Tableau Calculi for KLM Logics: extended version

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Abstract

We present tableau calculi for some logics of nonmonotonic reasoning, as defined by Kraus, Lehmann and Magidor. We give a tableau proof procedure for all KLM logics, namely preferential, loop-cumulative, cumulative and rational logics. Our calculi are obtained by introducing suitable modalities to interpret conditional assertions. We provide a decision procedure for the logics considered, and we study their complexity.

1 Introduction

In the early 90s [26] Kraus, Lehmann and Magidor (from now on KLM) proposed a formalization of nonmonotonic reasoning that was early recognized as a landmark. Their work stemmed from two sources: the theory of nonmonotonic consequence relations initiated by Gabbay [16] and the preferential semantics proposed by Shoham [33] as a generalization of Circumscription. Their work led to a classification of nonmonotonic consequence relations, determining a hierarchy of stronger and stronger systems. The so called *KLM properties* have been widely accepted as the "conservative core" of default reasoning. The role of KLM logics is similar to the role of AGM postulates in Belief Revision [17]: they give a set of postulates for default reasoning that any concrete reasoning mechanism should satisfy.

According to the KLM framework, defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals or assertions of the form

$$A \mid\!\sim B$$

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whose reading is *normally (or typically) the A's are B's.* The operator "$\mid\!\sim$" is nonmonotonic, in the sense that $A \mid\!\sim B$ does not imply $A \wedge C \mid\!\sim B$. For instance, a knowledge base K may contain the following set of conditionals:

$$adult \mid\!\sim worker, adult \mid\!\sim taxpayer, student \mid\!\sim adult, student \mid\!\sim \neg worker,$$

$$student \mid\!\sim \neg taxpayer, retired \mid\!\sim adult, retired \mid\!\sim \neg worker$$

whose meaning is that adults typically work, adults typically pay taxes, students are typically adults, but they typically do not work, nor do they pay taxes, and so on. Observe that if $\mid\!\sim$ were interpreted as classical (or intuitionistic) implication, we simply would get $student \mid\!\sim \bot$, $retired \mid\!\sim \bot$, i.e. typically there are not students, nor retired people, thereby obtaining a trivial knowledge base. One can derive new conditional assertions from the knowledge base by means of a set of inference rules.

In KLM framework, the set of adopted inference rules defines some fundamental types of inference systems, namely, from the weakest to the strongest: Cumulative (**C**), Loop-Cumulative (**CL**), Preferential (**P**) and Rational (**R**) logic. All these systems allow one to infer new assertions from a given knowledge base K without incurring in the trivializing conclusions of classical logic: regarding our example, in none of them, one can infer $student \mid\!\sim worker$ or $retired \mid\!\sim worker$. In cumulative logics (both **C** and **CL**) one can infer $adult \wedge student \mid\!\sim \neg worker$ (giving preference to more specific information), in Preferential logic **P** one can also infer that $adult \mid\!\sim \neg retired$ (i.e. typical adults are not retired). In the rational case **R**, if one further knows that $\neg(adult \mid\!\sim \neg married)$ (i.e. it is not the case that adults are typically unmarried), one can also infer that $adult \wedge married \mid\!\sim worker$.

From a semantic point of view, to each logic (**C**, **CL**, **P**, **R**) corresponds one kind of models, namely possible-world structures equipped with a preference relation among worlds or states. More precisely, for **P** we have models with a preference relation (an irreflexive and transitive relation) on worlds. For the stronger **R** the preference relation is further assumed to be *modular*. For the weaker logic **CL**, the transitive and irreflexive preference relation is defined on *states*, where a state can be identified, intuitively, with a set of worlds. In the weakest case of **C**, the preference relation is on states, as for **CL**, but it is no longer assumed to be transitive. In all cases, the meaning of a conditional assertion $A \mid\!\sim B$ is that B holds in the *most preferred* worlds/states where A holds.

In KLM framework the operator "$\mid\!\sim$" is considered as a meta-language operator, rather than as a connective in the object language. However, it has been readily observed that KLM systems **P** and **R** coincide to a large extent with the flat (i.e. unnested) fragments of well-known conditional logics, once we interpret the operator "$\mid\!\sim$" as a binary connective [9, 8, 25].

A recent result by Halpern and Friedman [14] has shown that preferential and rational logic are natural and general systems: surprisingly enough, the axiom system of preferential (likewise of rational logic) is complete with respect to a wide spectrum of semantics, from ranked models, to parametrized probabilistic structures, ϵ-semantics and possibilistic structures. The reason is that all these structures are examples of *plausibility structures* and the truth in them is captured by the axioms of preferential (or rational) logic. These results, and their extensions to the first order setting [15] are the source of a renewed interest in KLM framework. A considerable amount of research in the area has then concentrated in developing concrete mechanisms for plausible reasoning in accordance with KLM systems (**P** and **R** mostly). These mechanisms are defined by exploiting a variety of models of reasoning under uncertainty (ranked models, belief functions, possibilistic logic, etc. [6, 7, 34, 31, 29, 28]) that provide, as we remarked, alternative semantics to KLM systems. These mechanisms are based on the restriction of the semantics to preferred classes of models of KLM logics; this is also the case of Lehmann's notion of rational closure introduced in [27] (not to be confused with the logic **R**). More recent research has also explored the integration of KLM framework with paraconsistent logics [1]. Finally, there has been some recent investigation on the relation between KLM systems and decision-theory [12, 11].

Even if KLM was born as an inferential approach to nonmonotonic reasoning, curiously enough, there has not been much investigation on deductive mechanisms for these logics. In short, the state of the art is as follows:

- Lehmann and Magidor [27] have proved that validity in **P** is **coNP**-complete. Their decision procedure for **P** is more a theoretical tool than a practical algorithm, as it requires to guess sets of indexes and propositional evaluations. They have also provided another procedure for **P** that exploits its reduction to **R**. However, the reduction of **P** to **R** breaks down if boolean combinations of conditionals are allowed, indeed it is exactly when such combinations are allowed that the difference between **P** and **R** arises.

- A tableau proof procedure for **C** has been given in [2]. Their tableau procedure is fairly complicated; it uses labels and a complex unification mechanism. Moreover, the authors show how to extend the system to Loop-Cumulative logic **CL** and discuss some ways to extend it to the other logics.

- In [20] and [21] some labelled tableaux calculi have been defined for the conditional logic **CE** and its main extensions, including **CV**. The flat fragment (i.e. without nested conditionals) of **CE** and of **CV** corresponds respectively to **P** and to **R**. These calculi however need a fairly complicated loop-checking mechanism to ensure termination. It is not clear if they match complexity bounds and if they can be adapted in a simple way to **CL** and to **C**.

- Finally, decidability of **P** and **R** has also been obtained by interpreting them into standard modal logics, as it is done by Boutilier [8]. However, his mapping is not very direct and natural, as we discuss below.

- To the best of our knowledge, for **CL** no decision procedure and complexity bound was known before the present work.

In this work we introduce tableau procedures for all KLM logics, starting with the preferential logic **P**. Our approach is based on a novel interpretation of **P** into modal logics. As a difference with previous approaches (e.g. Crocco and Lamarre [9] and Boutillier [8]), that take S4 as the modal counterpart of **P**, we consider here Gödel-Löb modal logic of provability G (see for instance [24]). Our tableau method provides a sort of run-time translation of **P** into modal logic G.

The idea is simply to interpret the preference relation as an accessibility relation: a conditional $A \mid\sim B$ holds in a model if B is true in all minimal A-worlds w (i.e. worlds in which A holds and that are minimal). An A-world w is a minimal A-world if all smaller worlds are not A-worlds. The relation with modal logic G is motivated by the fact that we assume, following KLM, the so-called *smoothness condition*, which is related to the well-known *limit assumption*. This condition ensures that minimal A-worlds exist whenever there are A-worlds, by preventing infinitely descending chains of worlds. This condition therefore corresponds to the finite-chain condition on the accessibility relation (as in modal logic G). Therefore, our interpretation of conditionals is different from the one proposed by Boutilier, who rejects the smoothness condition and then gives a less natural (and more complicated) interpretation of **P** into modal logic S4.

As a further difference with previous approaches, we do not give a formal translation of **P** into G. Rather, we directly provide a tableau calculus for **P**. One can notice some similarities between some of the rules for **P** and some of the rules for G. This is due to the correspondence between the semantics of the two logics. For deductive purposes, we believe that our approach is more direct, intuitive, and efficient than translating **P** into G and then using a calculus for G.

We are able to extend our approach to the cases of **CL** and **C** by using a second modality which takes care of states. Regarding **CL**, we show that we can map **CL**-models into **P**-models with an additional modality. The very fact that one can interpret **CL** into **P** by means of an additional modality does not seem to be previously known and might be of independent interest. In both cases, **P** and **CL**, we can define a decision procedure and obtain also a complexity bound for these logics, namely that they are both **coNP**-complete. In case of **CL** this bound is new, to the best of our knowledge.

We treat **C** in a similar way: we can establish a mapping between Cumulative models and a kind of bi-modal models. However, because of the lack of transitivity, the target modal logic is no longer G. The reason is that the *smoothness condition* (for any formula A, if a state satisfies A, then either it is minimal or it admits a smaller minimal state satisfying A) can no longer be identified with the finite-chain condition of G. As a matter of fact, the smoothness condition for **C** cannot be identified with any property of the accessibility relation, as it involves unavoidably the evaluation of formulas in worlds. We can still derive a tableau calculus based on our semantic mapping. But we pay a price: as a difference with **P** and **CL** the calculus for **C** requires a sort of (analytic) cut rule to account for the smoothness condition. This calculus gives nonetheless a decision procedure for **C**.

Finally, we consider the case of the strongest logic **R**; as for the other weaker systems, our approach is based on an interpretation of **R** into an extension of modal logic G, including modularity of the preference relation (previous approaches [9, 8] take S4.3 as the modal counterpart of **R**). As a difference with the tableau calculi introduced for **P**, **CL**, and **C**, here we develop a *labelled* tableau system, which seems to be the most natural approach in order to capture the modularity of the preference relation. The calculus defines a systematic procedure which allows the satisfiability problem for **R** to be decided in nondeterministic polynomial time, in accordance with the known complexity results for this logic.

All the calculi presented in this paper have been implemented in SICStus Prolog. To the best of our knowledge, our theorem prover, called KLMLean, is the first one for KLM logics[1].

The plan of the paper is as follows: in section 2, we recall KLM logics (from the strongest to the weakest): **R**, **P**, **CL**, and **C**, and we show how their semantics can be represented by standard Kripke models. In section 3 we give a tableau calculus for **P**. We then elaborate this calculus in order to obtain a terminating procedure. We propose a further refinement that gives a **coNP** decision procedure. The latter is based on a tighter semantics of **P** in terms of *multi-linear models*. In section 4 we propose similar calculi for **CL**. In section 5, we give a tableau calculus for **C**. As mentioned above, the calculus requires a form of cut-rule. We prove however that we can restrict its application in an analytic way (namely, it is needed only for formulas A that are antecedents of conditionals $A \mid\sim B$ contained in the initial set of formulas). In section 6 we describe a labelled tableau calculus for **R**, then we refine it in order to obtain a terminating procedure and to describe a **coNP** decision procedure.

2 KLM Logics

We briefly recall the axiomatizations and semantics of the KLM systems. For the sake of exposition, we present the systems in the order from the strongest to the weakest: **R**, **P**, **CL**, and **C**. For a complete picture of KLM systems, see [26, 27]. The language of KLM logics consists just of conditional assertions $A \mid\sim B$. We consider a richer language allowing boolean combinations of assertions and propositional formulas. Our language $\mathcal{L}$ is defined from a set of propositional variables ATM, the boolean connectives and the conditional operator $\mid\sim$. We use $A, B, C, ...$ to denote propositional formulas (that do not contain conditional formulas), whereas $F, G, ...$ are used to denote all formulas (including conditionals); $\Gamma, \Delta, ...$ represent sets of formulas, whereas $X, Y, ...$

[1] The theorem prover KLMLean is not presented here. A description can be found in [30] and in [32]. KLMLean can be downloaded at `http://www.di.unito.it/∼pozzato/klmlean2.0`.

denote sets of sets of formulas. The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\sim B \in \mathcal{L}$; if F is a boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$.

2.1 Rational Logic R

The axiomatization of **R** consists of all axioms and rules of propositional calculus together with the following axioms and rules. We use $\vdash_{PC}$ to denote provability in the propositional calculus, whereas $\vdash$ is used to denote provability in **R**:

- REF. $A \mid\sim A$ (reflexivity)
- LLE. If $\vdash_{PC} A \leftrightarrow B$, then $\vdash (A \mid\sim C) \rightarrow (B \mid\sim C)$ (left logical equivalence)
- RW. If $\vdash_{PC} A \rightarrow B$, then $\vdash (C \mid\sim A) \rightarrow (C \mid\sim B)$ (right weakening)
- CM. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \wedge B \mid\sim C)$ (cautious monotonicity)
- AND. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \mid\sim B \wedge C)$
- OR. $((A \mid\sim C) \wedge (B \mid\sim C)) \rightarrow (A \vee B \mid\sim C)$
- RM. $((A \mid\sim B) \wedge \neg(A \mid\sim \neg C)) \rightarrow ((A \wedge C) \mid\sim B)$ (rational monotonicity)

REF states that A is always a default conclusion of A. LLE states that the syntactic form of the antecedent of a conditional formula is irrelevant. RW describes a similar property of the consequent. This allows to combine default and logical reasoning [14]. CM states that if B and C are two default conclusions of A, then adding one of the two conclusions to A will not cause the retraction of the other conclusion. AND states that it is possible to combine two default conclusions. OR states that it is allowed to reason by cases: if C is the default conclusion of two premises A and B, then it is also the default conclusion of their disjunction. RM is the rule of *rational monotonicity*, which characterizes the logic **R**[2]: if $A \mid\sim B$ and $\neg(A \mid\sim \neg C)$ hold, then one can infer $A \wedge C \mid\sim B$. This rule allows a conditional to be inferred from a set of conditionals in absence of other information. More precisely, "it says that an agent should not have to retract any previous defeasible conclusion when learning about a new fact the negation of which was not previously derivable" [27].

The semantics of **R** is defined by considering possible world structures with a preference relation (a strict partial order, i.e. an irreflexive and transitive relation) $w < w'$, whose meaning is that w is preferred to w'. The preference relation is also supposed to be *modular*: for all w, w_1 and w_2, if $w_1 < w_2$ then either $w_1 < w$ or $w < w_2$. We have that $A \mid\sim B$ holds in a model $\mathcal{M}$ if B holds in all *minimal worlds* (with respect to the relation $<$) where A holds. This definition makes sense provided minimal worlds for A exist whenever there are A-worlds. This is ensured by the *smoothness condition* in the next definition.

[2]As we will see in section 2.2, the axiom system of the weaker logic **P** can be obtained from the axioms of **R** without RM.

Definition 2.1 (Semantics of R, Definition 14 in [27]) *A rational model is a triple*

$$\mathcal{M} = \langle \mathcal{W}, <, V \rangle$$

where:

- $\mathcal{W}$ *is a non-empty set of items called worlds;*
- $<$ *is an irreflexive, transitive and modular relation on* $\mathcal{W}$*;*
- V *is a function* $V : \mathcal{W} \longmapsto pow(ATM)$*, which assigns to every world* w *the set of atoms holding in that world.*

We define the truth conditions for a formula F *as follows:*

- *If* F *is a boolean combination of formulas,* $\mathcal{M}, w \models F$ *is defined as for propositional logic;*
- *Let* A *be a propositional formula; we define* $Min_<(A) = \{w \in \mathcal{W} \mid \mathcal{M}, w \models A$ *and* $\forall w'$*,* $w' < w$ *implies* $\mathcal{M}, w' \not\models A\}$*;*
- $\mathcal{M}, w \models A \mid\sim B$ *if for all* w'*, if* $w' \in Min_<(A)$ *then* $\mathcal{M}, w' \models B$*.*

(Smoothness Condition). *The relation* $<$ *satisfies the following condition, called* smoothness*: if* $\mathcal{M}, w \models A$*, then* $w \in Min_<(A)$ *or* $\exists w' \in Min_<(A)$ *such that* $w' < w$*.*
We say that a formula F *is* valid in a model $\mathcal{M}$*, denoted with* $\mathcal{M} \models F$*, if* $\mathcal{M}, w \models F$ *for every* $w \in \mathcal{W}$*. A formula is* valid *if it is valid in every model* $\mathcal{M}$*. A formula* F *is satisfiable if there exists a model* $\mathcal{M}$ *such that* $\mathcal{M} \models F$*.*

Observe that the above definition of rational model extends the one given by KLM to boolean combinations of formulas.

Notice also that the truth conditions for conditional formulas are given with respect to single possible worlds for uniformity sake. Since the truth value of a conditional only depends on global properties of $\mathcal{M}$, we have that: $\mathcal{M}, w \models A \mid\sim B$ iff $\mathcal{M} \models A \mid\sim B$.

By the transitivity of $<$, the smoothness condition is equivalent to the following *Strong Smoothness Condition*, namely that for all A and w, if there is a world w' preferred to w that satisfies A (i.e. if $\exists w' : w' < w$ and $\mathcal{M}, w' \models A$), then there is also a *minimal* such world (i.e. $\exists w'' : w'' \in Min_<(A)$ and $w'' < w$). This follows immediately: by the smoothness condition, since $\mathcal{M}, w' \models A$, either $w' \in Min_<(A)$ (and the property immediately follows) or $\exists w''$ s.t. $w'' < w'$ and $w'' \in Min_<(A)$; in turn, by transitivity $w'' < w$. Observe that this holds for all A, whether $\mathcal{M}, w \models A$ or not. In turn, this entails that $<$ does not have infinite descending chains. Observe also that by the modularity of $<$ it follows that possible worlds of $\mathcal{W}$ are *clustered* into equivalence classes, each class consisting of worlds that are incomparable to one another; the classes are totally ordered[3]. In other words the property of modularity determines a *ranking* of worlds so that the semantics of **R** can be specified equivalently in terms of *ranked* models [27]. By means of the modularity condition on the preference relation, we can also prove the following theorem. We write $A \mid\sim B \in_+ \Gamma$ (resp. $A \mid\sim B \in_- \Gamma$) if $A \mid\sim B$ occurs positively (resp. negatively) in Γ, where positive and negative occurrences are defined in the standard way.

Theorem 2.2 (Small Model Theorem) *For any* $\Gamma \subseteq \mathcal{L}$*, if* Γ *is satisfiable in a rational model, then it is satisfiable in a rational model containing at most* n *worlds, where* n *is the size of* Γ*, i.e. the length of the string representing* Γ*.*

Proof. Let Γ be satisfiable in a rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, i.e. $\mathcal{M}, x_0 \models \Gamma$ for some $x_0 \in \mathcal{W}$. We build the model $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ as follows:

- We build the set of worlds $\mathcal{W}'$ by means of the following procedure:
  1. $\mathcal{W}' \longleftarrow \{x_0\}$;
  2. **for each** $A_i \mid\sim B_i \in_- \Gamma$ **do**
     - choose $x_i \in \mathcal{W}$ s.t. $x_i \in Min_<(A_i)$ and $\mathcal{M}, x_i \not\models B_i$;
     - $\mathcal{W}' \longleftarrow \mathcal{W}' \cup \{x_i\}$;
  3. **for each** $A_i \mid\sim B_i \in_+ \Gamma$ **do**
     if $Min_<(A_i) \neq \emptyset$, and there is no x_i s.t. $x_i \in Min_<(A_i)$ and x_i is already in $\mathcal{W}'$ **then**
     - choose any $x_i \in Min_<(A_i)$;
     - $\mathcal{W}' \longleftarrow \mathcal{W}' \cup \{x_i\}$;
- For all $x_i, x_j \in \mathcal{W}'$, we let $x_i <' x_j$ if $x_i < x_j$;
- For all $x_i \in \mathcal{W}'$, we let $V'(x_i) = V(x_i)$.

In order to show that $\mathcal{W}'$ is a rational model satisfying Γ, we can show the following Facts:

Fact 2.3 $|\mathcal{W}'| \leq n$.

Proof of Fact 2.3. The proof immediately follows by construction of $\mathcal{W}'$.

$\square$ *Fact 2.3*

Fact 2.4 $\mathcal{M}'$ *is a rational model, since* $<'$ *is irreflexive, transitive, modular and satisfies the Smoothness Condition.*

[3]Notice that the worlds themselves may be incomparable since the relation $<$ is not assumed to be (weakly) connected.

Proof of Fact 2.4. Irreflexivity, transitivity and modularity of $<'$ obviously follow from the definition of $<'$. The Smoothness Condition is ensured by the fact that $<'$ does not have infinite descending chains, since $<$ does not have.

□ *Fact 2.4*

Fact 2.5 *For all $x_i \in \mathcal{W}'$, for all propositional formulas A, $\mathcal{M}, x_i \models A$ iff $\mathcal{M}', x_i \models A$.*

Proof of Fact 2.5. By induction on the complexity of A.

