_s$, for any such w, $k *_s A(w) = k(w) - k(A) = 0$. It follows that there exists one w such that $k *_s A(w) = 0$. Therefore $Bel(k *_s A) \not\vdash \perp$. By the definition of $*_s$ and by lemma 1, we can conclude that $K * A \not\vdash_{PC} \perp$.
- $(\mathbf{B} * \mathbf{6})$ if $\vdash_{PC} A \leftrightarrow B$, then $K * A = K * B$.
  This follows directly by the fact that, by definition of$*_s$, we have that if $\vdash_{PC} A \leftrightarrow B$, then $k *_s A = k *_s B$.

– $(\mathbf{B*7})$ $[K * (A \wedge B)] \subseteq [(K * A) + B]$.
By definition of $k(A)$, we know that $k(A \wedge B) \geq k(A)$. By definition of $*_s$, it follows that for any w such that $w \models A \wedge B$, we have $k *_s (A \wedge B)(w) = k(w) - k(A \wedge B) \leq k(w) - k(A) = k *_s A(w)$. Therefore, for any w such that $w \models A \wedge B$, if $k *_s A(w) = 0$, then $k *_s (A \wedge B)(w) = 0$.
Since $\{w : k *_s A(w) = 0 \text{ and } w \models B\} \subseteq \{w : w \models A \wedge B\}$, it follows that $\{w : k *_s A(w) = 0 \text{ and } w \models B\} \subseteq \{w : k *_s (A \wedge B)(w) = 0\}$. Therefore $Bel(k *_s (A \wedge B)) \subseteq Bel(k * A) + B$ and, by lemma 2, it follows that $[K * (A \wedge B)] \subseteq [K * A] + B = [(K * A) + B]$.
– $(\mathbf{B*8})$ if $\neg B \notin [K * A]$, then $[(K * A) + B] \subseteq [K * (A \wedge B)]$.
If $\neg B \notin [K * A]$, then $\neg B \notin Bel(k * A)$, and by definition of Bel $\{w : k *_s A(w) = 0\} \cap [[B]] \neq \emptyset$. Let $w \in \{w : k *_s A(w) = 0\} \cap [[B]]$. From the fact that $k *_s A(w) = 0$ we can easily obtain that, first, $w \models A$, and therefore $w \models A \wedge B$; second, that $k(w) = k(A)$. Furthermore, since by definition of $k(A \wedge B)$ and $k(A)$ we know that $k(A \wedge B) \geq k(A)$, we have that $k(A \wedge B) \geq k(w)$, but since $w \models (A \wedge B)$, we conclude that $k(A \wedge B) = k(w) = k(A)$. It follows that $\{w : k *_s (A \wedge B)(w) = 0\} \subseteq \{w : k *_s A(w) = 0\}$. Moreover, since $\{w : k *_s (A \wedge B)(w) = 0\} \subseteq [[B]]$, we obtain that $\{w : k *_s (A \wedge B)(w) = 0\} \subseteq \{w : k *_s A(w) = 0\} \cap [[B]]$. From this, we can conclude that $Bel(k *_s (A)) + B \subseteq Bel(k *_s (A \wedge B))$. By lemma 2, it follows that $[K * A] + B \subseteq [K * (A \wedge B)]$ and therefore that $[K * A + B] \subseteq [K * (A \wedge B)]$.
– $(\mathbf{B} * \top)$ $K = K * \top$. For all k, $k * \top = k$, therefore $S(k) = S(k * \top)$ and $K = K * \top$.

We can now show that the belief revision system of definition 3 satisfies the Ramsey Test, as far as the formulas in $\mathcal{L}^*_>$ are concerned.

Theorem 2 *In the belief revision system of definition of 3, for all formulas $A > B \in \mathcal{L}*_>$, and for all $K \in \mathbf{K}$, we have that $A > B \in K$ iff $B \in K * A$.*

Proof. $\Rightarrow$ If $A > B \in \mathcal{L}*_>$ and $A > B \in K = Cn_{PC}S(k)$, by lemma 2, $A > B \in S(k)$. Then, by definition of S, $B \in S(k *_s A)$ and, by definition of $K * A$, $B \in K * A$.
$\Leftarrow$ If $B \in K * A = Cn_{PC}(S(k *_s A))$ and $B \in \mathcal{L}*_>$, then by lemma 2, $B \in S(k *_s A)$. It follows, by the definition of $S(k)$, that $A > B \in S(k)$ and therefore $A > B \in K$.

We collect the previous results to show that there is one non-trivial belief revision system satisfying the Ramsey Test.

Theorem 3 *There is a non-trivial belief revision system.*

Proof. (Sketch) As shown in [9], there is a non-trivial Spohn system. We sketch the proof. Let us consider the language $\mathcal{L}'$ containing only the propositional variables p_1, p_2, p_3, p_4. Let $A = \neg p_2 \wedge \neg p_3 \wedge p_4; B = p_2 \wedge \neg p_3 \wedge \neg p_4; C = \neg p_2 \wedge p_3 \wedge \neg p_3$.
Clearly, $\vdash_{PC} \neg(A \wedge B), \vdash_{PC} \neg(B \wedge C)$ and $\vdash_{PC} \neg(A \wedge C)$.