□ *Fact 2.5*

Fact 2.6 *For all $x_i \in \mathcal{M}'$, for all formulas A s.t. A is the antecedent of some conditional occurring in Γ, we have that $x_i \in Min_{<'}(A)$ iff $x_i \in Min_{<}(A)$.*

Proof of Fact 2.6. First, we prove that if $x_i \in Min_{<'}(A)$, then $x_i \in Min_{<}(A)$. Let $x_i \in Min_{<'}(A)$. Suppose that $x_i \notin Min_{<}(A)$. Since A is the antecedent of a conditional in Γ, by construction of $\mathcal{W}'$, $\mathcal{W}'$ contains x_j, $x_j \neq x_i$, s.t. $x_j \in Min_{<}(A)$ in $\mathcal{M}$. Since $\mathcal{M}, x_i \models A$ and $x_j \in Min_{<}(A)$, we have that $x_j < x_i$. By Fact 2.5, $\mathcal{M}', x_j \models A$, and by the definition of $<'$, $x_j <' x_i$, which contradicts the assumption that $x_i \in Min'_{<}(A)$. We conclude that $x_i \in Min_{<}(A)$ in $\mathcal{M}$.

Now we prove that if $x_i \in Min_{<}(A)$, then $x_i \in Min_{<'}(A)$. Let $x_i \in Min_{<}(A)$ in $\mathcal{M}$. Suppose that $x_i \notin Min_{<'}(A)$. Then there is x_j s.t. $\mathcal{M}', x_j \models A$ and $x_j <' x_i$. By Fact 2.5 (since A is a propositional formula), also $\mathcal{M}, x_j \models A$, and by definition of $<'$, $x_j < x_i$, which contradicts the assumption that $x_i \in Min_{<}(A)$. Hence, $x_i \in Min_{<'}(A)$.

□ *Fact 2.6*

Fact 2.7 *For all conditional formulas $(\neg)A \mid\!\sim B$ occurring in Γ, if $\mathcal{M}, x_0 \models (\neg)A \mid\!\sim B$, then $\mathcal{M}', x_0 \models (\neg)A \mid\!\sim B$.*

Proof of Fact 2.7. We distinguish the two cases:

- $\mathcal{M}, x_0 \models \neg(A \mid\!\sim B)$: by construction of $\mathcal{W}'$, there is $x_i \in \mathcal{W}'$ s.t. $x_i \in Min_{<}(A)$ and $\mathcal{M}, x_i \not\models B$. By Facts 2.5 and 2.6, $x_i \in Min_{<'}(A)$ and $\mathcal{M}', x_i \not\models B$, hence $\mathcal{M}', x_0 \models \neg(A \mid\!\sim B)$.
- $\mathcal{M}, x_0 \models A \mid\!\sim B$: consider any $x_i \in Min_{<'}(A)$, by Fact 2.6 $x_i \in Min_{<}(A)$, hence $\mathcal{M}, x_i \models B$, and, by Fact 2.5, $\mathcal{M}', x_i \models B$. We conclude that $\mathcal{M}', x_0 \models A \mid\!\sim B$.

□ *Fact 2.7*

By the Facts above, we have shown that Γ is satisfiable in a rational model containing at most n worlds, hence the Theorem follows.

■

In the calculus for **R**, that we will introduce in section 6, we need a slightly extended language $\mathcal{L}_R$. $\mathcal{L}_R$ extends $\mathcal{L}$ by formulas of the form $\Box A$, where A is propositional, whose intuitive meaning is that $\Box A$ holds in a world w if A holds in all the worlds preferred to w (i.e. in all w' such that $w' < w$). We extend the notion of rational model to provide an evaluation of boxed formulas as follows:

Definition 2.8 (Truth condition of modality $\Box$) *We define the truth condition of a boxed formula as follows:*

$$\mathcal{M}, w \models \Box A \textit{ if, for every } w' \in \mathcal{W}, \textit{ if } w' < w \textit{ then } \mathcal{M}, w' \models A$$

From definition of $Min_<(A)$ in Definition 2.1 above, and Definition 2.8, it follows that for any formula A, $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \land \Box\neg A$.

Notice that by the Strong Smoothness Condition, it holds that if $\mathcal{M}, w \not\models \Box\neg A$, then $\exists w' < w$: $\mathcal{M}, w' \models A \land \Box\neg A$. If we regard the relation $<$ as the inverse of the accessibility relation R (thus xRy if $y < x$), it immediately follows that the Strong Smoothness Condition is an instance of the property G (restricted to A propositional). Hence it turns out that the modality $\Box$ has the properties of modal system G, in which the accessibility relation is transitive and does not have infinite ascending chains.

Since we have introduced boxed formulas for capturing a notion of minimality among worlds, in the rest of the paper we will only use this modality in front of negated formulas. Hence, to be precise, the language $\mathcal{L}_R$ of our tableau extends $\mathcal{L}$ with modal formulas of the form $\Box\neg A$.

2.2 Preferential Logic P

The axiomatization of **P** can be obtained from the axiomatization of **R** by removing the axiom RM. As for **R**, the semantics of **P** is defined by considering possible world structures with a preference relation (an irreflexive and transitive relation), which is no longer assumed to be modular.

Definition 2.9 (Semantics of P, Definition 16 in [26]) *A preferential model is a triple*

$$\mathcal{M} = \langle \mathcal{W}, <, V \rangle$$

where $\mathcal{W}$ and V are defined as for rational models in Definition 2.1, and $<$ is an irreflexive and transitive relation on $\mathcal{W}$. The truth conditions for a formula F, the smoothness condition, and the notions of validity of a formula are defined as for rational models in Definition 2.1.

As for rational models, we have extended the definition of preferential models given by KLM in order to deal with boolean combinations of formulas.

Even in this case, we define the satisfiability of conditional formulas with respect to worlds rather than with respect to models for uniformity sake. As for **R**, by the transitivity of $<$, the smoothness condition is equivalent to the Strong Smoothness Condition. In turn, this entails that $<$ does not have infinite descending chains.

Here again, we consider the language $\mathcal{L}_P$ of the calculus introduced in section 3; $\mathcal{L}_P$ corresponds to the language $\mathcal{L}_R$, i.e. it extends $\mathcal{L}$ by boxed formulas of the form $\Box\neg A$. It follows that, even in **P**, we can prove that, for any formula A, $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \land \Box\neg A$.

2.2.1 Multi-linear models for P

In the following of the paper we will need a special kind of preferential models, that we call *multi-linear*. As we will see, these models will be useful in order to provide an optimal calculus for **P**. Indeed, as we will see in section 3.1, our calculus for **P** based on multi-linear models will allow us to define proof search procedures for testing the satisfiability of a set of formulas in **P** in nondeterministic polynomial time. This result matches the known complexity results for **P**, according to which the problem of validity for **P** is in coNP.

Definition 2.10 *A finite preferential model $\mathcal{M} = (\mathcal{W}, <, V)$ is multi-linear if the set of worlds $\mathcal{W}$ can be partitioned into a set of components $\mathcal{W}_i$ for $i = 1, \ldots, n$, that is $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$ and*

1. *the relation $<$ is a total order on each $\mathcal{W}_i$;*
2. *the elements in two different components $\mathcal{W}_i$ and $\mathcal{W}_j$ are incomparable with respect to $<$.*

The following theorem shows that we could restrict our consideration to multi-linear models and generalizes Lemma 8 in [27].

Theorem 2.11 *Let Γ be any set of formulas, if Γ is satisfiable then it has a multi-linear model.*

Proof. Let us make explicit the negated conditionals in Γ by rewriting it as

$$\Gamma = \Gamma', \neg(C_1 \mid\!\sim D_1), \ldots, \neg(C_k \mid\!\sim D_k).$$

Assume Γ is satisfiable, then there is a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $x \in \mathcal{W}$, such that $\mathcal{M}, x \models \Gamma$. We have that there are $y_1, \ldots, y_k \in \mathcal{W}$, such that for each $j = 1, \ldots, k$,

$$y_j \in Min_<(C_j) \text{ and } \mathcal{M}, y_j \not\models D_j$$

We define for x and each y_j:

$$\mathcal{W}_x = \{z \in \mathcal{W} \mid z < x\} \cup \{x\}$$

$$\mathcal{W}_{y_j} = \{z \in \mathcal{W} \mid z < y_j\} \cup \{y_j\}$$

Moreover, we consider for each $j = 1, \ldots, k$ a renaming function (i.e. a bijection) f_j whose domain is $\mathcal{W}_{y_j}$ that makes a copy $\mathcal{W}_{f_j(y_j)}$ of $\mathcal{W}_{y_j}$ which is (i) disjoint from $\mathcal{W}_x$, (ii) disjoint from any $\mathcal{W}_{y_l}$, and (iii) disjoint from any other $\mathcal{W}_{f_l(y_l)}$ with $l \neq j$. Observe that we make k disjoint sets $\mathcal{W}_{f_j(y_j)}$ even if some y_j's coincide among themselves or coincide with x. We define a model $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ as follows:

$$\mathcal{W}' = \mathcal{W}_x \cup \mathcal{W}_{f_1(y_1)} \ldots \mathcal{W}_{f_k(y_k)}$$

The relation $<'$ is defined as follows:

$$u <' v \text{ iff (i) } u, v \in \mathcal{W}_x \text{ and } u < v,$$
$$\text{or (ii) } u, v \in \mathcal{W}_{f_j(y_j)} \text{ so that } u = f_j(z) \text{ and } v = f_j(w)$$

where $z, w \in \mathcal{W}_{y_j}$ and $z < w$. Observe that elements in different components (i.e. $\mathcal{W}_x$ or $\mathcal{W}_{y_j}$) are incomparable w.r.t. $<'$.

Finally, we let $V'(z) = V(z)$ for $z \in \mathcal{W}_x$ and for $u \in \mathcal{W}_{f_j(y_j)}$ with $u = f_j(w)$, we let $V'(u) = V(w)$.

We prove that $\mathcal{M}', x \models \Gamma$. The claim is obvious for propositional formulas and for (negated) boxed formulas by definition of $\mathcal{W}_x$.

For any negated conditional $\neg(C_j \mid\!\sim D_j)$, we have that $y_j \in Min_<(C_j)$ and $\mathcal{M}, y_j \not\models D_j$. By definition of $\mathcal{M}'$ we get that $\mathcal{M}', f_j(y_j) \models C_j$ and $\mathcal{M}', f_j(y_j) \not\models D_j$; we have to show that there is no $u \in \mathcal{W}_{f_j(y_j)}$ such that $u <' f_j(y_j)$ and $\mathcal{M}', u \models C_j$. But if there were a such u, we would get that $u = f_j(z)$ for some $z \in \mathcal{W}_{y_j}$ with $z < y_j$ and $\mathcal{M}, z \models C_j$ against the minimality of y_j.

For any positive conditional in Γ, say $E \mid\!\sim F$, let $u \in Min_<(E)$: if $u \in \mathcal{W}_x$ it must be $u \in Min_<(E)$ thus $\mathcal{M}', u \models F$. If $u \in \mathcal{W}_{f_j(y_j)}$, then $u = f_j(z)$ for some $z \in \mathcal{W}_{y_j}$; it must be $z \in Min_<(E)$, for otherwise if it were $z' < z$ with $\mathcal{M}, z' \models E$, since $z' \in \mathcal{W}_{y_j}$ we would have $f_j(z') < u$ and $\mathcal{M}, f_j(z') \models E$ against the minimality of u. Thus $z \in Min_<(E)$ and then $\mathcal{M}, z \models F$, and this implies $\mathcal{M}', u \models F$.

We now define a multi-linear model $\mathcal{M}_1 = \langle \mathcal{W}', <_1, V' \rangle$ as follows: we let $<_1$ be any total order on $\mathcal{W}_x$ and on each $\mathcal{W}_{f_j(y_j)}$ which respects $<'$; the elements in different components remain incomparable. More precisely $<_1$ satisfies:

- if $u <' v$ then $u <_1 v$
- for each $u, v \in \mathcal{W}_x$ $(u, v \in \mathcal{W}_{f_j(y_j)})$ with $u \neq v$, $u <_1 v$ or $v <_1 u$
- for each $u \in \mathcal{W}_x$, $v \in \mathcal{W}_{f_j(y_j)}, u \not<_1 v$ and $v \not<_1 u$
- for each $u \in \mathcal{W}_{f_i(y_i)}, v \in \mathcal{W}_{f_j(y_j)}$, with $i \neq j$ $u \not<_1 v$ and $v \not<_1 u$

In Figure 2 we show an example of multi-linear model, obtained by applying the above construction to the model represented in Figure 1.

We show that $\mathcal{M}_1, x \models \Gamma$. For propositional formulas the claim is obvious. For positive boxed-formulas we have: if $\Box\neg A \in \Gamma$, and $z <_1 x$, then $z \in \mathcal{W}_x$, thus $z <' x$, and the result follows by $\mathcal{M}', x \models \Gamma$. For negated boxed formulas, we

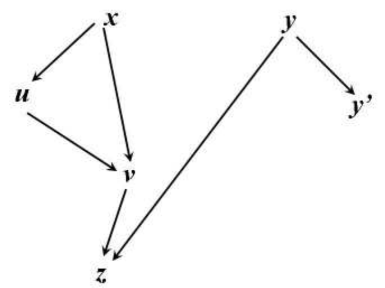

Figure 1: A preferential model satisfying a set of formulas Γ. Edges represent the preference relation $<$ ($u < x, v < u$, and so on).

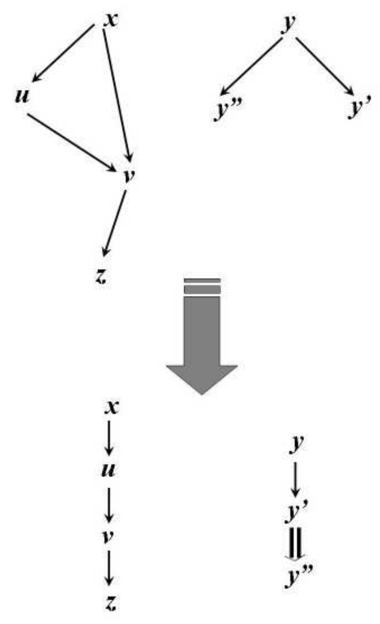

Figure 2: A multi-linear model obtained by means of the construction described in the proof of Theorem 2.11. Edges represent the preference relation $<_1$. The empty edge has been added to let $<_1$ be a total order. In order to have that elements in different components are incomparable w.r.t. $<_1$, in the rightmost component the world z has been renamed in y''.

similarly have: $\neg\Box\neg A \in \Gamma$, then $\mathcal{M}', x \models \neg\Box\neg A$, thus there exists $z <' x$ such that $\mathcal{M}', z \models A$, but $z <' x$ implies $z <_1 x$ and we can conclude.

For negated conditionals, let $\neg(C_j \mid\sim D_j)$, we know that $\mathcal{M}', x \models \neg(C_j \mid\sim D_j)$, witnessed by the C_j-minimal element $f_j(y_j)$. Since the propositional evaluation is the same, we only have to check that $f_j(y_j)$ is also minimal w.r.t. $<_1$. Suppose it is not, then there is $z \in \mathcal{W}_{f_j(y_j)}$ with $z <_1 f_j(y_j)$ such that $\mathcal{M}_1, z \models C_j$, but we would get $z <' f_j(y_j)$ against the minimality of $f_j(y_j)$.

For positive conditionals in Γ, say $E \mid\sim F$, let $u \in Min_{<_1}(E)$. It must be also $u \in Min_{<_1}(E)$, for otherwise, if there were $v <' u$, such that $\mathcal{M}', v \models E$ then we would get also $v <_1 u$ and $\mathcal{M}_1, v \models E$, against the minimality of u in $\mathcal{M}_1$.

■

2.3 Loop Cumulative Logic CL

The next KLM logic we consider is **CL**, weaker than **P**. The axiomatization of **CL** can be obtained from the axiomatization of **P** by removing the axiom OR and by adding the following infinite set of LOOP axioms:

LOOP. $(A_0 \mid\sim A_1) \land (A_1 \mid\sim A_2)...(A_{n-1} \mid\sim A_n) \land (A_n \mid\sim A_0) \rightarrow (A_0 \mid\sim A_n)$

and the following axiom CUT:

$$\text{CUT. } ((A \mid\sim B) \land (A \land B \mid\sim C)) \rightarrow (A \mid\sim C)$$

Notice that these axioms are derivable in **P** (and therefore in **R**).

The following Definition is essentially the same as Definition 13 in [26], but it is extended to boolean combinations of conditionals.

Definition 2.12 (Semantics of CL) *A loop-cumulative model is a tuple*

$$\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$$

where:

- *S is a set, whose elements are called states;*
- *$\mathcal{W}$ is a set of possible worlds;*
- *$l : S \mapsto pow(\mathcal{W})$ is a function that labels every state with a nonempty set of worlds;*
- *$<$ is an irreflexive and transitive relation on S;*
- *V is a valuation function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the atoms holding in that world.*

For $s \in S$ and A propositional, we let $\mathcal{M}, s \models\!\equiv A$ if $\forall w \in l(s)$, $\mathcal{M}, w \models A$, where $\mathcal{M}, w \models A$ is defined as for propositional logic. Let $Min_<(A)$ be the set of minimal states s such that $\mathcal{M}, s \models\!\equiv A$. We define $\mathcal{M}, s \models\!\equiv A \mid\!\sim B$ if $\forall s' \in Min_<(A)$, $\mathcal{M}, s' \models\!\equiv B$. The relation $\models\!\equiv$ can be extended to boolean combinations of conditionals in the standard way. We assume that $<$ satisfies the smoothness condition.

The above notion of cumulative model extends the one given by KLM to boolean combinations of conditionals. A further extension to arbitrary boolean combinations will be provided by the notion of CL-preferential model below.

Here again, we define satisfiability of conditionals with respect to states rather than with respect to models for uniformity reasons. Indeed, a conditional is satisfied by a state of a model only if and only if it is satisfied by all the states of that model, hence by the whole model.

As for **P** and **R**, by the transitivity of $<$, the smoothness condition is equivalent to the Strong Smoothness Condition. In turn, this entails that $<$ does not have infinite descending chains.

We show that we can map loop-cumulative models into preferential models extended with an additional accessibility relation R. We call these preferential models *CL-preferential models*. The idea is to represent states as sets of possible worlds related by R in such a way that a formula is satisfied in a state s just in case it is satisfied in all possible worlds w' accessible from a world w corresponding to s. The syntactic counterpart of the extra accessibility relation R is a modality L. Given a loop-cumulative model $\mathcal{M}$ and the corresponding CL-preferential model $\mathcal{M}'$, $\mathcal{M}, s \models\!\equiv A$ iff for a world $w \in \mathcal{M}'$ corresponding to s, we have that $\mathcal{M}', w \models LA$. As we will see, this mapping enables us to use a variant of the tableau calculus for **P** to deal with system **CL**. As for **P**, the tableau calculus for **CL** will use boxed formulas. In addition, it will also use L-formulas. Thus, the formulas that appear in the tableaux for **CL** belong to the language $\mathcal{L}_L$ obtained from $\mathcal{L}$ as follows: (i) if A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$; (ii) if A, B are propositional, then $A \mid\!\sim B \in \mathcal{L}_L$; (iii) if F is a boolean combination of formulas of $\mathcal{L}_L$, then $F \in \mathcal{L}_L$. Observe that the only allowed combination of $\Box$ and L is in formulas of the form $\Box\neg LA$ where A is propositional.

We can map loop-cumulative models into preferential models with an additional accessibility relation as defined below:

Definition 2.13 (CL-preferential models) *A CL-preferential model has the form*

$$\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$$

where:

- *$\mathcal{W}$ and V are defined as for preferential models in Definition 2.9;*
- *$<$ is an irreflexive and transitive relation on $\mathcal{W}$;*

- *R is a serial accessibility relation;*

We add to the truth conditions for preferential models in Definition 2.9 the following clause:

$$\mathcal{M}, w \models LA \text{ if, for all } w', \ wRw' \text{ implies } \mathcal{M}, w' \models A$$

The relation $<$ satisfies the following Smoothness Condition*: if $\mathcal{M}, w \models LA$, then $w \in Min_<(LA)$ or $\exists w' \in Min_<(LA)$ such that $w' < w$.*
Moreover, we need to change the truth condition for conditional formulas as follows: $\mathcal{M}, w \models A \mid\!\sim B$ if for all $w' \in Min_<(LA)$ we have $\mathcal{M}, w' \models LB$.

We can prove that, given a loop-cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V\rangle$ satisfying a boolean combination of conditional formulas, one can build a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$ satisfying the same combination of conditionals. We build a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$ as follows:

- $\mathcal{W}' = \{(s, w) : s \in S \text{ and } w \in l(s)\}$;
- $(s, w)R(s, w')$ for all $(s, w), (s, w') \in \mathcal{W}'$;
- $(s, w) <' (s', w')$ if $s < s'$;
- $V'(s, w) = V(w)$.

Viceversa, given a CL-preferential model $\mathcal{M} = \langle \mathcal{W}, R, <, V\rangle$ satisfying a boolean combination of conditional formulas, one can build a loop-cumulative model $\mathcal{M}' = \langle S, \mathcal{W}, l, <', V'\rangle$ satisfying the same combination of conditional formulas. The model $\mathcal{M}'$ is defined as follows (we define $Rw = \{w' \in \mathcal{W} \mid (w, w') \in R\}$):

- $S = \{(w, Rw) \mid w \in \mathcal{W}\}$;
- $l((w, Rw)) = Rw$;
- $(w, Rw) <' (w', Rw')$ if $w < w'$;
- $V'(w) = V(w)$.

This is stated in a rigorous manner by the following proposition:

Proposition 2.14 *A boolean combination of conditional formulas is satisfiable in a loop-cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V\rangle$ iff it is satisfiable in a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$.*

Proof. The Proposition immediately follows from the following Lemma:

Lemma 2.15 *A set of conditional formulas $\{(\neg)A_1 \mid\!\sim B_1, \ldots, (\neg)A_n \mid\!\sim B_n\}$ is satisfiable in a loop-cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V\rangle$ iff it is satisfiable in a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$.*

First, we prove the *only if* direction. Let $\mathcal{M} = \langle S, \mathcal{W}, l, <, V\rangle$ be a loop-cumulative model, and $s \in S$ s.t. $(\mathcal{M}, s) \models\equiv \{(\neg)A_i \mid\sim B_i\}$.

We build a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$ as follows:

- $\mathcal{W}' = \{(s, w) : s \in S \text{ and } w \in l(s)\}$;
- $(s, w)R(s, w')$ for all $(s, w), (s, w') \in \mathcal{W}'$;
- $(s, w) <' (s', w')$ if $s < s'$;
- $V'(s, w) = V(w)$.

Observe that for each $s \in S$ there is at least one corresponding $(s, w) \in \mathcal{W}'$, since $l(s) \neq \emptyset$. From the fact that $<$ in $\mathcal{M}$ is irreflexive and transitive it immediately follows by construction that also $<'$ in $\mathcal{M}'$ satisfies the same properties. We show in Fact 2.20 below that $<'$ satisfies the smoothness condition on L-formulas.

The relation R is serial, since it is reflexive.

Fact 2.16 *For every propositional formula A we have that $(\mathcal{M}, s) \models\equiv A$ iff $(\mathcal{M}', (s, w)) \models LA$.*

Proof of Fact 2.16. $(\Rightarrow)$ Let $(\mathcal{M}, s) \models\equiv A$. By definition, for all $w \in l(s)$, $(\mathcal{M}, w) \models A$. By induction on the complexity of A, we can easily show that $(\mathcal{M}', (s, w)) \models A$. Since $R(s, w) = \{(s, w') \mid w' \in l(s)\}$, it follows that $(\mathcal{M}', (s, w)) \models LA$.
$(\Leftarrow)$ Let $(\mathcal{M}', (s, w)) \models LA$. Then, for all $(s, w') \in R(s, w)$, $(\mathcal{M}', (s, w')) \models A$. By definition of $\mathcal{M}'$ it follows that for all $w' \in l(s)$, $(\mathcal{M}, w') \models A$. Hence, $(\mathcal{M}, s) \models\equiv A$.

□ *Fact 2.16*

Fact 2.17 *$s \in Min_<(A)$ in $\mathcal{M}$ iff $(s, w) \in Min_{<'}(LA)$ in $\mathcal{M}'$.*

Proof of Fact 2.17. $(\Rightarrow)$ Let $s \in Min_<(A)$ in $\mathcal{M}$. Consider (s, w) in $\mathcal{M}'$. By Fact 2.16, $(\mathcal{M}', (s, w)) \models LA$. By absurd, suppose there exists a (s', w') s.t. $(\mathcal{M}', (s', w')) \models LA$, and $(s', w') < (s, w)$. By Fact 2.16, $(\mathcal{M}, s') \models\equiv A$, and $s' < s$ by construction, which contradicts the fact that $s \in Min_<(A)$ in $\mathcal{M}$. Hence $(s, w) \in Min_{<'}(LA)$ in $\mathcal{M}'$.
$(\Leftarrow)$ Let $(s, w) \in Min_{<'}(LA)$ in $\mathcal{M}'$. Consider s in $\mathcal{M}$. By Fact 2.16, $(\mathcal{M}, s) \models\equiv A$. Furthermore there is no $s' < s$ s.t. $(\mathcal{M}, s') \models\equiv A$. By absurd suppose there was such a s'. By construction of $\mathcal{M}'$, and by Fact 2.16, $(\mathcal{M}', (s', w')) \models LA$, and $(s', w') < (s, w)$, which is a contradiction. Hence $s \in Min_<(A)$.

□ *Fact 2.17*

Fact 2.18 *For every conditional formula $A \mid\sim B$ we have that* $(\mathcal{M}, s) \models\equiv A \mid\sim B$ *iff* $(\mathcal{M}', (s,w)) \models A \mid\sim B$.

Proof of Fact 2.18. ($\Rightarrow$) Let $(\mathcal{M}, s) \models\equiv A \mid\sim B$. Then for all $s' \in Min_<(A)$, $(\mathcal{M}, s') \models\equiv B$. By Facts 2.16 and 2.17, it follows that for all $(s', w') \in Min_{<'}(LA)$, $(\mathcal{M}', (s', w')) \models LB$, hence $(\mathcal{M}', (s,w)) \models A \mid\sim B$.
($\Leftarrow$) Let $(\mathcal{M}', (s,w)) \models A \mid\sim B$. Then, for all $(s', w') \in Min'_<(LA)$, $(\mathcal{M}', (s', w')) \models LB$. By Facts 2.16 and 2.17 it follows that for all $s' \in Min_<(A)$ in $\mathcal{M}$, $(\mathcal{M}, s') \models\equiv B$. Hence, $(\mathcal{M}, s) \models\equiv A \mid\sim B$.

□ *Fact 2.18*

Fact 2.19 *For every negated conditional formula $\neg(A \mid\sim B)$ we have that $(\mathcal{M}, s) \models\equiv \neg(A \mid\sim B)$ iff $(\mathcal{M}', (s,w)) \models \neg(A \mid\sim B)$.*

Proof of Fact 2.19. ($\Rightarrow$) Let $(\mathcal{M}, s) \models\equiv \neg(A \mid\sim B)$. Then there is an $s' \in Min_<(A)$ s.t. $\mathcal{M}, s' \not\models\equiv B$. Consider (s', w') in $\mathcal{M}'$. By Facts 2.16 and 2.17 $(s', w') \in Min_{<'}(LA)$ and $(\mathcal{M}', (s', w')) \not\models LB$. Hence, $(\mathcal{M}', (s,w)) \models \neg(A \mid\sim B)$.
($\Leftarrow$) Let $(\mathcal{M}', (s,w)) \models \neg(A \mid\sim B)$. Then, there is a $(s', w') \in Min'_<(LA)$ s.t. $(\mathcal{M}', (s', w')) \not\models LB$. Consider s' in $\mathcal{M}$. From Facts 2.16 and 2.17 we conclude that $s' \in Min_<(A)$, and $(\mathcal{M}, s') \not\models\equiv B$. Hence $(\mathcal{M}, s) \models\equiv \neg(A \mid\sim B)$.

□ *Fact 2.19*

From Facts 2.18 and 2.19 we conclude that $(\mathcal{M}', (s,w)) \models \{(\neg)A_i \mid\sim B_i\}$. Furthermore, we show that $<'$ satisfies the smoothness condition on L-formulas.

Fact 2.20 *$<'$ satisfies the smoothness condition on L-formulas.*

Proof of Fact 2.20. Let $(\mathcal{M}', (s,w)) \models LA$, and $(s,w) \notin Min_{<'}(LA)$. By Fact 2.16 $(\mathcal{M}, s) \models\equiv A$, and by Fact 2.17, $s \notin Min_<(A)$ in $\mathcal{M}$. By the smoothness condition in $\mathcal{M}$ there is s' such that $s' \in Min_<(A)$ and $s' < s$. Consider any $(s', w') \in \mathcal{M}'$. By Fact 2.17 $(s', w') \in Min_{<'}(LA)$, and by definition of $<'$, $(s', w') <' (s, w)$.

□ *Fact 2.20*

Let us now consider the *if* direction. Let the set of conditionals $\{(\neg)A_i \mid\sim B_i\}$ be satisfied in a possible world w in the CL-preferential model $\mathcal{M} = \langle \mathcal{W}, R, <, V\rangle$. We build a Loop-Cumulative model $\mathcal{M}' = \langle S, \mathcal{W}, l, <', V'\rangle$ as follows (Rw is defined as $Rw = \{w' \in \mathcal{W} \mid (w, w') \in R\}$):

- $S = \{(w, Rw) \mid w \in \mathcal{W}\}$;

- $l((w, Rw)) = Rw$;
- $(w, Rw) <' (w', Rw')$ if $w < w'$;
- $V'(w) = V(w)$.

From the fact that $<$ in $\mathcal{M}$ is transitive and irreflexive, it follows by construction that $<'$ in $\mathcal{M}'$ is transitive and irreflexive. As far as the smoothness condition, see Fact 2.24 below. Furthermore, for all $(w, Rw) \in S$, $l(w, Rw) \neq \emptyset$, since R is serial.

We now show that $(\mathcal{M}', (w, Rw)) \models\equiv \{(\neg)A_i \mid\!\sim B_i\}$.

Fact 2.21 *For A propositional, $(\mathcal{M}, w) \models LA$ iff $(\mathcal{M}', (w, Rw)) \models\equiv A$.*

Proof of Fact 2.21. ($\Rightarrow$) Let $(\mathcal{M}, w) \models LA$. Then for all $w' \in Rw$, $\mathcal{M}, w' \models A$. By definition of $\mathcal{M}'$, it follows that for all $w' \in l(w, Rw)$, $\mathcal{M}', w' \models A$. Hence $(\mathcal{M}', (w, Rw)) \models\equiv A$.
($\Leftarrow$) Let $(\mathcal{M}', (w, Rw)) \models\equiv A$. Then, for all $w' \in l(w, Rw)$, $(\mathcal{M}', w') \models A$. By induction on A, we show that for all $w' \in Rw$, $(\mathcal{M}, w') \models A$, hence $(\mathcal{M}, w) \models LA$.

$\square$ *Fact 2.21*

Fact 2.22 *$w \in Min_<(LA)$ in $\mathcal{M}$ iff $(w, Rw) \in Min_{<'}(A)$ in $\mathcal{M}'$.*

Proof of Fact 2.22. ($\Rightarrow$) Let $w \in Min_<(LA)$ in $\mathcal{M}$. Consider (w, Rw). By Fact 2.21 $(\mathcal{M}', (w, Rw)) \models\equiv A$. Furthermore, suppose by absurd there was (w', Rw') s.t. $(\mathcal{M}', (w', Rw')) \models\equiv A$, and $(w', Rw') <' (w, Rw)$. Then in $\mathcal{M}$, $\mathcal{M}, w' \models LA$ and $w' < w$, which contradicts the fact that $w \in Min_<(LA)$. It follows that in $\mathcal{M}'$, $(w, Rw) \in Min_{<'}(A)$.
($\Leftarrow$) Let $(w, Rw) \in Min_{<'}(A)$ in $\mathcal{M}'$. Consider w in $\mathcal{M}$. By Fact 2.21, $(\mathcal{M}, w) \models LA$. By absurd, suppose there was a w' s.t. $\mathcal{M}, w' \models LA$ and $w' < w$. By Fact 2.21, $(\mathcal{M}', (w', Rw')) \models\equiv A$, and $(w', Rw') <' (w, Rw)$, which contradicts the fact that $(w, Rw) \in Min_{<'}(A)$. It follows that $w \in Min_<(LA)$ in $\mathcal{M}$.

$\square$ *Fact 2.22*

We can reason similarly to what done in Facts 2.18 and 2.19 above to prove the following Fact:

Fact 2.23 *For every conditional formula $(\neg)A \mid\!\sim B$ we have that $(\mathcal{M}, w) \models (\neg)A \mid\!\sim B$ iff $(\mathcal{M}', (w, Rw)) \models\equiv (\neg)A \mid\!\sim B$.*

We conclude that $(\mathcal{M}', (w, Rw)) \models\equiv \{(\neg)A_i \mid\!\sim B_i\}$.
Furthermore, we show that $<'$ satisfies the smoothness condition:

Fact 2.24 *$<'$ satisfies the smoothness condition on L-formulas.*

Proof of Fact 2.24. Let $(\mathcal{M}', (w, Rw)) \models\equiv A$ and $(w, Rw) \notin Min_{<'}(A)$ in $\mathcal{M}'$. By Facts 2.21 and 2.22, $(\mathcal{M}, w) \models LA$ and $w \notin Min_{<}(LA)$ in $\mathcal{M}$. By the smoothness condition on L-formulas in $\mathcal{M}$, it follows that in $\mathcal{M}$ there is $w' < w$ s.t. $(\mathcal{M}, w') \models LA$ and $w' \in Min_{<}(LA)$. Consider (w', Rw') in $\mathcal{M}'$. By Facts 2.21 and 2.22 $(\mathcal{M}', (w', Rw')) \models\equiv A$ and $(w', Rw') \in Min_{<'}(A)$ in $\mathcal{M}'$.

□ *Fact 2.24*

■

Similarly to what done for **P**, we define multi-linear CL-preferential models as follows:

Definition 2.25 *A finite CL-preferential model $\mathcal{M} = (\mathcal{W}, R, <, V)$ is multi-linear if the set of worlds $\mathcal{W}$ can be partitioned into a set of components $\mathcal{W}_i$ for $i = 1, \ldots, n$ (that is $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$), and in each $\mathcal{W}_i$:*

1. *we can distinguish a chain of worlds $w_1, w_2, \ldots, w_h$ totally ordered w.r.t. $<$ (i.e. $w_1 < w_2 < \ldots < w_h$) such that all other worlds $w \in \mathcal{W}_i$ are R-accessible from some w_l in the chain, i.e. $\forall w \in \mathcal{W}_i$ such that $w \neq w_1, w_2, \ldots, w_h$, we have that $w_1Rw \vee w_2Rw \vee \ldots \vee w_hRw$;*
2. *for all w_l, w_j, w_k, if $w_l < w_j$ and w_jRw_k, then $w_l < w_k$.*

Moreover, the elements of different $\mathcal{W}_i$ are incomparable w.r.t. $<$.

We can easily prove the following Theorem:

Theorem 2.26 *Let Γ be any set of formulas, if Γ is satisfiable in a CL-preferential model, then it has a multi-linear model.*

2.4 Cumulative Logic C

The weakest logical system considered by KLM [26] is Cumulative Logic **C**. System **C** is weaker than **CL** considered above since it does not have the set of (LOOP) axioms. At a semantic level, the difference between **CL** models and **C** models is that in **CL** models the relation $<$ is transitive, whereas in **C** it is not. Thus, cumulative **C** models are defined as follows :

Definition 2.27 (Semantics of C, Definitions 5, 6, 7 in [26]) *A cumulative model is a tuple*

$$\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$$

where S, $\mathcal{W}$, l, and V are defined as for loop-cumulative models in Definition 2.12, whereas $<$ is an irreflexive relation on S. The truth definitions of formulas are as for loop-cumulative models in Definition 2.12. We assume that $<$ satisfies the smoothness condition.

Since $<$ is no longer transitive, the smoothness condition is no longer equivalent to the Strong Smoothness Condition; hence, in this case, we cannot show that $<$ does not have infinite descending chains. Indeed, the relation $<$ might have cycles (leading to infinite descending chains): it can be easily seen that in **C** we may have sequences of worlds such as: a minimal A-world followed by a minimal B-world followed by a minimal A-world and so on. This sequence respects the smoothness condition. However, one can legitimately wonder what minimal means in this case, the notion having lost its intuitive meaning.

In order to be convinced that (1) the Strong Smoothness Condition and (2) the smoothness condition are not equivalent, consider the following set of formulas: $\{\neg(C \mid\sim B), C \mid\sim A, A \mid\sim B, B \mid\sim C\}$. This set of formulas is unsatisfiable in a model satisfying (1) whereas it is satisfiable in a model only satisfying (2), hence it is satisfiable in **C**.

Similarly to what done for loop-cumulative models, we can establish a correspondence between cumulative models and preferential models augmented with an accessibility relation in which the preference relation $<$ is an irreflexive relation satisfying the smoothness condition. We call these models C-preferential models.

Definition 2.28 (C-preferential models) *A C-preferential model has the form $\mathcal{M} = \langle \mathcal{W}, R, <, V\rangle$ where: $\mathcal{W}$ is a non-empty set of items called worlds; R is a serial accessibility relation; $<$ is an irreflexive relation on $\mathcal{W}$ satisfying the smoothness condition for L-formulas; V is a function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the atomic formulas holding in that world. The truth conditions for the boolean cases are defined in the obvious way. Truth conditions for modal and conditional formulas are the same as in CL-preferential models in Definition 2.13, thus:*

- $M, w \models LA$ *if for all* w', wRw' *implies* $\mathcal{M}, w' \models A$
- $\mathcal{M}, w \models A \mid\sim B$ *if for all* $w' \in Min_<(LA)$, *we have* $\mathcal{M}, w' \models LB$.

The correspondence between cumulative and preferential models is established by the following proposition. Its proof is the same as the proof of Proposition 2.14 (except for transitivity) and is therefore omitted.

Proposition 2.29 *A boolean combination of conditional formulas is satisfiable in a cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V\rangle$ iff it is satisfiable in a C-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V'\rangle$.*

In the following sections we present the tableaux calculi for the logics introduced. We start by presenting the calculus for **P**, which is the simpler and more general one. The calculi for **CL**, **C**, and **R** will become more understandable once the calculus for **P** is known.

3 The Tableau Calculus for Preferential Logic P

In this section we present a tableau calculus for **P** called $\mathcal{T}\mathbf{P}$, then we analyze it in order to obtain a decision procedure for this logic. We also give an explicit complexity bound for **P**.

As already mentioned in section 2.2, we consider the language $\mathcal{L}_P$, which extends $\mathcal{L}$ by boxed formulas of the form $\Box\neg A$.

Definition 3.1 (The calculus $\mathcal{T}\mathbf{P}$) *The rules of the calculus manipulate sets of formulas Γ. We write Γ, F as a shorthand for $\Gamma \cup \{F\}$. Moreover, given Γ we define the following sets:*

- $\Gamma^{\Box} = \{\Box\neg A \mid \Box\neg A \in \Gamma\}$
- $\Gamma^{\Box^{\downarrow}} = \{\neg A \mid \Box\neg A \in \Gamma\}$
- $\Gamma^{\mid\sim^{+}} = \{A \mid\sim B \mid A \mid\sim B \in \Gamma\}$
- $\Gamma^{\mid\sim^{-}} = \{\neg(A \mid\sim B) \mid \neg(A \mid\sim B) \in \Gamma\}$
- $\Gamma^{\mid\sim^{\pm}} = \Gamma^{\mid\sim^{+}} \cup \Gamma^{\mid\sim^{-}}$

The tableau rules are given in Figure 3. A tableau is a tree whose nodes are sets of formulas Γ. Therefore, a branch is a sequence of sets of formulas $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ Each node Γ_i is obtained by its immediate predecessor Γ_{i-1} by applying a rule of $\mathcal{T}\mathbf{P}$, having Γ_{i-1} as the premise and Γ_i as one of its conclusions. A branch is closed if one of its nodes is an instance of (**AX**)*, otherwise it is open. We say that a tableau is closed if all its branches are closed.*

The rues for the boolean propositions are the usual ones. According to the rule ($\mid\sim^{-}$), if a negated conditional $\neg(A \mid\sim B)$ holds in a world, then there is a minimal A-world (i.e. in which A and $\Box\neg A$ hold) which falsifies B. According to the rule ($\mid\sim^{+}$), if a positive conditional $A \mid\sim B$ holds in a world, then either the world falsifies A or it is not minimal for A (i.e. $\neg\Box\neg A$ holds) or it is a B-world. According to the rule ($\Box^{-}$), if a world satisfies $\neg\Box\neg A$, by the strong smoothness condition there must be a preferred minimal A-world, i.e. a world in which A and $\Box\neg A$ hold. In our calculus $\mathcal{T}\mathbf{P}$, axioms are restricted to atomic formulas only. It is easy to extend axioms to a generic formula F, as stated by the following Proposition:

Proposition 3.2 *Given a formula F and a set of formulas Γ, then $\Gamma, F, \neg F$ has a closed tableau.*

Proof. By an easy inductive argument on the structure of the formula F.