Notice that the set W of all the classical interpretations of $\mathcal{L}'$ will be the set $\{w : w \in 2^{\{p_1,p_2,p_3,p_4\}}$. Consider now Spohn system S' that contains a ranking k' such that $k'(w) = 0$ iff $w \models p_1$.
Clearly, $Bel(k') = \{p_1\}$, and $\neg A \notin Bel(k')$, $\neg B \notin Bel(k')$, $\neg C \notin Bel(k')$. Therefore S' is non-trivial.

Consider now the belief revision system $\langle \mathbf{K}, *, [\ ] \rangle$ obtained from S' by the construction described in definition 3. By construction, there will be in $\mathbf{K}$ an epistemic state $K' = Cn_{PC}(S(k'))$. By the fact that $\neg A \notin Bel(k')$, $\neg B \notin Bel(k')$, $\neg C \notin Bel(k')$, by definition of $S(k')$, and by lemma 2, we can conclude that $\neg A \notin K'$, $\neg B \notin K'$, and $\neg C \notin K'$.

We have shown that there is a non-trivial belief revision system which satisfies the Ramsey Test. Therefore the triviality result does not apply to our belief revision systems. The conflict between the Minimal Change Principle and the Monotonicity Principle has been solved by giving up the first one as far as conditional formulas are concerned (while retaining it for non conditional formulas). As we have seen in section 2, this has been obtained by weakening some AGM postulates.

As a final remark, we have shown that there are non-trivial belief revision systems that satisfy the Ramsey Test with respect to formulas in $\mathcal{L}*_>$, i.e. formulas which do not have nested conditionals (on the left). Since we consider the revision operation as defined only for propositional formulas, a more general form of Ramsey Test involving arbitrary $\mathcal{L}_>$-formulas would have been meaningless. In [4] Gärdenfors does not consider this limitation, as he allows revision by conditional formulas as well. He shows that there is no non-trivial belief revision system which satisfies the Ramsey Test for arbitrary formulas of $\mathcal{L}_>$. However, the restriction on formulas of $\mathcal{L}*_>$ does not affect Gardenfors' argument, for the triviality proof works exactly the same even if RT is restricted to $\mathcal{L}*_>$ formulas.

4 Conclusions

In this paper we have shown how the triviality result by Gärdenfors can be avoided by restricting some of the revision postulates to propositional formulas. In the introduction we have argued that these restrictions have an intuitive support.

The triviality result by Gärdenfors has been widely investigated in the literature. A possible way out to the triviality result consists in considering a different notion of belief change, called "belief update" [11,10], which does not enforce the Minimal Change Principle (not even for propositional formulas). Grahne [10] has proposed a conditional logic which combines updates and counterfactuals and which does not entail triviality.

Recently Ryan and Schobbens [18] have established a link between updates and counterfactuals, by regarding them as existential and universal modalities. The Ramsey rule is an axiomatization of the inverse relationship between the two sets of modalities.

Makinson [14] has analyzed the triviality result in the case the inference operator is non-monotonic and he has proved that triviality still holds in this case.

Rott has suggested that triviality could perhaps be avoided by weakening the Ramsey rule which, in the formulation used by Gärdenfors in the proof of the triviality result, leads to the counterintuitive conclusion that if $A \in K$ and $B \in K$ then $A > B \in K$. However, Gärdenfors [5] has shown that this is not the case, as he has considered several ways of weakening the Ramsey rule that avoid the counterintuitive conclusion but still lead to triviality.

Lindström and Rabinowicz in [13] discuss some alternative solutions to Gärdenfors negative result such as questioning preservation, weakening the Ramsey rule, leaving the conditional formulas out of the belief sets, and making the evaluation of conditionals dependent on the epistemic state.

Levi [12] has proposed a way out to the triviality problem based on a strict separation between conditional and non-conditional beliefs. The triviality is avoided by assuming that conditional sentences cannot be members of belief sets, consequently the belief operator only applies to sets of propositional beliefs and therefore revision postulates are naturally restricted to propositional belief sets. In this respect our approach has some similiarity with Levi's one. As a difference, we only restrict *some* of the postulates and we allow the occurrence of conditionals in epistemic states, including iterated conditionals whose acceptance (whence meaning) is not defined in Levi's framework.

The last approach is close to the one proposed in [9] to define a conditional logic for revision, as well as to the one developed in [3], which defines a logical framework for modeling both revision and update. The fact that in both [9] and [3] triviality does not occur seems to be explainable by the fact that, besides leaving the conditional formulas out of the belief sets (so that postulates are only required to hold for propositional formulas), epistemic states are considered to be complete with respect to some conditional formulas and that, in essence, a stronger version of the Ramsey Test is adopted. In this paper we have shown that a solution to the Triviality Result can be found also in the case in which epistemic states are not assumed to be complete with respect to conditional formulas.

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