Definition 3.3 *Given a set of formulas Γ, Γ is consistent if no tableau for Γ is closed.*

$$(\mathbf{AX})\ \Gamma, P, \neg P \quad \text{with } P \in ATM \qquad\qquad (\neg)\ \frac{\Gamma, \neg\neg F}{\Gamma, F}$$

$$(\wedge^{+})\ \frac{\Gamma, F \wedge G}{\Gamma, F, G} \qquad\qquad (\wedge^{-})\ \frac{\Gamma, \neg(F \wedge G)}{\Gamma, \neg F \qquad \Gamma, \neg G}$$

$$(\vee^{+})\ \frac{\Gamma, F \vee G}{\Gamma, F \qquad \Gamma, G} \qquad\qquad (\vee^{-})\ \frac{\Gamma, \neg(F \vee G)}{\Gamma, \neg F, \neg G}$$

$$(\rightarrow^{+})\ \frac{\Gamma, F \rightarrow G}{\Gamma, \neg F \qquad \Gamma, G} \qquad\qquad (\rightarrow^{-})\ \frac{\Gamma, \neg(F \rightarrow G)}{\Gamma, F, \neg G}$$

$$(\mathrel{|\!\sim}^{+})\ \frac{\Gamma, A \mathrel{|\!\sim} B}{\Gamma, \neg A, A \mathrel{|\!\sim} B \qquad \Gamma, \neg\Box\neg A, A \mathrel{|\!\sim} B \qquad \Gamma, B, A \mathrel{|\!\sim} B}$$

$$(\mathrel{|\!\sim}^{-})\ \frac{\Gamma, \neg(A \mathrel{|\!\sim} B)}{A, \Box\neg A, \neg B, \Gamma^{\mathrel{|\!\sim}^{\pm}}} \qquad\qquad (\Box^{-})\ \frac{\Gamma, \neg\Box\neg A}{\Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mathrel{|\!\sim}^{\pm}}, A, \Box\neg A}$$

Figure 3: Tableau system $\mathcal{TP}$.

As an example, we show that $adult \mathrel{|\!\sim} \neg retired$ can be inferred from a knowledge base containing the following assertions: $adult \mathrel{|\!\sim} worker, retired \mathrel{|\!\sim} adult, retired \mathrel{|\!\sim} \neg worker$. Figure 4 shows a derivation for the initial set of formulas $adult \mathrel{|\!\sim} worker, retired \mathrel{|\!\sim} adult, retired \mathrel{|\!\sim} \neg worker, \neg(adult \mathrel{|\!\sim} \neg retired)$.
Our tableau calculus $\mathcal{TP}$ is based on a runtime translation of conditional assertions into modal logic G. As we have seen in section 2.2, this allows a characterization of the minimal worlds satisfying a formula A (i.e., the worlds in $Min_{<}(A)$) as the worlds w satisfying the formula $A \wedge \Box\neg A$. It is tempting to provide a full translation of the conditionals in the logic G, and then to use the standard tableau calculus for G. To this purpose, we can exploit the transitivity properties of G frames in order to capture the fact that conditionals are global to all worlds by the formula $\Box(A \wedge \Box\neg A \rightarrow B)$. Hence, the overall translation of a conditional formula $A \mathrel{|\!\sim} B$ could be the following one: $(A \wedge \Box\neg A \rightarrow B) \wedge \Box(A \wedge \Box\neg A \rightarrow B)$. However, there are significant differences between the calculus resulting from the translation and our calculus.

Using the standard tableau rules for G on the translation, we get the rule $(\mathrel{|\!\sim}^{+})$ as a derived rule. Instead, the rule for dealing with negated conditionals (which would be translated in G as a disjunction of two formulas, namely $(A \wedge \Box\neg A \wedge \neg B) \vee \Diamond(A \wedge \Box\neg A \wedge \neg B))$, is rather different.

Let us first observe that the rule $(\mathrel{|\!\sim}^{-})$ we have introduced precisely captures the intuition that conditionals are global, hence (1) all conditionals are kept in

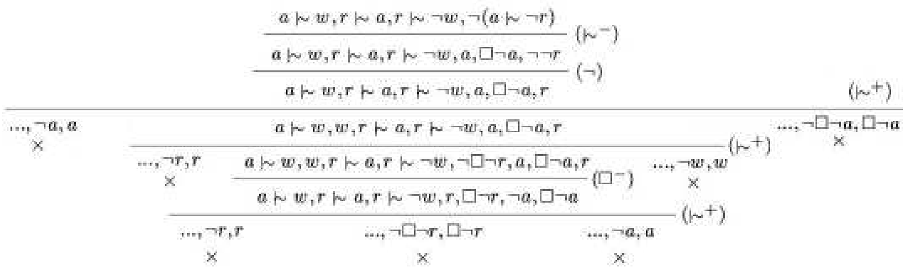

Figure 4: A derivation of $adult \mid\!\sim worker, retired \mid\!\sim adult, retired \mid\!\sim \neg worker, \neg(adult \mid\!\sim \neg retired)$. For readability, we use a to denote *adult*, r for *retired*, and w for *worker*.

the conclusion of the rule and (2) when moving to a new minimal world, all the boxed formulas (positive and negated) are removed. Conversely, when the tableau rules for G are applied to the translation of the negated conditionals, we get two branches (due to the disjunction). None of the branches can be eliminated. In both branches all the boxed formulas are kept, while negated conditionals are erased. This is quite different from our rule $(\mid\!\sim^-)$, and it is not that obvious that the calculus obtained by the translation of **P** conditionals in G is equivalent to $\mathcal{T}\mathbf{P}$. Roughly speaking, point (2) can be explained as follows: when a negated conditional $\neg(A \mid\!\sim B)$ is evaluated in a world w, this corresponds to finding a minimal A-world w' satisfying $\neg B$ (a world satisfying $A, \Box\neg A, \neg B$). w' does not depend from w (since conditionals are global), hence boxed formulas, keeping information about w, can be removed.

Also observe that, from the semantic point of view, the model extracted from an open tableau has the structure of a forest, while the model constructed by applying the tableau for G to the translation of conditionals has the structure of a tree. This difference is due to the fact that the above translation of **P** in G uses the same modality $\Box$ both for capturing the minimality condition and for modelling the fact that conditionals are global. For this reason, a translation to G as the one proposed above for **P**, would not be applicable to the cumulative logic **C**, as the relation $<$ is not transitive in **C**. Moreover, the treatment of both the logics **C** and **CL** would anyhow require the addition to the language of a new modality to deal with states. The advantage of the runtime translation we have adopted is that of providing a uniform approach to deal with the different logics.

The system $\mathcal{T}\mathbf{P}$ is sound and complete with respect to the semantics.

Theorem 3.4 (Soundness of $\mathcal{T}$P) *The system $\mathcal{T}\mathbf{P}$ is sound with respect to preferential models, i.e. if there is a closed tableau for a set Γ, then Γ is unsatisfiable.*

Proof. As usual, we proceed by induction on the structure of the closed tableau having the set Γ as a root. The base case is when the tableau consists of a single node; in this case, both P and $\neg P$ occur in Γ, therefore Γ is obviously unsatisfiable. For the inductive step, we have to show that, for each rule r, if all the conclusions of r are unsatisfiable, then the premise is unsatisfiable too. We show the contrapositive, i.e. we prove that if the premise of r is satisfiable, then at least one of the conclusions is satisfiable. Boolean cases are easy and left to the reader. We present the cases for conditional and box rules:

- $(\mid\!\sim^+)$: if $\Gamma, A \mid\!\sim B$ is satisfiable, then there exists a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ with some world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \Gamma, A \mid\!\sim B$. We distinguish the two following cases:
  - $\mathcal{M}, w \not\models A$, thus $\mathcal{M}, w \models \neg A$: in this case, the left conclusion of the $(\mid\!\sim^+)$ rule is satisfied ($\mathcal{M}, w \models \Gamma, A \mid\!\sim B, \neg A$);
  - $\mathcal{M}, w \models A$: we consider two subcases:
    * $w \in Min_<(A)$: by the definition of $\mathcal{M}, w \models A \mid\!\sim B$, we have that for all $w' \in Min_<(A)$, $\mathcal{M}, w' \models B$. Therefore, we have that $\mathcal{M}, w \models B$ and the right conclusion of $(\mid\!\sim^+)$ is satisfiable;
    * $w \notin Min_<(A)$: by the smoothness condition, there exists a world $w' < w$ such that $w' \in Min_<(A)$; therefore, $\mathcal{M}, w \models \neg\Box\neg A$ by the definition of $\Box$. The central conclusion of the $(\mid\!\sim^+)$ rule is then satisfiable.
- $(\mid\!\sim^-)$: if $\Gamma, \neg(A \mid\!\sim B)$ is satisfiable, then $\mathcal{M}, w \models \Gamma$ and $(*)\mathcal{M}, w \not\models A \mid\!\sim B$ for some world w. By $(*)$, there is a world w' in the model $\mathcal{M}$ such that $w' \in Min_<(A)$ (i.e. $(1)\mathcal{M}, w' \models A$ and $(2)\mathcal{M}, w' \models \Box\neg A$) and $(3)\mathcal{M}, w' \not\models B$. By $(1), (2)$ and (3), we have that $\mathcal{M}, w' \models A, \Box\neg A, \neg B$[4]. We conclude $\mathcal{M}, w' \models A, \Box\neg A, \neg B, \Gamma^{\mid\sim\pm}$, since conditionals are "global" in a model.
- $(\Box^-)$: if $\Gamma, \neg\Box\neg A$ is satisfiable, then there is a model $\mathcal{M}$ and some world w such that $\mathcal{M}, w \models \Gamma, \neg\Box\neg A$, then $\mathcal{M}, w \not\models \Box\neg A$. By the truth definition of $\Box$, there exists a world w' such that $w' < w$ and $\mathcal{M}, w' \models A$. By the Strong Smoothness Condition, we can assume that w' is a *minimal* A-world. Therefore, $\mathcal{M}, w' \models \Box\neg A$ by the truth definition of $\Box$. It is easy to conclude that $\mathcal{M}, w' \models A, \Box\neg A, \Gamma^{\mid\sim\pm}, \Gamma^{\Box^\downarrow}, \Gamma^{\Box}$, since 1. conditionals are global in a model, then $\mathcal{M}, w' \models \Gamma^{\mid\sim\pm}$, 2. formulas in $\Gamma^{\Box^\downarrow}$ are true in w' since $w' < w$ and 3. the $<$ relation is transitive, thus boxed formulas holding in w (i.e. $\Gamma^{\Box}$) also hold in w'.

■

To prove the completeness of $\mathcal{TP}$ we have to show that if F is unsatisfiable, then there is a closed tableau starting with F. We prove the contrapositive, that is: if there is no closed tableau for F, then there is a model satisfying F. This proof

[4] We use $\mathcal{M}, w' \models F_1, F_2, \ldots, F_n$ to denote that $\mathcal{M}, w' \models F_1$, $\mathcal{M}, w' \models F_2$, ..., and $\mathcal{M}, w' \models F_n$.

is inspired by [22]. First of all, we distinguish between *static* and *dynamic* rules. The rules ($\mid\!\sim^-$) and ($\Box^-$) are called *dynamic*, since their conclusion represents another world with respect to the premise; the other rules are called *static*, since the world represented by premise and conclusion(s) is the same. Moreover, we have to introduce the *saturation* of a set of formulas Γ. Given a set of formulas Γ, we say that it is saturated if all the rules have been applied.

Definition 3.5 (Saturated sets) *A set of formulas Γ is saturated with respect to the static rules if the following conditions hold:*

- *if $F \wedge G \in \Gamma$ then $F \in \Gamma$ and $G \in \Gamma$;*
- *if $\neg(F \wedge G) \in \Gamma$ then $\neg F \in \Gamma$ or $\neg G \in \Gamma$;*
- *if $F \vee G \in \Gamma$ then $F \in \Gamma$ or $G \in \Gamma$;*
- *if $\neg(F \vee G) \in \Gamma$ then $\neg F \in \Gamma$ and $\neg G \in \Gamma$;*
- *if $F \rightarrow G \in \Gamma$ then $\neg F \in \Gamma$ or $G \in \Gamma$;*
- *if $\neg(F \rightarrow G) \in \Gamma$ then $F \in \Gamma$ and $\neg G \in \Gamma$;*
- *if $\neg\neg F \in \Gamma$ then $F \in \Gamma$;*
- *if $A \mid\!\sim B \in \Gamma$ then $\neg A \in \Gamma$ or $\neg\Box\neg A \in \Gamma$ or $B \in \Gamma$.*

It is easy to observe that the following Lemma holds:

Lemma 3.6 *Given a consistent finite set of formulas Γ, there is a consistent, finite, and saturated set $\Gamma' \supseteq \Gamma$.*

Proof. Consider the set $\Gamma^\star$ of complex formulas in Γ such that there is a static rule that has not yet been applied to that formula in Γ. For instance, if $\Gamma = \{P \mid\!\sim Q, \neg(R \rightarrow S), R, \neg S, R \vee T, \neg\Box\neg Q\}$, then $\Gamma^\star = \{P \mid\!\sim Q\}$, since neither $\neg P$ nor $\neg\Box\neg P$ nor Q, resulting from an application of the static rule ($\mid\!\sim^+$) to $P \mid\!\sim Q$, belong to Γ.

If $\Gamma^\star$ is empty, we are done. Otherwise, we construct the saturated set Γ' as follows: 1. initialize Γ' with Γ; 2. choose a complex formula F in $\Gamma^\star$ and apply the *static* rule corresponding to its principal operator; 3. add to Γ' the formula(s) of (one of) the *consistent* conclusions obtained by applying the static rule; 4. update $\Gamma^\star$[5] and repeat from 2. until $\Gamma^\star$ is empty. This procedure terminates, since in all static rules the conclusions have a lower complexity than the premise; a brief discussion on the ($\mid\!\sim^+$) rule: from the premise $A \mid\!\sim B$ we have the following possible conclusions: $\neg A$, $\neg\Box\neg A$ and B. If $\neg\Box\neg A$ is introduced, then no other static rule will be applied to it. Since A and B are boolean combinations of formulas, then the other applications of static rules to them will decrease the complexity.

[5] The complex formula analyzed at the current step must be removed from $\Gamma^\star$ and formulas obtained by the application of the static rules that fulfill the definition of $\Gamma^\star$ must be added.

Furthermore, each step of the procedure preserves the consistency of Γ. Indeed, each conclusion of the rule applied to Γ corresponds to a branch of a tableau for Γ. If all the branches were inconsistent, Γ would have a closed tableau, hence would be inconsistent, against the hypothesis.

■

By Lemma 3.6, we can think of having a function which, given a consistent set Γ, returns one fixed consistent saturated set, denoted by $\texttt{SAT}(\Gamma)$. Moreover, we denote by $\texttt{APPLY}(\Gamma, F)$ the result of applying to Γ the rule for the principal connective in F. In case the rule for F has several conclusions (the case of a branching), we suppose that the function APPLY chooses one consistent conclusion in an arbitrary but fixed manner.

Theorem 3.7 (Completeness of $\mathcal{T}$P) *$\mathcal{T}$**P** is complete w.r.t. preferential models, i.e. if a set of formulas Γ is unsatisfiable, then it has a closed tableau in $\mathcal{T}$**P**.*

Proof. We assume that no tableau for Γ_0 is closed, then we construct a model for Γ_0. We build X, the set of worlds of the model, as follows:

1. initialize $X = \{\texttt{SAT}(\Gamma_0)\}$; mark $\texttt{SAT}(\Gamma_0)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved Γ from X;
  3. **for** each formula $\neg(A \mid\!\sim B) \in \Gamma$
    3a. let $\Gamma_{\neg(A \mid\sim B)} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg(A \mid\!\sim B)))$;
    3b. **if** $\Gamma_{\neg(A \mid\sim B)} \notin X$ **then** $X = X \cup \{\Gamma_{\neg(A \mid\sim B)}\}$;
  4. **for** each formula $\neg\Box\neg A \in \Gamma$, let $\Gamma_{\neg\Box\neg A} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg\Box\neg A))$;
    4a. add the relation $\Gamma_{\neg\Box\neg A} < \Gamma$;
    4b. **if** $\Gamma_{\neg\Box\neg A} \notin X$ **then** $X = X \cup \{\Gamma_{\neg\Box\neg A}\}$.
  5. mark Γ as resolved;
endWhile;

This procedure terminates, since the number of possible sets of formulas that can be obtained by applying $\mathcal{T}$**P**'s rules to an initial finite set Γ is finite. We construct the model $\mathcal{M} = \langle X, <_X, V\rangle$ for Γ as follows:

- $<_X$ is the transitive closure of the relation $<$;
- $V(\Gamma) = \{P \mid P \in \Gamma \cap ATM\}$

In order to show that $\mathcal{M}$ is a preferential model for Γ, we prove the following facts:

Fact 3.8 *The relation $<_X$ is acyclic.*

Proof of Fact 3.8. If there were a loop, there would be Γ_1 and Γ_3 in X, s.t. $\Gamma_3 <_X \Gamma_1$, and Γ_1 is obtained again from Γ_3 by applying step 4 (i.e. $\Gamma_1 <_X \Gamma_3$). However, this situation, presented in Figure 5, will never happen. Indeed, since $\Gamma_3 <_X \Gamma_1$, Γ_3 has been generated by a sequence of applications of $(\Box^-)$, starting from an initial application of $(\Box^-)$ to some formula $\neg\Box\neg A$ in Γ_1. By the $(\Box^-)$

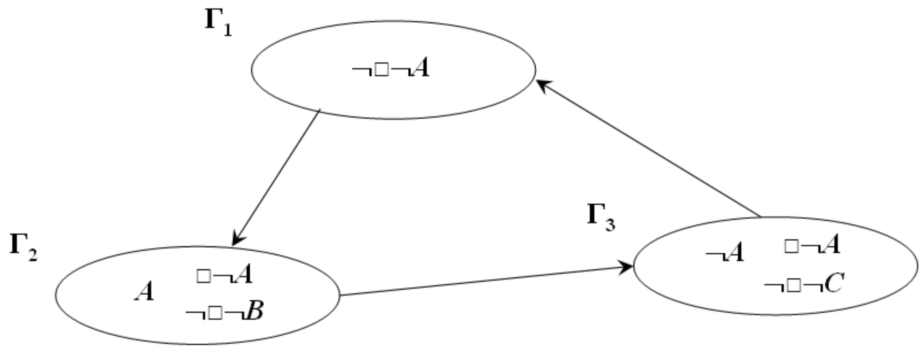

Figure 5:

rule, $\Box\neg A \in \Gamma_3$. If Γ_1 were to be generated again from Γ_3 by an application of $(\Box^-)$, then $\Box\neg A \in \Gamma_1$, which contradicts the fact that Γ_1 is consistent. We can reason in the same way for loops of any length.

$\Box$ *Fact 3.8*

Fact 3.9 *The relation $<_X$ is irreflexive, transitive, and satisfies the smoothness condition.*

Proof of Fact 3.9. Transitivity follows by construction. Irreflexivity follows the acyclicity. As there are finitely many worlds, and the relation $<_X$ is acyclic, it follows that there cannot be infinitely descending chains. This fact, together with the transitivity of $<_X$, entails that $<_X$ satisfies the smoothness condition.

$\Box$ *Fact 3.9*

The only rules introducing a new world in X in the procedure above are $(\mid\!\sim^-)$ and $(\Box^-)$. Since these two rules keep positive conditionals in their conclusions, it follows that any positive conditional $A \mid\!\sim B$ belonging to $\texttt{SAT}(\Gamma_0)$, where Γ_0 is the initial set of formulas, also belongs to each world introduced in X. Furthermore, it can be easily shown that *only* the conditionals in $\texttt{SAT}(\Gamma_0)$ belong to possible worlds in X. Indeed, all worlds in X are generated by the application of a dynamic rule, followed by the application of static rules for saturation. It can be shown that this combination of rules does never introduce a new conditional. This gives the following Fact:

Fact 3.10 *Given a world $\Delta \in X$ and any positive conditional $A \mid\!\sim B$, we have that $A \mid\!\sim B \in \Delta$ iff $A \mid\!\sim B \in \texttt{SAT}(\Gamma_0)$.*

We conclude by proving the following Fact:

Fact 3.11 *For all formulas F and for all sets $\Gamma \in X$ we have that: (i) if $F \in \Gamma$ then $\mathcal{M}, \Gamma \models F$; (ii) if $\neg F \in \Gamma$ then $\mathcal{M}, \Gamma \not\models F$.*

Proof of Fact 3.11. By induction on the structure of F. If F is an atom P, then $P \in \Gamma$ implies $\mathcal{M}, \Gamma \models P$ by definition of V. Moreover, $\neg P \in \Gamma$ implies that $P \notin \Gamma$ as Γ is consistent; thus, $\mathcal{M}, \Gamma \not\models P$ (by definition of V). For the inductive step we only consider the case of $(\neg) \mid\!\sim$ and $(\neg)\Box$:

- $\Box\neg A \in \Gamma$. Then, for all $\Gamma_i <_X \Gamma$ we have $\neg A \in \Gamma_i$ by definition of $(\Box^-)$, since Γ_i has been generated by a sequence of applications of $(\Box^-)$. By inductive hypothesis $\mathcal{M}, \Gamma_i \not\models A$ for all $\Gamma_i <_X \Gamma$, whence $\mathcal{M}, \Gamma \models \Box\neg A$.

- $\neg\Box\neg A \in \Gamma$. By construction there is a Γ' s.t. $\Gamma' <_X \Gamma$ and $A \in \Gamma'$. By inductive hypothesis $\mathcal{M}, \Gamma' \models A$. Thus, $\mathcal{M}, \Gamma \not\models \Box\neg A$.

- $A \mid\!\sim B \in \Gamma$. Let $\Delta \in Min_{<_X}(A)$; one can observe that (1)$\neg A \in \Delta$ or (2)$\neg\Box\neg A \in \Delta$ or (3)$B \in \Delta$, since $A \mid\!\sim B \in \Delta$ by Fact 3.10, and since Δ is saturated. (1) cannot be the case, since otherwise by inductive hypothesis $\mathcal{M}, \Delta \not\models A$, which contradicts the definition of $Min_{<_X}(A)$. If (2), by construction of $\mathcal{M}$ there exists a set $\Delta' <_X \Delta$ such that $A \in \Delta'$. By inductive hypothesis $\mathcal{M}, \Delta' \models A$, which contradicts $\Delta \in Min_{<_X}(A)$. Thus it must be that (3)$B \in \Delta$, and by inductive hypothesis $\mathcal{M}, \Delta \models B$. Hence, we can conclude $\mathcal{M}, \Gamma \models A \mid\!\sim B$.

- $\neg(A \mid\!\sim B) \in \Gamma$: by construction of X, there exists $\Gamma' \in X$ such that $A, \Box\neg A, \neg B \in \Gamma'$. By inductive hypothesis we have that $\mathcal{M}, \Gamma' \models A$ and $\mathcal{M}, \Gamma' \models \Box\neg A$. It follows that $\Gamma' \in Min_{<_X}(A)$. Furthermore, always by induction, $\mathcal{M}, \Gamma' \not\models B$. Hence, $\mathcal{M}, \Gamma \not\models A \mid\!\sim B$.

□ *Fact 3.11*

By the above Facts the proof of the completeness of $\mathcal{T}\mathbf{P}$ is over, since $\mathcal{M}$ is a model for the initial set Γ_0.

■

By Theorem 3.4 above and by the construction of the model done in the proof of Theorem 3.7 just above, we can show the following Corollary.

Corollary 3.12 (Finite model property) **P** *has the finite model property.*

Proof. By Theorem 3.4, if Γ is satisfiable, then there is no closed tableau for Γ. By the construction in the proof of Theorem 3.7, if there is no closed tableau for Γ, then Γ is satisfiable in a finite model.

■

A relevant property of the calculus that will be useful to estimate the complexity of logic **P** is the so-called *disjunction property* of conditional formulas:

Proposition 3.13 (Disjunction property) *If there is a closed tableau for* $\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)$*, then there is a closed tableau either for* $\Gamma, \neg(A \mid\!\sim B)$ *or for* $\Gamma, \neg(C \mid\!\sim D)$*.*

Proof. Consider a closed tableau for $\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)$. If the tableau does not contain any application of $(\mid\!\sim^-)$, then the property immediately follows. The same holds if either $\neg(A \mid\!\sim B)$ or $\neg(C \mid\!\sim D)$ are not used in the tableau. Consider the case in which there is an application of $(\mid\!\sim^-)$ first to $\neg(C \mid\!\sim D)$, and then to $\neg(A \mid\!\sim B)$. We show that in this case there is also a closed tableau for $\Gamma, \neg(A \mid\!\sim B)$. We can build a tableau of the form:

$$\begin{array}{c}
\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D) \\
\Pi_1 \\
\dfrac{\Gamma', \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)}{\Gamma'^{\mid\sim^{\pm}}, \neg(A \mid\!\sim B), C, \Box\neg C, \neg D}\ (\mid\!\sim^-) \\
\Pi_2 \\
\dfrac{\Gamma'', \neg(A \mid\!\sim B)}{\Gamma''^{\mid\sim^{\pm}}, A, \Box\neg A, \neg B}\ (\mid\!\sim^-)
\end{array}$$

Since C and D are propositional formulas, $C, \Box\neg C, \neg D$ and, eventually, their subformulas introduced by the application of some boolean rules in Π_2, will be removed by the application of $(\mid\!\sim^-)$ on $\neg(A \mid\!\sim B)$. Therefore, one can obtain a closed tableau of $\Gamma, \neg(A \mid\!\sim B)$ as follows:

$$\begin{array}{c}
\Gamma, \neg(A \mid\!\sim B) \\
\Pi'_1 \\
\Gamma', \neg(A \mid\!\sim B) \\
\Pi'_2 \\
\dfrac{\Gamma^*, \neg(A \mid\!\sim B)}{\Gamma''^{\mid\sim^{\pm}}, A, \Box\neg A, \neg B}\ (\mid\!\sim^-)
\end{array}$$

Π'_1 is obtained by removing $\neg(C \mid\!\sim D)$ from all the nodes of Π_1;Π'_2 is obtained by removing from Π_2 the application of rules on $C, \neg D$ and their subformulas. The symmetric case, corresponding to the case in which $(\mid\!\sim^-)$ is applied first on $\neg(A \mid\!\sim B)$ and then on $\neg(C \mid\!\sim D)$, can be proved in the same manner, thus we can conclude that $\Gamma, \neg(C \mid\!\sim D)$ has a closed tableau.

■

The reason why this property holds is that the $(\mid\!\sim^-)$ rule discards all the other formulas that could have been introduced by its previous application.

3.1 Decidability and Complexity of P

3.1.1 Terminating procedure for P

In general, non-termination in tableau calculi can be caused by two different reasons: 1. some rules copy their principal formula in the conclusion, and can thus be reapplied over the same formula without any control; 2. dynamic rules may generate infinitely-many worlds, creating infinite branches.

Concerning the second source of non-termination (point 2.), notice that infinitely-many worlds cannot be generated on a branch by $(\mid\!\sim^-)$ rule, since this rule can be applied only once to a given negated conditional on a branch. Another possible source of infinite branches could be determined by the interplay between rules $(\mid\!\sim^+)$ and $(\Box^-)$. We show that this cannot occur, once we introduce the following standard restriction on the order of application of the rules.

Definition 3.14 (Restriction on the calculus) *Building a tableau for a set of formulas* Γ*, the application of the* $(\Box^-)$ *rule must be postponed to the application of the propositional rules and to the verification that* Γ *is an instance of* (**AX**).

It is easy to observe that, without the restriction above, point 2. could occur; for instance, consider the following trivial example, showing a branch of a tableau starting with $P \mid\!\sim Q$, with $P, Q \in ATM$:

$$
\begin{array}{cl}
P \mid\!\sim Q & \\ \hline
\neg\Box\neg P, P \mid\!\sim Q \qquad \ldots & (\mid\!\sim^+) \\ \hline
P, \Box\neg P, P \mid\!\sim Q & (\Box^-) \\ \hline
P, \Box\neg P, \neg\Box\neg P, P \mid\!\sim Q \qquad \ldots & (\mid\!\sim^+) \\ \hline
(*)\neg P, P, \Box\neg P, P \mid\!\sim Q & (\Box^-) \\ \hline
\neg P, P, \Box\neg P, \neg\Box\neg P, P \mid\!\sim Q \qquad \ldots & (\mid\!\sim^+) \\ \hline
\neg P, P, \Box\neg P, P \mid\!\sim Q & (\Box^-) \\ \hline
\neg P, P, \Box\neg P, \neg\Box\neg P, P \mid\!\sim Q \qquad \ldots & (\mid\!\sim^+) \\
\ldots &
\end{array}
$$

In the above example, the $(\Box^-)$ rule is applied systematically before the other rules, thus generating an infinite branch. However, if the restriction in Definition 3.14 is adopted, as it is easy to observe, the procedure terminates at the step marked as $(*)$. Indeed, the test that $\neg P, P, \Box\neg P, P \mid\!\sim Q$ is an instance of the axiom (**AX**) succeeds before applying $(\Box^-)$ again, and the branch is considered to be closed.

As already mentioned, with the above restriction at hand, we can show (Lemma 3.20 and Theorem 3.21) that the interplay between $(\mid\!\sim^+)$ and $(\Box^-)$ does not generate branches containing infinitely-many worlds. Intuitively, the application of $(\Box^-)$ to a formula $\neg\Box\neg A$ (introduced by $(\mid\!\sim^+)$) adds the formula $\Box\neg A$ to the conclusion, so that $(\mid\!\sim^+)$ can no longer consistently introduce $\neg\Box\neg A$. This is due to the properties of $\Box$ in G, and would not hold if $\Box$ had weaker properties (e.g. K4 properties).

Concerning point 1. the above calculus $\mathcal{T}\mathbf{P}$ does not ensure a terminating proof search due to $(\mid\!\sim^+)$, which can be applied without any control. We ensure

$$(\mid\!\sim^{+})\ \frac{\Gamma, A \mid\!\sim B; \Sigma}{\Gamma, \neg A; \Sigma, A \mid\!\sim B \qquad \Gamma, \neg\Box\neg A; \Sigma, A \mid\!\sim B \qquad \Gamma, B; \Sigma, A \mid\!\sim B}$$

$$(\mid\!\sim^{-})\ \frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{\Sigma, A, \Box\neg A, \neg B, \Gamma^{\mid\sim\pm}; \emptyset} \qquad (\Box^{-})\ \frac{\Gamma, \neg\Box\neg A; \Sigma}{\Sigma, \Gamma^{\Box}, \Gamma^{\Box\downarrow}, \Gamma^{\mid\sim\pm}, A, \Box\neg A; \emptyset}$$

Figure 6: The calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$. Propositional rules are as in Figure 3 adding Σ.

the termination by putting some constraints on $\mathcal{T}\mathbf{P}$. The intuition is as follows: one does not need to apply $(\mid\!\sim^{+})$ on the same conditional formula $A \mid\!\sim B$ *more than once in the same world*, therefore we keep track of positive conditionals already used by moving them in an additional set Σ in the conclusions of $(\mid\!\sim^{+})$, and restrict the application of this rule to unused conditionals only. The dynamic rules re-introduce formulas from Σ in order to allow further applications of $(\mid\!\sim^{+})$ in the other worlds. This machinery is standard.

Theorem 3.21 below shows that no additional machinery is needed to ensure termination. Notice that this would not work in other systems (for instance, in K4 one needs a more sophisticated loop-checking as described in [23]).

The terminating calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ is presented in Figure 6. Observe that the tableau nodes are now pairs $\Gamma; \Sigma$. The calculus $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ is sound and complete with respect to the semantics:

Theorem 3.15 (Soundness and completeness of $\mathcal{T}\mathbf{P}^{\mathbf{T}}$) *Given a set of formulas Γ, it is unsatisfiable iff it has a closed tableau in $\mathcal{T}\mathbf{P}^{\mathbf{T}}$.*

Proof. The soundness is immediate and left to the reader. The completeness easily follows from the fact that two applications of $(\mid\!\sim^{+})$ to the same conditional in the same world are useless. Indeed, given a proof in $\mathcal{T}\mathbf{P}$, if $(\mid\!\sim^{+})$ is applied *twice* to $\Gamma, A \mid\!\sim B$ in the same world, then we can assume, without loss of generality, that the two applications are consecutive. Therefore, the second application of $(\mid\!\sim^{+})$ is useless, since each of the conclusions has already been obtained after the first application, and can be removed.

∎

Let us introduce a property of the tableau which will be crucial in many of the following proofs. Let us first define the notion of regular node.

Definition 3.16 *A node $\Gamma; \Sigma$ is called regular if the following condition holds:*

$$\textit{if } \neg\Box\neg A \in \Gamma, \textit{ then there is } A \mid\!\sim B \in \Gamma \cup \Sigma$$

It is easy to see that all nodes in a tableau starting from a pair $\Gamma_0; \emptyset$ are regular, when Γ_0 is a set of formulas of $\mathcal{L}$. This is stated by the following Proposition:

Proposition 3.17 *Given a pair $\Gamma_0; \emptyset$, where Γ_0 is a set of formulas of $\mathcal{L}$, all the tableaux obtained by applying $\mathcal{TP}^{\mathbf{T}}$'s rules only contain regular nodes.*

Proof. Given a regular node $\Gamma; \Sigma$ and any rule of $\mathcal{TP}^{\mathbf{T}}$, we have to show that each conclusion of the rule is still a regular node. The proof is immediate for all the propositional rules, for $(\Box^-)$ and for $(\mid\!\sim^-)$, since no negated box formula not belonging to their premise is introduced in their conclusion(s). Consider now an application of $(\mid\!\sim^+)$ to a regular node $\Gamma', A \mid\!\sim B; \Sigma$: one of the conclusions is $\Gamma', \neg\Box\neg A; \Sigma, A \mid\!\sim B$, and it is a regular node since there is $A \mid\!\sim B$ in the auxiliary set of used conditional in correspondence of the negated boxed formula $\neg\Box\neg A$.

■

From now on, we can assume without loss of generality that only regular nodes may occur in a tableau.
In order to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search, we define a complexity measure on a set of formulas Γ and the corresponding set of positive conditionals already used Σ, denoted by $m(\Gamma; \Sigma)$, which consists of four measures c_1, c_2, c_3 and c_4 in a lexicographic order. We denote by $cp(F)$ the complexity of a formula F, defined as follows:

Definition 3.18 (Complexity of a formula)

- $cp(P) = 1$, *where* $P \in ATM$
- $cp(\neg F) = 1 + cp(F)$
- $cp(F \bigotimes G) = 1 + cp(F) + cp(G)$, *where* $\bigotimes$ *is any binary boolean operator*
- $cp(\Box\neg A) = 1 + cp(\neg A)$
- $cp(A \mid\!\sim B) = 3 + cp(A) + cp(B)$.

Definition 3.19 *We define* $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$ *where:*

- $c_1 = \mid \{A \mid\!\sim B \in_- \Gamma\} \mid$
- $c_2 = \mid \{A \mid\!\sim B \in_+ \Gamma \cup \Sigma \mid \Box\neg A \not\in \Gamma\} \mid$
- $c_3 = \mid \{A \mid\!\sim B \in_+ \Gamma\} \mid$
- $c_4 = \sum_{F \in \Gamma} cp(F)$

We consider the lexicographic order *given by* $m(\Gamma; \Sigma)$, *that is to say: given* $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$ *and* $m(\Gamma'; \Sigma') = \langle c'_1, c'_2, c'_3, c'_4 \rangle$, *we say that* $m(\Gamma; \Sigma) < m(\Gamma'; \Sigma')$ *iff there exists* i, $i = 1, 2, 3, 4$, *such that the following conditions hold:*

- $c_i < c'_i$
- *for all* j, $0 < j < i$, *we have that* $c_j = c'_j$

Intuitively, c_1 is the number of negated conditionals to which the ($\mid\!\sim^-$) rule can still be applied. An application of ($\mid\!\sim^-$) reduces c_1. c_2 represents the number of positive conditionals *which can still create a new world.* The application of ($\Box^-$) reduces c_2: indeed, if ($\mid\!\sim^+$) is applied to $A \mid\!\sim B$, this application introduces a branch containing $\neg\Box\neg A$; when a new world is generated by an application of ($\Box^-$) on $\neg\Box\neg A$, it contains A and $\Box\neg A$. If ($\mid\!\sim^+$) is applied to $A \mid\!\sim B$ once again, then the conclusion where $\neg\Box\neg A$ is introduced leads to a closed branch, by the presence of $\Box\neg A$ in that branch. c_3 is the number of positive conditionals not yet considered in that branch. c_4 is the sum of the complexities of the formulas in Γ; an application of a boolean rule reduces c_4.

To prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search, we show that the tableau cannot contain an open branch of infinite length. To this purpose we need the following Lemma:

Lemma 3.20 *Let $\Gamma'; \Sigma'$ be obtained by an application of a rule of $\mathcal{TP}^{\mathbf{T}}$ to a premise $\Gamma; \Sigma$. Then, we have that either $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$ or $\mathcal{TP}^{\mathbf{T}}$ leads to the construction of a closed tableau for $\Gamma'; \Sigma'$.*

Proof. We consider each rule of the calculus $\mathcal{TP}^{\mathbf{T}}$:

- ($\mid\!\sim^-$): one can easily observe that the conditional formula $\neg(A \mid\!\sim B)$ to which this rule is applied does not belong to the only conclusion. Hence the measure c_1 in $m(\Gamma'; \Sigma')$, say $c_{1'}$, is smaller than c_1 in $m(\Gamma, \Delta)$, say c_1;

- ($\Box^-$): no negated conditional is added nor deleted in the conclusions, thus $c_1 = c'_1$. Suppose we are considering an application of ($\Box^-$) on a formula $\neg\Box\neg A$. We can observe the following facts:

  - the formula $\neg\Box\neg A$ has been introduced by an application of ($\mid\!\sim^+$), being this one the only rule introducing a boxed formula in the conclusion; more precisely, it derives from an application of ($\mid\!\sim^+$) on a conditional formula $A \mid\!\sim B$;
  - $A \mid\!\sim B$ belongs to both $\Gamma; \Sigma$ and $\Gamma'; \Sigma'$, since no rule of $\mathcal{TP}^{\mathbf{T}}$ removes positive conditionals (at most, the ($\mid\!\sim^+$) rule *moves* conditionals from Γ to Σ);
  - $A \mid\!\sim B$ does not "contribute" to $c_{2'}$, since the application of ($\Box^-$) introduces $\Box\neg A$ in the conclusion Γ' (remember that $c_{2'} = \mid \{A \mid\!\sim B \in_+ \Gamma' \cup \Sigma' \mid \Box\neg A \not\in \Gamma'\} \mid$).

  We distinguish two cases:

  1. $\Box\neg A$ does *not* belong to the premise of ($\Box^-$): in this case, by the above facts, we can easily conclude that $c_{2'} < c_2$, since $\Box\neg A$ belongs only to the conclusion;
  2. $\Box\neg A$ belongs to the premise of ($\Box^-$): we are considering a derivation of the following type:

$$\frac{\Gamma, \Box\neg A, \neg\Box\neg A}{\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \neg A, A, \Box\neg A} \ (\Box^-)$$

In this case, $c_{2'} = c_2$; however, we can conclude that the tableau built for $\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \neg A, A, \Box\neg A$ is closed, since:

- A is a propositional formula
- the restriction in Definition 3.14 leads to a proof in which the propositional rules and (**AX**) are applied to A and $\neg A$ *before* $(\Box^-)$ is further applied. The resulting tableau is closed;

- $(\mid\sim^+)$: we have that $c_1 = c_1'$, since we have the same negated conditionals in the premise as in all the conclusions. The same for c_2, since the formula $A \mid\sim B$ to which the rule is applied is also maintained in the conclusions (it moves from unused to already used conditionals). We conclude that $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$, since $c_{3'} < c_3$. Indeed, the $(\mid\sim^+)$ rule moves $A \mid\sim B$ from Γ to the set Σ of already considered conditionals;

- rules for the boolean connectives: it is easy to observe that c_1, c_2 and c_3 are the same in the premise and in any conclusion, since conditional formulas are side formulas in the application of these rules. We conclude that $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$ since $c_{4'} < c_4$. Indeed, the complexity of the formula to which the rule is applied is greater than (the sum of) the complexity of its subformula(s) introduced in the conclusion(s).

■

Now we have all the elements to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures termination in a proof search:

Theorem 3.21 (Termination of $\mathcal{TP}^{\mathbf{T}}$) *$\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search.*

Proof. By Lemma 3.20 we know that, starting from $\Gamma_0; \emptyset$, the value of $m(\Gamma; \Sigma)$ decreases each time a tableau rule is applied or leads to a closed tableau. Therefore, a finite number of applications of the rules leads either to build a closed tableau or to nodes $\Gamma; \Sigma$ such that $m(\Gamma; \Sigma)$ is *minimal*. In particular, we observe that, when the branch does not close, $m(\Gamma; \Sigma) = \langle 0, 0, 0, c_{4_{min}} \rangle$, and the following facts hold:

- no negated conditional belongs to Γ, since $c_1 = 0$;
- for each $A \mid\sim B \in \Gamma \cup \Sigma$, we have that $\Box\neg A \in \Gamma$, since $c_2 = 0$;
- all positive conditionals $A \mid\sim B$ have been moved in Σ since $c_3 = 0$;
- Γ is saturated with respect to the propositional rules, since c_4 assumes its minimal value $c_{4_{min}}$.

By the above facts it is easy to see that, in this case, either $\Gamma; \Sigma$ is closed or no rule, with the exception of ($\Box^-$), is applicable to $\Gamma; \Sigma$. Indeed, ($\mid\!\sim^-$) rule is not applicable, since no negated conditional belongs to Γ. If ($\Box^-$) is applicable, then there is $\neg\Box\neg A \in \Gamma$, to which the rule is applied. However, since $c_2 = 0$, we have also that $\Box\neg A \in \Gamma$. Therefore, the conclusion of an application of ($\Box^-$) contains both A and $\neg A$ and, by the restriction in Definition 3.14 and since A is propositional, the procedure terminates building a closed tableau. ($\mid\!\sim^+$) is not applicable, since no positive conditionals belong to Γ (all positive conditionals $A \mid\!\sim B$ have been moved in Σ). Last, no rule for the boolean connectives is applicable. For a contradiction, suppose one boolean rule is still applicable: by Lemma 3.20, the sum of the complexity of the formulas in the conclusion(s) decreases, i.e. c_4 in the conclusion(s) is smaller than in the premise $\Gamma; \Sigma$, against the minimality of this measure in $\Gamma; \Sigma$.

■

3.1.2 Optimal Proof Search Procedure for P

We conclude this section with a complexity analysis of $\mathcal{TP}^{\mathbf{T}}$, in order to prove that validity in **P** is **coNP**-complete. Intuitively, we can obtain a **coNP** decision procedure, by taking advantage of the following facts:

1. Negated conditionals do not interact with the current world, nor they interact among themselves (by the disjunction property). Thus they can be handled separately and eliminated always as a first step.

2. We can replace the ($\Box^-$) which is responsible of backtracking in the tableau construction by a stronger rule that does not need backtracking.

Regarding (1), by the disjunction property we can reformulate the ($\mid\!\sim^-$) rule as follows:

$$\frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{\Sigma, A, \Box\neg A, \neg B, \Gamma^{\mid\sim^+}; \emptyset} \, (\mid\!\sim^-)$$

This rule reduces the length of a branch at the price of making the proof search more non-deterministic.

Regarding (2), we can adopt the following strengthened version of ($\Box^-$). We use $\Gamma^{\Box^-}_{-i}$ to denote $\{\neg\Box\neg A_j \vee A_j \mid \neg\Box\neg A_j \in \Gamma \wedge j \neq i\}$:

$$\frac{\Gamma, \neg\Box\neg A_1, \neg\Box\neg A_2, ..., \neg\Box\neg A_n; \Sigma}{\Sigma, \Gamma^{\mid\sim^\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, A_1, \Box\neg A_1, \Gamma^{\Box^-}_{-1}; \emptyset \mid \ldots \mid \Sigma, \Gamma^{\mid\sim^\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, A_n, \Box\neg A_n, \Gamma^{\Box^-}_{-n}; \emptyset} \, (\Box^-_s)$$

The advantage of this rule over the original ($\Box^-$) rule is that no backtracking on the choice of the formula $\neg\Box\neg A_i$ is needed. The reason is that all alternatives are kept in the conclusion. As we will see below, by using this rule we can provide a tableau construction algorithm with no backtracking.

We call $L\mathcal{TP}^{\mathbf{T}}$ the calculus obtained by replacing in $\mathcal{TP}^{\mathbf{T}}$ the initial rules $(\mid\!\sim^-)$ and $(\Box^-)$ with the ones reformulated above. We can prove that $L\mathcal{TP}^{\mathbf{T}}$ is sound and complete w.r.t. the preferential models. To prove soundness, we consider the multi-linear models introduced in section 2.2.1.

Theorem 3.22 *The rule* $(\Box_s^-)$ *is sound.*

Proof. Let $\Gamma = \Gamma', \neg\Box\neg A_1, \neg\Box\neg A_2, \ldots, \neg\Box\neg A_n$. We omit Σ for readability reasons. We prove that if Γ is satisfiable then also one conclusion of the rule

$$\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, A_i, \Box\neg A_i, \Gamma^{\Box^-}_{-i}$$

is satisfiable. By Theorem 2.11, we can assume that Γ is satisfiable in a multi-linear model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, let $\mathcal{M}, x \models \Gamma$. Then there are $z_1 < x, \ldots, z_n < x$, such that $z_i \in Min_<(A_i)$; thus $\mathcal{M}, z_i \models A_i \wedge \Box\neg A_i$; we easily have also that $\mathcal{M}, z_i \models \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}$. Being $\mathcal{M}$ a multi-linear model, the z_i, $i = 1, 2, \ldots, n$, whenever distinct, are totally ordered: we have that $z_i < x$, so that they must belong to the same component. Let z_k be the maximum of z_i, for a certain $1 \leq k \leq n$. We have that for each z_i $(i \neq k)$ either (i) $z_i = z_k$, so that $\mathcal{M}, z_k \models A_i$, or (ii) $z_i < z_k$, so that $\mathcal{M}, z_k \models \neg\Box\neg A_i$. We have shown that for each $i \neq k$, $\mathcal{M}, z_k \models A_i \vee \neg\Box\neg A_i$. We can conclude that $\mathcal{M}, z \models \Gamma^{\Box^-}_{-k}$. Thus

$$\mathcal{M}, z_k \models \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, A_k, \Box\neg A_k, \Gamma^{\Box^-}_{-k}$$

which is one of the conclusions of the rule.

■

We can prove that the calculus obtained by replacing the $(\Box^-)$ rule with its stronger version $(\Box_s^-)$ is complete w.r.t. the semantics:

Theorem 3.23 *The calculus* $L\mathcal{TP}^{\mathbf{T}}$ *is complete.*

Proof. We repeat the same construction as in the proof of Theorem 3.7, in order to build a preferential model, more precisely a multi-linear model, of a set of formulas Γ_0 for which there is no closed tableau. We denote by $\mathtt{APPLY}(\Gamma, \mathit{RuleName})$ the result of applying the rule corresponding to *RuleName* to Γ. As a difference with the construction in Theorem 3.7, we replace point 4. by the points 4_{strong}, $4a_{strong}$, $4b_{strong}$, and $4c_{strong}$, obtaining the following procedure (X is the set of worlds of the model):

1. initialize $X = \{\mathtt{SAT}(\Gamma_0)\}$; mark $\mathtt{SAT}(\Gamma_0)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved Γ from X;
  3. **for** each formula $\neg(A \mid\!\sim B) \in \Gamma$
    3a. let $\Gamma_{\neg(A\mid\sim B)} = \mathtt{SAT}(\mathtt{APPLY}(\Gamma, \neg(A \mid\!\sim B)))$;
    3b. **if** $\Gamma_{\neg(A\mid\sim B)} \notin X$ **then** $X = X \cup \{\Gamma_{\neg(A\mid\sim B)}\}$;
  4_{strong}. **if** there is $\neg\Box\neg A \in \Gamma$ **then**

$4a_{strong}$. let $\Gamma' =$`SAT(APPLY(`$\Gamma, \Box_s^-$`))`;
$4b_{strong}$. add the relation $\Gamma' < \Gamma$;
$4c_{strong}$. **if** $\Gamma' \notin X$ **then** $X = X \cup \{\Gamma'\}$;
5. mark Γ as resolved;
endWhile;

Facts 3.8 and 3.9 can be proved as in Theorem 3.7. This holds also for Fact 3.11 with one difference, for what concerns the case in which $\neg\Box\neg A \in \Gamma$. In this case, by construction there is a Γ' such that $\Gamma' <_X \Gamma$. We can prove by induction on the length n of the chain $<_X$ starting from Γ that if $\neg\Box\neg A \in \Gamma$, then $\mathcal{M}, \Gamma \not\models \Box\neg A$. If $n = 1$, it must be the case that $A \in \Gamma'$; hence, by inductive hypothesis on the structure of the formula, $\mathcal{M}, \Gamma' \models A$, thus $\mathcal{M}, \Gamma \not\models \Box\neg A$. If $n > 1$, by the $(\Box_s^-)$ rule, either $A \in \Gamma'$ and we conclude as in the previous case, or $\neg\Box\neg A \in \Gamma'$ and the Fact holds by inductive hypothesis on the length.

■

We give a non-deterministic algorithm for testing satisfiability in **P** that: (i) takes a set of formulas Γ as input; (ii) returns `SAT` iff Γ is satisfiable. By using the new version of $(\mid\!\sim^-)$ rule, we can consider a negated conditional at a time. Indeed, for $\Gamma, \neg(A \mid\!\sim B), \neg(C \mid\!\sim D)$ to be satisfiable, it is sufficient that both $\Gamma, \neg(A \mid\!\sim B)$ and $\Gamma, \neg(C \mid\!\sim D)$, separately considered, are satisfiable. For each negated conditional, the algorithm `GENERAL-CHECK` applies the rule $(\mid\!\sim^-)$ to it, and calls the algorithm `CHECK` on the resulting set of formulas. `CHECK` is a non-deterministic algorithm that tests satisfiability in **P** of a set of formulas not containing negated conditionals.

Let `EXPAND`(Γ) be a procedure that returns one saturated expansion of Γ w.r.t. all static rules. In case of a branching rule, `EXPAND` nondeterministically selects (guesses) one conclusion of the rule. The algorithm below allows the satisfiability of a set of formulas (not containing negated conditionals) to be decided. In brackets we give the complexity of each operation, considering that $n = \mid \Gamma \mid$.

`CHECK`(Γ)
1. $\Gamma \longleftarrow$ `EXPAND`(Γ); $(O(n))$
2. **if** Γ contains an axiom **then return** `UNSAT`; $(O(n^2))$
3. **if** $\{\neg\Box\neg A \mid \neg\Box\neg A \in \Gamma\} = \emptyset$ **then return** `SAT`;
4. **else return** `CHECK(APPLY(`$\Gamma, \Box_s^-$`))`;

Notice that the execution of `APPLY(`$\Gamma, \Box_s^-$`)` chooses the branch generated by the application of $(\Box_s^-)$ to Γ.

To see that `CHECK` is a nondeterministic polynomial procedure to decide the satisfiability of a set of formulas (not containing negated conditionals), observe that: (1) the complexity of each call to the procedure `EXPAND` is polynomial. Indeed, as the number of different subformulas is at most $O(n)$, `EXPAND` makes at most $O(n)$ applications of the static rules. (2) The test that a set Γ (of size $O(n)$) of formulas contains an axiom has at most complexity $O(n^2)$. (3) The number of recursive calls to the procedure CHECK is at most $O(n)$, since in a branch the rule $(\Box_s^-)$ can be applied only once for each formula $\neg\Box\neg A_i$, and the number of different negated box formulas is at most $O(n)$.

Let us now define a procedure to decide whether an arbitrary set of formulas Γ (possibly containing negated conditionals) is satisfiable:

GENERAL-CHECK(Γ)
1. $\Gamma \longleftarrow$ EXPAND(Γ); ($O(n)$)
2. let $\neg(A_1 \mid\sim B_1), \ldots, \neg(A_k \mid\sim B_k)$ be all negated conditionals in Γ;
   2.1. **for all** $i = 1, \ldots, k$ result[i] $\longleftarrow$ CHECK(APPLY($\Gamma, \neg(A_i \mid\sim B_i)$)) ;
3. **if for all** $i = 1, \ldots, k$ result[i]==SAT **then return** SAT;
   else return UNSAT;

By the subformula property, the number of negated conditionals which can occur in Γ is at most $O(n)$. Hence, the procedure GENERAL-CHECK calls to the algorithm CHECK at most $O(n)$ times.

Theorem 3.24 (Complexity of P) *The problem of deciding validity for preferential logic* **P** *is* **coNP***-complete.*

Proof. The procedure GENERAL-CHECK allows the satisfiability of a set of formulas of logic **P** to be decided in nondeterministic polynomial time. The validity problem for **P** is therefore in **coNP**. As **coNP**-hardness is immediate (this logic includes classical propositional logic), we conclude that the validity problem for logic **P** is **coNP**-complete.

■

This result matches the known complexity results for logic **P** [27]. Due to the **coNP** lower bound, the above method provides a computationally optimal reasoning procedure for logic **P**.

4 The Tableau Calculus for Loop Cumulative Logic CL

In this section we develop a tableau calculus $\mathcal{T}$**CL** for **CL**, and we show that it can be turned into a terminating calculus. This provides a decision procedure for **CL** and a **coNP**-membership upper bound for validity in **CL**.

The calculus $\mathcal{T}$**CL** can be obtained from the calculus $\mathcal{T}$**P** for preferential logics, by adding a suitable rule (L^-) for dealing with the modality L introduced in section 2.3. As already mentioned in section 2.3, the formulas that appear in the tableaux for **CL** belong to the language $\mathcal{L}_L$ obtained from $\mathcal{L}$ as follows: (*i*) if A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$; (*ii*) if A, B are

$$(\mid\!\sim^+)\ \frac{\Gamma, A \mid\!\sim B}{\Gamma, \neg LA, A \mid\!\sim B \qquad \Gamma, \neg\Box\neg LA, A \mid\!\sim B \qquad \Gamma, LB, A \mid\!\sim B}$$

$$(\mid\!\sim^-)\ \frac{\Gamma, \neg(A \mid\!\sim B)}{LA, \Box\neg LA, \neg LB, \Gamma^{\mid\sim\pm}} \qquad (\Box^-)\ \frac{\Gamma, \neg\Box\neg LA}{\Gamma^{\Box}, \Gamma^{\Box\downarrow}, \Gamma^{\mid\sim\pm}, LA, \Box\neg LA}$$

$$(L^-)\ \frac{\Gamma, \neg LA}{\Gamma^{L^\downarrow}, \neg A}\ ;\ \frac{\Gamma}{\Gamma^{L^\downarrow}}\ \text{if } \Gamma \text{ does not contain negated } L\text{-formulas}$$

Figure 7: Tableau system $\mathcal{T}\mathbf{CL}$. If there are no negated L-formulas $\neg LA$ in the premise of (L^-), then the rule allows to step from Γ to $\Gamma^{L^\downarrow}$. To save space, the boolean rules are omitted.

propositional, then $A \mid\!\sim B \in \mathcal{L}_L$; (iii) if F is a boolean combination of formulas of $\mathcal{L}_L$, then $F \in \mathcal{L}_L$. Observe that the only allowed combination of $\Box$ and L is in formulas of the form $\Box\neg LA$ where A is propositional.

We define:

$$\Gamma^{L^\downarrow} = \{A \mid LA \in \Gamma\}$$

Our tableau system $\mathcal{T}\mathbf{CL}$ is shown in Figure 7. Observe that rules $(\mid\!\sim^+)$ and $(\mid\!\sim^-)$ have been changed as they introduce the modality L in front of the propositional formulas A and B in their conclusions. This straightforwardly corresponds to the semantics of conditionals in CL preferential models (see Definition 2.13). The new rule (L^-) is a dynamic rule.

Theorem 4.1 (Soundness of $\mathcal{T}$CL) *The system $\mathcal{T}\mathbf{CL}$ is sound with respect to CL-preferential models, i.e. if there is a closed tableau for a set of formulas Γ, then Γ is unsatisfiable.*

Proof. We show that for all the rules in $\mathcal{T}\mathbf{CL}$, if the premise is satisfiable by a CL-preferential model then also one of the conclusions is. As far as the rules already present in $\mathcal{T}\mathbf{P}$ are concerned, the proof is very similar, with the only exception that we have to substitute A in the proof by LA.

We consider now the new rule (L^-). Let $\mathcal{M}, w \models \Gamma, \neg LA$ where $\mathcal{M} = \langle \mathcal{W}, R, <, V\rangle$ is a CL-preferential model. Then there is $w' : wRw'$ and $\mathcal{M}, w' \models \neg A$. Furthermore, $\mathcal{M}, w' \models \Gamma^{L^\downarrow}$. It follows that the conclusion of the rule is satisfiable. If Γ does not contain negated L-formulas, since R is serial, we still have that $\exists w' : wRw'$, and $\mathcal{M}, w' \models \Gamma^{L^\downarrow}$, hence the conclusion is still satisfiable.

■

Soundness with respect to loop-cumulative models in Definition 2.12 follows from the correspondence established by Proposition 2.14.

The proof of the completeness of the calculus can be done as for the preferential case, provided we suitably modify the procedure for constructing a model for a finite consistent set of formulas Γ of $\mathcal{L}_L$. First of all, we modify the definition of saturated sets as follows:

- if $A \mid\sim B \in \Gamma$ then $\neg LA \in \Gamma$ or $\neg\Box\neg LA \in \Gamma$ or $LB \in \Gamma$

For this notion of saturated set of formulas we can still prove Lemma 3.6 for language $\mathcal{L}_L$.

Theorem 4.2 (Completeness of $\mathcal{T}$CL) *$\mathcal{T}$**CL** is complete with respect to CL-preferential models, i.e. if a set of formulas Γ is unsatisfiable, then it has a closed tableau in $\mathcal{T}$**CL**.*

Proof. We define a procedure for constructing a model satisfying a consistent set of formulas $\Gamma_0 \in \mathcal{L}_L$ by modifying the procedure for the preferential logic **P**. We add to the procedure used in the proof of Theorem 3.7 the new steps 4' and 4" between step 4 and step 5, obtaining the following procedure:

1. initialize $X = \{\texttt{SAT}(\Gamma_0)\}$; mark $\texttt{SAT}(\Gamma_0)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved Γ from X;
  3. **for** each formula $\neg(A \mid\sim B) \in \Gamma$
    3a. let $\Gamma_{\neg(A\mid\sim B)} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg(A \mid\sim B)))$;
    3b. **if** $\Gamma_{\neg(A\mid\sim B)} \notin X$ **then** $X = X \cup \{\Gamma_{\neg(A\mid\sim B)}\}$;
  4. **for** each formula $\neg\Box\neg LA \in \Gamma$, let $\Gamma_{\neg\Box\neg LA} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg\Box\neg LA))$;
    4a. add the relation $\Gamma_{\neg\Box\neg LA} < \Gamma$;
    4b. **if** $\Gamma_{\neg\Box\neg LA} \notin X$ **then** $X = X \cup \{\Gamma_{\neg\Box\neg LA}\}$.
  4'. **if** $\{\neg LA \mid \neg LA \in \Gamma\} \neq \emptyset$ **then**
      for each $\neg LA \in \Gamma$, let $\Gamma_{\neg LA} = \texttt{SAT}(\texttt{APPLY}(\Gamma, \neg LA))$;
        4' a. add the relation $\Gamma\ R\ \Gamma_{\neg LA}$;
        4' b. $X = X \cup \{\Gamma_{\neg LA}\}$;
  4". **else if** $\Gamma^{L^{\downarrow}} \neq \emptyset$ **then** let $\Gamma' = \texttt{SAT}(\texttt{APPLY}(\Gamma, L^-))$;
      4" a. add the relation $\Gamma\ R\ \Gamma'$;
      4" b. $X = X \cup \{\Gamma'\}$;
  5. mark Γ as resolved;
endWhile;

This procedure terminates. Observe that, although an application of (L^-) may introduce in X several copies of the same world (set) of propositional formulas, each of these worlds cannot lead to generate any further world by means of a dynamic rule.

We construct the model $\mathcal{M} = \langle X, R_X, <_X, V\rangle$ by defining X and V as in the case of **P**. We then define R_X as the relation obtained from R augmented with the following conditions:

(i) all the pairs (Γ, Γ) such that $\Gamma \in X$ and Γ has no R-successor.

(ii) all the pairs (Γ, Γ') such that $(\Gamma'', \Gamma) \in R$ and $(\Gamma'', \Gamma') \in R$ for some Γ'';

Last, we define $<_X$ as follows:

(iii) if $\Gamma' < \Gamma$, then $\Gamma' <_X \Gamma$;

(iv) if $\Gamma' < \Gamma$, and $\Gamma R_X \Gamma^{''}$, then $\Gamma' <_X \Gamma^{''}$;

(v) if $\Gamma' <_X \Gamma$ and $\Gamma <_X \Gamma^{''}$, then $\Gamma' <_X \Gamma^{''}$, i.e. $<_X$ is transitive.

Notice that the above conditions on R_X and $<_X$ are needed since the procedure builds two different kinds of worlds:

- *bad* worlds, obtained by an application of (L^-);
- *good* worlds: the other ones.

Bad worlds are those obtained by an application of (L^-). These worlds "forget" the positive conditionals in the initial set of formulas; for instance, if $\Gamma = \{\neg C, D\}$ is a bad world obtained from $\Gamma' = \{A \mid\!\sim B, \neg LC, LD\}$, then it is "incomplete" by the absence of $A \mid\!\sim B$. The above supplementary conditions on R_X and $<_X$ are needed in order to prove that, even in presence of bad worlds, we can build a CL-preferential model satisfying the initial set of formulas, as shown below by Fact 4.3.

It is easy to show that the following properties hold for $\mathcal{M}$:

Fact 4.3 *For all* $\Gamma, \Gamma^{'} \in X$*, if* $(\Gamma, \Gamma^{'}) \in R_X$ *and* $LA \in \Gamma$ *then* $A \in \Gamma^{'}$*.*

Proof of Fact 4.3. In case $\Gamma \neq \Gamma'$, then $(\Gamma, \Gamma') \in R$ and it has been added to R by step 4' or step 4" of the procedure above. Indeed, for all (Γ, Γ') that have been introduced because $(\Gamma^{''}, \Gamma) \in R$ and $(\Gamma^{''}, \Gamma') \in R$, both Γ and $\Gamma^{'}$ derive from the application of (L^-) to $\Gamma^{''}$, hence they only contain propositional formulas and do not contain any LA.

Hence, we have two different cases:

- the relation $(\Gamma, \Gamma^{'})$ has been added to R by step 4': in this case, we have that $\neg LB \in \Gamma$. We can conclude that $A \in \Gamma'$ by construction, since for each $LA \in \Gamma$ we have that $A \in \Gamma^{'}$ as a result of the application of $\texttt{SAT(APPLY(}\Gamma, \neg LB\texttt{))}$;
- the relation $(\Gamma, \Gamma^{'})$ has been added to R by step 4": similarly to previous case, for each $LA \in \Gamma$, we have that A is added to $\Gamma^{'}$ by construction.

In the case $\Gamma = \Gamma'$, then it must be that (Γ, Γ) has been added to R_X, as Γ has no R-successors. This means that Γ does not contain formulas of the form $LA, \neg LA$, otherwise it would have an R-successor.

$\Box$ *Fact 4.3*

Fact 4.4 *For all formulas* F *and for all sets* $\Gamma \in X$ *we have that:*
(i) if $F \in \Gamma$ *then* $\mathcal{M}, \Gamma \models F$*; (ii) if* $\neg F \in \Gamma$ *then* $\mathcal{M}, \Gamma \not\models F$*.*

Proof of Fact 4.4. The proof is similar to the one for the preferential case. If F is an atom or a boolean combination of formulas, the proof is the same as the proof for Fact 3.11 of the preferential case. We consider the following cases:

- $LA \in \Gamma$: we have to show that $\mathcal{M}, \Gamma \models LA$, that is, we must show that, for all $\Delta \in X$, if $(\Gamma, \Delta) \in R_X$ then $\mathcal{M}, \Delta \models A$. Let Δ be such that $(\Gamma, \Delta) \in R_X$. Then, by Fact 4.3, as $LA \in \Gamma$, we can conclude $A \in \Delta$. By inductive hypothesis, then $\mathcal{M}, \Delta \models A$.

- $\neg LA \in \Gamma$: we have to show that $\mathcal{M}, \Gamma \not\models LA$, that is, we must show that there exists $\Delta \in X$ such that $(\Gamma, \Delta) \in R_X$ and $\mathcal{M}, \Delta \not\models A$. As $\neg LA \in \Gamma$, by construction (step $4'$ in the procedure) there must be a $\Delta \in X$ such that $\neg A \in \Delta$. By inductive hypothesis, $\mathcal{M}, \Delta \not\models A$, which concludes the proof that $\mathcal{M}, \Gamma \not\models LA$.

- $\Box\neg LA \in \Gamma$. Then, for all $\Gamma_i <_X \Gamma$ we have $\neg LA \in \Gamma_i$ by the definition of $(\Box^-)$, since Γ_i has been generated by a sequence of applications of $(\Box^-)$ (notice that point (iv) in the definition of $<_X$ above does not play any role here, since this point only concerns sets of formulas Γ that are propositional and do not contain boxed or negated box formulas). By inductive hypothesis $\mathcal{M}, \Gamma_i \not\models LA$ for all $\Gamma_i <_X \Gamma$, whence $\mathcal{M}, \Gamma \models \Box\neg LA$.

- $\neg\Box\neg LA \in \Gamma$. By construction there is a $\Gamma^{'}$ s.t. $\Gamma^{'} <_X \Gamma$ and $LA \in \Gamma^{'}$. By inductive hypothesis $\mathcal{M}, \Gamma^{'} \models LA$. Thus, $\mathcal{M}, \Gamma \not\models \Box\neg LA$.

- $A \mid\sim B \in \Gamma$. Let $\Delta \in Min_{<_X}(LA)$. We distinguish two cases:

  - $A \mid\sim B \in \Delta$, one can observe that $(1)\neg LA \in \Delta$ or $(2)\neg\Box\neg LA \in \Delta$ or $(3)LB \in \Delta$, since Δ is saturated. (1) cannot be the case, since by inductive hypothesis $\mathcal{M}, \Delta \not\models LA$, which contradicts the definition of $Min_{<_X}(LA)$. If (2), by construction of $\mathcal{M}$ there exists a set $\Delta^{'} <_X \Delta$ such that $LA \in \Delta^{'}$. By inductive hypothesis $\mathcal{M}, \Delta^{'} \models LA$, which contradicts $\Delta \in Min_{<_X}(LA)$. Therefore, it must be that $(3)LB \in \Delta$, and by inductive hypothesis $\mathcal{M}, \Delta \models LB$.
  - $A \mid\sim B \notin \Delta$. Since all the rules apart from (L^-) preserve the conditionals, Δ must have been generated by applying (L^-) to Δ', i.e. Δ is a *bad world*. Hence, $\Delta' R_X \Delta$. In turn, it can be easily shown that Δ' itself cannot have been generated by (L^-), hence $A \mid\sim B \in \Delta'$, and, since Δ' is saturated, either $(1)\neg LA \in \Delta'$ or $(2)\neg\Box\neg LA \in \Delta'$ or $(3)LB \in \Delta'$. (1) is not possible, since by inductive hypothesis, it would entail that $\mathcal{M}, \Delta' \not\models LA$, i.e. there is $\Delta^{''}$ such that $\Delta^{'} R_X \Delta^{''}$ and $\mathcal{M}, \Delta^{''} \not\models A$. By point (ii) in the definition of R_X above, also $\Delta R_X \Delta^{''}$, hence also $\mathcal{M}, \Delta \not\models LA$, which contradicts $\Delta \in Min_{<_X}(LA)$. If (2), by construction of $\mathcal{M}$ there exists a set $\Delta^{''} <_X \Delta'$ such that $LA \in \Delta^{''}$. By point (iv) in the definition of $<_X$ above, $\Delta^{''} <_X \Delta$, which contradicts $\Delta \in Min_{<_X}(LA)$, since by inductive hypothesis $\mathcal{M}, \Delta^{''} \models LA$. It follows that $LB \in \Delta'$. By

inductive hypothesis $\mathcal{M}, \Delta' \models LB$, hence also $\mathcal{M}, \Delta \models LB$ (indeed, since Δ does not contain any L-formula, by construction of the model and by point (ii) in the definition of R_X above, $\Delta R_X \Delta''$ just in case $\Delta' R_X \Delta''$, from which the result follows).

Hence, we can conclude $\mathcal{M}, \Gamma \models A \mid\sim B$.

- $\neg(A \mid\sim B) \in \Gamma$: by construction of X, there exists $\Gamma' \in X$ such that $LA, \Box\neg LA, \neg LB \in \Gamma'$. By inductive hypothesis we have that $\mathcal{M}, \Gamma' \models LA$ and $\mathcal{M}, \Gamma' \models \Box\neg LA$. It follows that $\Gamma' \in Min_{<_X}(LA)$. Furthermore, always by induction, $\mathcal{M}, \Gamma' \not\models LB$. Hence, $\mathcal{M}, \Gamma \not\models A \mid\sim B$.

$\Box$ *Fact 4.4*

Similarly to the case of **P**, it is easy to prove the following Fact:

Fact 4.5 *The relation $<_X$ is irreflexive, transitive, and satisfies the smoothness condition.*

Moreover:

Fact 4.6 *The relation R_X is serial.*

From the above Facts, we can conclude that $\mathcal{M} = \langle X, R_X, <_X, V \rangle$ is a CL-preferential model satisfying Γ_0, which concludes the proof of completeness.

■

From the above Theorem 4.2, together with Proposition 2.14, it follows that for any boolean combination of conditionals Γ_0, if it does not have any closed tableau, then it is satisfiable in a loop-cumulative model.

Similarly to what done for **P**, we can prove the following Corollary.

Corollary 4.7 (Finite model property) **CL** *has the finite model property.*

4.1 Decision Procedure for CL

Let us now analyze the calculus $\mathcal{T}$**CL** in order to obtain a decision procedure for **CL** logic. First of all, we reformulate the calculus as we have done for **P**, obtaining a system called $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$: we reformulate the $(\mid\sim^+)$ rule so that it applies only once to each conditional in each world, by adding an extra set Σ. We reformulate the other rules accordingly. Moreover, we adopt the same restriction on the order of application of the rules in Definition 3.14.

Notice that the rule (L^-) can only be applied a finite number of times. Indeed, if it is applied to a premise $\Gamma, \neg LA$, then the conclusion only contains propositional formulas $\Gamma^{L^\downarrow}, \neg A$, and the rule (L^-) is no further applicable. The same in the case (L^-) is applied to a premise Γ, LA. Notice that (L^-) can

$$(L^-)\ \frac{\Gamma, \neg LA; \Sigma}{\Gamma^{L^{\downarrow}}, \neg A; \emptyset}\ ;\ \frac{\Gamma; \Sigma}{\Gamma^{L^{\downarrow}}; \emptyset}\ \text{if } \Gamma \text{ does not contain negated } L\text{-formulas}$$

Figure 8: The rule (L^-) reformulated in $\mathcal{T}\mathbf{CL^T}$.

also be applied to a set of formulas *not containing* any formula LA or $\neg LA$: in this case, the conclusion of the rule corresponds to an empty set of formulas. Therefore, we reformulate (L^-) only by adding the extra set Σ of conditionals; the reformulated rule is shown in Figure 8.

Exactly as we made for **P**, we consider a lexicographic order given by $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$, and easily prove that each application of the rules of $\mathcal{T}\mathbf{CL^T}$ reduces this measure, as stated by the following Lemma:

Lemma 4.8 *Consider an application of any rule of* $\mathcal{T}\mathbf{CL^T}$ *to a premise* $\Gamma; \Sigma$ *and be* $\Gamma'; \Sigma'$ *any conclusion obtained; we have that either* $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$ *or* $\mathcal{T}\mathbf{CL^T}$ *leads to the construction of a closed tableau for* $\Gamma'; \Sigma'$.

Proof. Identical to the proof of Lemma 3.20. Just observe that if (L^-) is applied, then c_1, c_2, and c_3 become 0, since conditional formulas are not kept in the conclusion. If the premise contains only L-formulas, then c_1, c_2 and c_3 are already equal to 0 in both the premise and the conclusion, but c_4 decreases, since (at least) one formula $(\neg)LA$ in the premise is removed, and a formula with a lower complexity ($\neg A$ or A) is introduced in the conclusion.

■

Thus, $\mathcal{T}\mathbf{CL^T}$ ensures termination. Furthermore, the decision algorithm for **P** described in section 3.1.2 can be adapted to **CL**. To this aim, we observe that the disjunction property holds for **CL**, and this allows us to change the rule for negated conditionals in order to treat them independently as we have done for **P**. Moreover, we can replace the $(\Box^-)$ rule by a stronger rule that does not require backtracking in the tableau construction. The rule is the following ($\Gamma^{\Box^-}_{-i}$ is used to denote $\{\neg\Box\neg LA_j \vee LA_j \mid \neg\Box\neg LA_j \in \Gamma \wedge j \neq i\}$):

$$\frac{\Gamma, \neg\Box\neg LA_1, \neg\Box\neg LA_2, ..., \neg\Box\neg LA_n; \Sigma}{\Sigma, \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, LA_1, \Box\neg LA_1, \Gamma^{\Box^-}_{-1}; \emptyset \mid \ldots \mid \Sigma, \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, LA_n, \Box\neg LA_n, \Gamma^{\Box^-}_{-n}; \emptyset}\ (\Box^-_s)$$

By reasoning similarly to what done for **P**, we can show that the calculus in which $(\Box^-)$ is replaced by $(\Box^-_s)$ is sound and complete w.r.t. multi-linear CL-preferential models introduced in Definition 2.25. We get a decision procedure as for **P**, structured in a top-level `GENERAL-CHECK` procedure, taking care of multiple negated conditionals, and in a `CHECK` procedure of tableau expansion. The procedure `CHECK` has to be modified by introducing steps 2' and 2" between steps 2 and 3 in the procedure for **P** as follows:

```
CHECK(Γ)
1. Γ ⟵ EXPAND(Γ); (O(n))
2. if Γ contains an axiom then return UNSAT; (O(n^2))
2'. if {¬LA | ¬LA ∈ Γ} ≠ ∅ then
    2'a. for all ¬LA_i ∈ Γ do result[i] ⟵ CHECK(APPLY(Γ, ¬LA_i));
    2'b. if for some i result[i]==UNSAT then return UNSAT;
2". else if {LA | LA ∈ Γ} ≠ ∅ then
    2"a. if CHECK(APPLY( Γ, L^-))==UNSAT then return UNSAT;
3. if {¬□¬LA | ¬□¬LA ∈ Γ} = ∅ then return SAT;
4. else return CHECK(APPLY(Γ, □_s^-));
```

The top-level procedure GENERAL-CHECK is the same as the one in section 3.1.2. For a better readability, we rewrite this procedure here below:

```
GENERAL-CHECK(Γ)
1. Γ ⟵ EXPAND(Γ); (O(n))
2. let ¬(A_1 |~ B_1), ..., ¬(A_k |~ B_k) be all negated conditionals in Γ;
    2.1. for all i = 1, ..., k result[i] ⟵ CHECK(APPLY(Γ, ¬(A_i |~ B_i))) ;
3. if for all i = 1, ..., k result[i]==SAT then return SAT;
    else return UNSAT;
```

Observe that the two recursive calls of CHECK in 2'a and 2"a do not generate further recursive calls. By this reason, we can argue similarly to what done for **P**, then we obtain the following result:

Theorem 4.9 (Complexity of CL) *The problem of deciding validity for* **CL** *is* **coNP***-complete.*

5 The Tableau Calculus for Cumulative Logic C

In order to provide a calculus for the weaker logic **C**, we have to replace the rule $(\Box^-)$ with the weaker $(\Box^{\mathbf{C}-})$:

$$(\Box^{\mathbf{C}-})\frac{\Gamma, \neg\Box\neg LA}{\Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim\pm}, LA}$$

Observe that, if we ignore conditionals, this rule is nothing else than the standard rule of modal logic K. This rule is weaker than the corresponding rule of the two other systems in two respects: (i) transitivity is not assumed thus we no longer have $\Gamma^{\Box}$ in the conclusion; (ii) the smoothness condition does no longer ensure that if $\neg\Box\neg LA$ is true in one world, then there is a smaller *minimal* world satisfying LA, this only happens if the world satisfies LA; thus $\Box\neg LA$ is dropped from the conclusion as well.

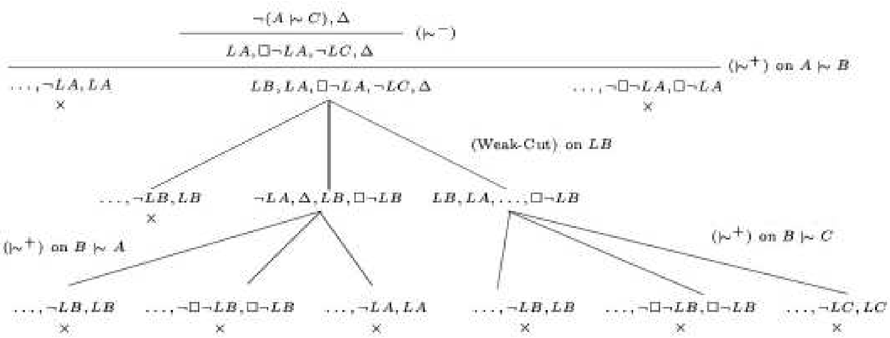

Figure 9: A derivation of $\neg(A \mid\!\sim C), A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C$ in $\mathcal{T}\mathbf{C}$. To save space, we use Δ to denote the set of positive conditionals $A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C$.

Moreover, we add the following form of cut:

$$\text{(Weak-Cut)}\frac{\Gamma}{\Gamma, \neg LA \qquad \Gamma^{\mid\sim\pm}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad \Gamma, \Box\neg LA}$$

Intuitively, this rule takes care of enforcing the smoothness condition, and it can be applied to all L-formulas.
The (Weak-Cut) rule is not completely eliminable, as shown by the following example. Let $\Gamma = \{\neg(A \mid\!\sim C), A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C\}$: the set Γ is unsatisfiable in $\mathbf{C}$. Γ has a closed tableau only if we use (Weak-Cut) in the calculus, as shown by the derivation in Figure 9. Without (Weak-Cut), the above set of formulas does not have any closed tableau.
The (Weak-Cut) rule makes the resulting calculus not analytic. However, the rule can be restricted so that it only applies to formulas LA such that A is the antecedent of a positive conditional formula in Γ, thus making the resulting calculus analytic. In order to prove this, we simplify the calculus by incorporating the application of $(\Box^{C-})$ and the restricted form of (Weak-Cut) in the $(\mid\!\sim^{+})$ rule. The resulting calculus, called $\mathcal{T}\mathbf{C}$ and given in Figure 10, is equivalent to the calculus that would be obtained from the calculus $\mathcal{T}\mathbf{CL}$ by replacing $(\Box^{-})$ with $(\Box^{C-})$ and by introducing (Weak-Cut) restricted to antecedents of positive conditionals. The advantage of the adopted formulation is that, being more compact, it allows a simpler proof of the admissibility of the non-restricted (Weak-Cut) (see Theorem 5.8 below).
Observe that the calculus does not contain any rule for negated box formulas, as the modified $(\mid\!\sim^{+})$ rule does no longer introduce formulas of the form $\neg\Box\neg LA$. The resulting language $\mathcal{L}_{L'}$ of the formulas appearing in a tableau for $\mathbf{C}$ extends $\mathcal{L}$ by formulas LA and $\Box\neg LA$ where A is propositional, whereas negated box formulas of the form $\neg\Box\neg LA$ are not allowed. Therefore, $\mathcal{L}_{L'}$ is a restriction of $\mathcal{L}_L$ used in $\mathcal{T}\mathbf{CL}$.

Notice also that, as a difference with $\mathcal{T}\mathbf{CL}$, the $(\mid\!\sim^{+})$ rule is neither a *static*

$$(\mid\!\sim^+)\ \frac{\Gamma, A \mid\!\sim B}{\Gamma, A \mid\!\sim B, \neg LA \qquad \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, A \mid\!\sim B, LA, \Box\neg LA \qquad \Gamma, A \mid\!\sim B, LA, \Box\neg LA, LB}$$

$$(\mid\!\sim^-)\ \frac{\Gamma, \neg(A \mid\!\sim B)}{LA, \Box\neg LA, \neg LB, \Gamma^{\mid\sim^{\pm}}} \qquad (L^-)\ \frac{\Gamma, \neg LA}{\Gamma^{L^{\downarrow}}, \neg A}\ ;\ \frac{\Gamma}{\Gamma^{L^{\downarrow}}}\ \text{if } \Gamma \text{ does not contain negated } L\text{-formulas}$$

Figure 10: Tableau system $\mathcal{TC}$. The boolean rules are omitted.

nor a *dynamic* rule. Indeed, the leftmost and rightmost conclusions of the rule represent the same world as the world represented by the premise, whereas the inner conclusion represents a world which is different from the one represented by the premise.

We prove that $\mathcal{TC}$ is sound and complete w.r.t. the semantics.

Theorem 5.1 (Soundness of $\mathcal{TC}$) *The system $\mathcal{TC}$ is sound with respect to C-preferential models, i.e. if there is a closed tableau for a set of formulas Γ, then Γ is unsatisfiable.*

Proof. Given a set of formulas Γ, if there is a closed tableau for Γ, then Γ is unsatisfiable in C-preferential models. For the rules already present in $\mathcal{TCL}$, the proof is the same as the proof for Theorem 4.1 (notice that in the proof the transitivity of $<$ does not play any role). We only consider here the rule $(\mid\!\sim^+)$. We show that if the premise is satisfiable by a C-preferential model, then also one of the conclusions is.

Let $\mathcal{M}, w \models \Gamma, A \mid\!\sim B$. We distinguish the two following cases:

- $\mathcal{M}, w \not\models LA$, thus $\mathcal{M}, w \models \neg LA$: in this case, the left conclusion of the $(\mid\!\sim^+)$ rule is satisfied ($\mathcal{M}, w \models \Gamma, A \mid\!\sim B, \neg LA$);
- $\mathcal{M}, w \models LA$: we consider two subcases:
  - $w \in Min_<(LA)$, hence $\mathcal{M}, w \models LA, \Box\neg LA$. By the definition of $\mathcal{M}, w \models A \mid\!\sim B$, we have that $\mathcal{M}, w \models LB$. Therefore, the right conclusion of $(\mid\!\sim^+)$ is satisfiable;
  - $w \notin Min_<(LA)$: by the smoothness condition, there exists a world $w^{'} < w$ such that $w^{'} \in Min_<(LA)$. It follows that $(\mathcal{M}, w') \models LA, \Box\neg LA$. Furthermore, by the semantics of $\mid\!\sim$, $(\mathcal{M}, w') \models \Gamma^{\mid\sim^{\pm}}$ and since $w' < w$, $(\mathcal{M}, w') \models \Gamma^{\Box^{\downarrow}}$.

■

Soundness with respect to cumulative models follows from the correspondence established by Proposition 2.29.

We can prove that the (Weak-Cut) rule is admissible in $\mathcal{T}\mathbf{C}$; this is stated by the following Theorem 5.8 below. In order to prove it, we need to prove some Lemmas.

First of all, we prove that weakening is height-preserving admissible in our tableau calculi, i.e. if there is a closed tableau for a set of formulas Γ, then there is also a closed tableau for Γ, F (for any formula F of the language) of height no greater than the height of a tableau for Γ. Moreover, weakening is cut-preserving admissible, in the sense that the closed tableau for Γ, F does not add any application of (Weak-Cut) to the closed tableau for Γ. Furthermore, we prove that the rules for the boolean connectives are height-preserving and cut-preserving invertible, i.e. if there is a closed tableau for Γ, then there is a closed tableau for any set of formulas that can be obtained from Γ as a conclusion of an application of a boolean rule.

Lemma 5.2 (Height-preserving and cut-preserving admissibility of weakening) *Given a formula F, if there is a closed tableau of height h for Γ, then there is also a closed tableau for Γ, F of no greater height than h, i.e. weakening is height-preserving admissible. Moreover, the closed tableau for Γ, F does not add any application of (Weak-Cut) to the closed tableau for Γ, i.e. weakening is cut-preserving admissible.*

Proof. By induction on the height h of the closed tableau for Γ.

Lemma 5.3 (Height-preserving and cut-preserving invertibility of boolean rules) *The rules for the boolean connectives are height-preserving invertible, i.e. given a set of formulas Γ and given any conclusion Γ', obtained by applying a boolean rule to Γ, if there is a closed tableau of height h for Γ, then there is a closed tableau for Γ' of height no greater than h. Moreover, the closed tableau for Γ' does not add any application of (Weak-Cut) to the closed tableau for Γ, i.e. the boolean rules are cut-preserving invertible.*

Proof. For each boolean rule (R), we proceed by induction on the height of the closed tableau for the premise. As an example, consider the $(\vee^+)$ rule. We show that, if $\Gamma, F \vee G$ has a closed tableau, also Γ, F and Γ, G have. If $\Gamma, F \vee G$ is an instance of the axiom (AX), then there is an atom P such that $P \in \Gamma$ and $\neg P \in \Gamma$, since axioms are restricted to atomic formulas only. Therefore, Γ has a closed tableau (it is an instance of (AX) too), and we conclude that both Γ, F and Γ, G have a closed tableau, since weakening is height-preserving admissible (Lemma 5.2 above). For the inductive step, we consider the first rule in the tableau for $\Gamma, F \vee G$. If $(\vee^+)$ is applied to $F \vee G$, then we are done, since we have closed tableaux for both Γ, F and Γ, G of a lower height than the premise's. If $(\mid\!\sim^-)$ is applied, then $F \vee G$ is removed from the conclusion, then Γ has a closed tableau; we conclude since weakening is height-preserving admissible (Lemma 5.2). If a boolean rule is applied, then $F \vee G$ is copied into the conclusion(s); in these cases, we can apply the inductive hypothesis and then conclude by re-applying the same rule. As an example, consider the case in which $(\rightarrow^-)$ is applied to $\Gamma', \neg(H \rightarrow I), F \vee G$ as follows:

$$\frac{\Gamma', \neg(H \rightarrow I), F \vee G}{(*)\Gamma', H, \neg I, F \vee G}\ (\rightarrow^-)$$

By the inductive hypothesis, there is a closed tableau (of no greater height than the height of $(*)$) for $(**)\Gamma', H, \neg I, F$ and for $(***)\Gamma', H, \neg I, G$. We conclude as follows:

$$\frac{\Gamma', \neg(H \rightarrow I), F}{(**)\Gamma', H, \neg I, F}\ (\rightarrow^-)$$

$$\frac{\Gamma', \neg(H \rightarrow I), G}{(***)\Gamma', H, \neg I, G}\ (\rightarrow^-)$$

If the first rule of the closed tableau for $\Gamma, F \vee G$ is $(\mid\!\sim^+)$ applied to $A \mid\!\sim B \in \Gamma$, then we have the following situation:

$$\frac{\Gamma, F \vee G}{(1)\Gamma, F \vee G, \neg LA \qquad (2)\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad (3)\Gamma, F \vee G, LA, \Box\neg LA, LB}\ (\mid\!\sim^+)$$

By the inductive hypothesis on (1), we have closed tableaux for $(1')\Gamma, F, \neg LA$ and for $(1'')\Gamma, G, \neg LA$. By the inductive hypothesis on (3), we have closed tableaux for $(3')\Gamma, F, LA, \Box\neg LA, LB$ and for $(3'')\Gamma, G, LA, \Box\neg LA, LB$. We conclude as follows:

$$\frac{\Gamma, F}{(1')\Gamma, F, \neg LA \qquad (2)\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad (3')\Gamma, F, LA, \Box\neg LA, LB}\ (\mid\!\sim^+)$$

$$\frac{\Gamma, G}{(1'')\Gamma, G, \neg LA \qquad (2)\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad (3'')\Gamma, G, LA, \Box\neg LA, LB}\ (\mid\!\sim^+)$$

If F (resp. G) were a conditional formula (even negated), we conclude as above, replacing the inner conclusion with $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, F, LA, \Box\neg LA$ (resp. $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, G, LA, \Box\neg LA$), for which there is a closed tableau since weakening is height-preserving admissible (Lemma 5.2).

■

Now we prove that we can assume, without loss of generality, that the conclusions of (Weak-Cut) are never derived by an application of $(\mid\!\sim^-)$, as stated by the following Lemma:

Lemma 5.4 *If Γ has a closed tableau, then there is a closed tableau for Γ in which all the conclusions of each application of (Weak-Cut) are derived by a rule different from $(\mid\!\sim^-)$.*

Proof. Consider an application of (Weak-Cut) in Γ in which one of its conclusions is obtained by an application of ($\mid\!\sim^-$). The application of (Weak-Cut) is useless, since the premise of the cut can be obtained by applying directly ($\mid\!\sim^-$) without (Weak-Cut). For instance, consider the following derivation, in which the inner conclusion of (Weak-Cut) is obtained by an application of ($\mid\!\sim^-$):

$$\dfrac{\dfrac{\Gamma', \neg(C \mid\!\sim D)}{\Gamma', \neg(C \mid\!\sim D), \neg LA \qquad \Gamma'^{\mid\sim\pm}, \Gamma'^{\Box\downarrow}, \neg(C \mid\!\sim D), LA, \Box\neg LA \qquad \Gamma', \neg(C \mid\!\sim D), \Box\neg LA} \text{(Weak-Cut)}}{(*)\Gamma'^{\mid\sim\pm}, LC, \Box\neg LC, \neg LD} (\mid\!\sim^-)$$

We can remove the application of (Weak-Cut), obtaining the following closed tableau:

$$\dfrac{\Gamma', \neg(C \mid\!\sim D)}{(*)\Gamma'^{\mid\sim\pm}, LC, \Box\neg LC, \neg LD} (\mid\!\sim^-)$$

Obviously, the proof can be concluded in the same way in the case the leftmost (resp. the rightmost) conclusion of (Weak-Cut) has a derivation starting with ($\mid\!\sim^-$).

■

Now we prove that cut is admissible on propositional formulas and on formulas of the form LA. By cut we mean the following rule:

$$\dfrac{\Gamma}{\Gamma, F \qquad \Gamma, \neg F} (Cut)$$

and we show that it can be derived if F is a propositional formula or a formula of kind LA.

Lemma 5.5 *Given a set of propositional formulas* Γ *and a propositional formula* A*, if there is a closed tableau for both* $(1)\Gamma, \neg A$ *and* $(2)\Gamma, A$ *without (Weak-Cut), then there is also a closed tableau for* Γ *without (Weak-Cut).*

Proof. Since Γ is propositional, the only applicable rules are the propositional rules. The result follows by admissibility of cut in tableaux systems for propositional logic.

■

Lemma 5.6 *If there is a closed tableau without (Weak-Cut) for* $(1)\Gamma, \neg LA$ *and for* $(2)\Gamma, LA$*, then there is also a closed tableau without (Weak-Cut) for* Γ*.*

Proof. Let $h1$ be the height of the tableau for (1), and $h2$ the height of the tableau for (2). We proceed by induction on $h1 + h2$.
Base Case: $h1 + h2 = 0$. In this case, $h1 = 0$ and $h2 = 0$. In this case, (1) and (2) contain an axiom, and since axioms are restricted to atomic formulas, it can only be that $P, \neg P \in \Gamma$, where $P \in ATM$. Therefore, we conclude that there is a closed tableau for Γ without (Weak-Cut).

For the inductive step, we show that if the property holds in case $h1 + h2 = n - 1$, then it also holds in case $h1 + h2 = n$. We reason by cases according to which is the first rule applied to (1) or to (2). If the first rule applied to (1) is ($\mid\!\sim^-$), applied to a conditional $\neg(C \mid\!\sim D) \in \Gamma$, let $\Gamma_{\neg(C\mid\sim D)}$ be the set of formulas so obtained. It can be easily verified that the same set can be obtained by applying the same rule to the same conditional in Γ, hence Γ has a closed tableau without (Weak-Cut). If the first rule applied to (1) is ($\mid\!\sim^+$), applied to a conditional $(C \mid\!\sim D) \in \Gamma$, then we have that $(1a)\Gamma, \neg LC, \neg LA$, $(1b)\Gamma^{\mid\sim^\pm}, \Gamma^{\Box^\downarrow}, LC, \Box\neg LC$, and $(1c)\Gamma, LC, \Box\neg LC, LD, \neg LA$ have a closed tableau with height smaller than $h1$. We can then apply weakening and the inductive hypothesis first over (2) and $(1a)$ and then over (2) and $(1c)$, to obtain that (i)$\Gamma, \neg LC$, and (ii) $\Gamma, LC, \Box\neg LC, LD$ respectively have a closed tableau without (Weak-Cut). Since (i),(1b),(ii) are obtained from Γ by applying ($\mid\!\sim^+$) on $C \mid\!\sim D$, we conclude that also Γ has a closed tableau without (Weak-Cut). The case in which the first rule applied is a propositional rule immediately follows from the height-preserving invertibility of the boolean rules (see Lemma 5.3 above). For instance, suppose $(1)\Gamma', F \wedge G, \neg LA$ is derived by an application of $(\wedge^+)$, i.e. $(1')\Gamma', F, G, \neg LA$ has a closed tableau of height smaller than $h1$. Since $(2)\Gamma', F \wedge G, LA$ has a closed tableau, and $(\wedge^+)$ is height-preserving invertible, then also $(2')\Gamma', F, G, LA$ has a closed tableau of height no greater than $h2$, and we can conclude as follows:

$$\dfrac{\dfrac{\Gamma', F \wedge G}{\Gamma', F, G}\,(\wedge^+)}{(1')\Gamma', F, G, \neg LA \qquad (2')\Gamma', F, G, LA}\,(cut)$$

If the first rule applied to (2) is either a propositional rule, or ($\mid\!\sim^-$), or ($\mid\!\sim^+$), we can reason as for (1). We are left with the case in which the first rule applied both to (1) and to (2) is (L^-).

If the first rule applied to (1) is (L^-), applied to $\neg LB \in \Gamma$, let $\Gamma_{\neg LB}$ be the set obtained. The same set can be obtained by applying (L^-) to $\neg LB$ in Γ, hence Γ has a closed tableau without (Weak-Cut). If (L^-) is applied to $\neg LA$ itself, then $(*)\Gamma^{L^\downarrow}, \neg A$ has a closed tableau. In this case, we have to consider the tableau for (2). We distinguish two cases: in the first case (L^-) in (2) has been applied to some $\neg LB \in \Gamma$, in the second case Γ does not contain any negated L formula, hence (L^-) has not been applied to any specific $\neg LB$. First case: $(**)\Gamma^{L^\downarrow}, \neg B, A$ has a closed tableau. By weakening from $(*)$, also $(*')\Gamma^{L^\downarrow}, \neg B, \neg A$ has a closed tableau. By Lemma 5.5 applied to $(*')$ and $(**)$, also $\Gamma^{L^\downarrow}, \neg B$ has a closed tableau, and since this set can be obtained from Γ by applying (L^-) to $\neg LB$, it follows that Γ has a closed tableau, without (Weak-Cut). Second case: $(***)\Gamma^{L^\downarrow}, A$ has a closed tableau. By Lemma 5.5 applied to $(*)$ and $(***)$, also $\Gamma^{L^\downarrow}$ has a closed tableau, and since this set can be obtained by applying (L^-) to Γ, it follows that Γ has a closed tableau, without (Weak-Cut).

■

Lemma 5.7 *Let LA and LB be such that there is a closed tableau without (Weak-Cut) for $\{LA, \neg LB\}$. Then for all sets of formulas Γ, if $\Gamma, \Box\neg LA, \Box\neg LB$ has a closed tableau without (Weak-Cut), also $\Gamma, \Box\neg LB$ has.*

Proof. By induction on the height h of the tableau for $\Gamma, \Box\neg LA, \Box\neg LB$. If $h = 0$, then $\Gamma, \Box\neg LA, \Box\neg LB$ contains an axiom, hence (since axioms only concern atoms), also Γ does, and there is a tableau for $\Gamma, \Box\neg LB$ without (Weak-Cut). We prove that if the property holds for all tableaux of height $h-1$, then it also holds for tableaux of height h. We proceed by considering all possible cases corresponding to the first rule applied to $\Gamma, \Box\neg LA, \Box\neg LB$.
The case in which the first rule is boolean is easy and left to the reader.
If the first rule is (L^-), it can be easily verified that the same set of formulas can be obtained by applying (L^-) to $\Gamma, \Box\neg LB$ that hence has a closed tableau without (Weak-Cut).
If the first rule is $(\mid\!\sim^-)$, again it can be easily verified that the same set of formulas can be obtained by applying $(\mid\!\sim^-)$ to $\Gamma, \Box\neg LB$, that hence has a closed tableau without (Weak-Cut).
If the first rule is $(\mid\!\sim^+)$ applied to a conditional $C \mid\!\sim D \in \Gamma$, then $(1)\Gamma, \Box\neg LA, \Box\neg LB, \neg LC$; $(2)\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LA, \neg LB, \Box\neg LC, LC$; $(3)\Gamma, \Box\neg LA, \Box\neg LB, \Box\neg LC, LC, LD$ have a closed tableau with height smaller than h. By the inductive hypothesis from (1) and (3), we infer that $(1')\Gamma, \Box\neg LB, \neg LC$ and $(3')\Gamma, \Box\neg LB, \Box\neg LC, LC, LD$ have a closed tableau without (Weak-Cut). Furthermore, since by hypothesis $\{LA, \neg LB\}$ has a closed tableau without (Weak-Cut), from (2), weakening (Lemma 5.2), and Lemma 5.6, we infer that also $(2')\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LC, \Box\neg LC, \neg LB$ has a closed tableau without (Weak-Cut). Since $(1'), (2'), (3')$ can be obtained from $\Gamma, \Box\neg LB$ by applying $\mid\!\sim^+$ to $C \mid\!\sim D$ in Γ, we conclude that $\Gamma, \Box\neg LB$ has a closed tableau without (Weak-Cut).
There are no other cases, hence the result follows.

■

Now we are able to prove that the (Weak-Cut) is admissible in $\mathcal{T}\mathbf{C}$, as stated by Theorem 5.8:

Theorem 5.8 *Given a set of formulas Γ and a propositional formula A, if there is a closed tableau for each of the following sets of formulas:*

(1) $\Gamma, \neg LA$

(2) $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA$

(3) $\Gamma, \Box\neg LA$

then there is also a closed tableau for Γ, i.e. the (Weak-Cut) *rule is admissible.*

Proof. We prove that for all sets of formulas Γ, if there is a closed tableau without (Weak-Cut) for each of the following sets of formulas:

(1) $\Gamma, \neg LA$

(2) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA$

(3) $\Gamma, \Box\neg LA$

then there is also a closed tableau without (Weak-Cut) for Γ. By this property, given a closed tableau for a starting set of formulas Γ_0, a closed tableau without (Weak-Cut) for Γ_0 can be obtained by eliminating all applications of (Weak-Cut), starting from the leafs towards the root of the tableau.
First of all, notice that in general there can be several closed tableaux for Γ_0. We only consider closed tableaux for Γ_0 that are *minimal* (in the number of nodes). Moreover, by Lemma 5.4 we can restrict our concern to applications of (Weak-Cut) whose conclusions are not obtained by applications of ($\mid\!\sim^-$).
Let $h1, h2, h3$ be the heights of the tableaux for (1), (2), and (3) respectively. We proceed by induction on $h1 + h2 + h3$. The reader might be surprised by the fact that the proof is carried on by single induction on the heights of the premises of (Weak-Cut). Indeed, the proof relies on the proof of admissibility of ordinary cut at the propositional level (Lemma 5.5), that is proved as usual by double induction on the complexity of the cut formula and on the heights of the derivation of the two conclusions.

For the base case, notice that always $h2 > 0$, since axioms are restricted to atomic formulas, and $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}$ by definition does not contain any atomic formula. Our base case will hence be the case in which $h1 = 0$ or $h3 = 0$, i.e. (1) or (3) are axioms, and $h2$ is minimal.
Base Case: $h1 = 0$, $h3 = 0$, and $h2$ is minimal. If $h1 = 0$, then (1) is an axiom. In this case, since axioms restricted to atomic formulas, it must be that $P, \neg P \in \Gamma$ with $P \in ATM$. Therefore we can conclude that there is a closed tableau for Γ without (Weak-Cut).
For the inductive step, we distinguish the two following cases:

1. one of the conclusions of (Weak-Cut) is obtained by an application of a rule for the boolean connectives;
2. all the conclusions of (Weak-Cut) are obtained by ($\mid\!\sim^+$) or by (L^-).

The list is exhaustive; indeed, by Lemma 5.4, we can consider, without loss of generality, a closed tableau in which all the conclusions of each application of (Weak-Cut) are obtained by a rule different from ($\mid\!\sim^-$).

We consider the two cases above:

1. rules for the boolean connectives: first, notice that a boolean rule cannot be applied to the inner conclusion of (Weak-Cut), since it only contains conditional formulas (even negated), L- formulas (even negated), and a positive box formula. In these cases, we conclude by permuting the (Weak-Cut) rule over the boolean rule, i.e. we first cut the conclusion(s) of the boolean rule with the other conclusions of (Weak-Cut), then we conclude by applying the boolean rule on the sets of formulas obtained. As an

example, consider the following closed tableau, where the rightmost conclusion of (Weak-Cut) is obtained by an application of ($\vee^+$), and where F and G are conditional formulas:

$$\cfrac{\cfrac{\Gamma', F \vee G}{(1)\Gamma', F \vee G, \neg LA \quad (2)\Gamma'^{\mid\!\sim^{\pm}}, \Gamma'^{\Box^{\downarrow}}, LA, \Box\neg LA \quad (3)\Gamma', F \vee G, \Box\neg LA}\ (\text{Weak-Cut})}{(3a)\Gamma', F, \Box\neg LA \quad (3b)\Gamma', G, \Box\neg LA}\ (\vee^+)$$

Since ($\vee^+$) is height-preserving and cut-preserving invertible (see Theorem 5.3), there is a closed tableau of no greater height than (1), having no applications of (Weak-Cut) (since the closed tableau starting with (1) does not contain any application of it), of $(1')\Gamma', F, \neg LA$ and $(1'')\Gamma', G, \neg LA$. By the height-preserving admissibility of weakening (see Lemma 5.2), we have also a closed tableau, of no greater height than (2), for $(2')\Gamma'^{\mid\!\sim^{\pm}}, \Gamma'^{\Box^{\downarrow}}, F, LA, \Box\neg LA$ and for $(2'')\Gamma'^{\mid\!\sim^{\pm}}, \Gamma'^{\Box^{\downarrow}}, G, LA, \Box\neg LA$. We can apply the inductive hypothesis to $(1')$, $(2')$, and $(3a)$ to obtain a closed tableau, without applications of (Weak-Cut), of $(*)\Gamma', F$. Notice that we can apply the inductive hypothesis since the tableaux for $(1')$ and $(2')$ do not contain applications of (Weak-Cut) and have no greater height than (1) and (2), respectively; however, the closed tableau for $(3a)$, not containing any application of (Weak-Cut), has obviously a smaller height than the one for (3). In the same way, we can apply the inductive hypothesis to $(1'')$, $(2'')$, and $(3b)$ to obtain a closed tableau for $(**)\Gamma', G$. We can conclude by applying ($\vee^+$) to $(*)$ and $(**)$ to obtain a proof without (Weak-Cut) for $\Gamma', F \vee G$.

2. ($\mid\!\sim^+$) or (L^-): let us denote with a triple $< R_1, R_2, R_3 >$ the rules applied, respectively, to the three conclusions (1), (2), and (3) of (Weak-Cut), where $R_i \in \{(L^-), (\mid\!\sim^+), (\text{ANY})\}$. We use (ANY) to represent either (L^-) and ($\mid\!\sim^+$). As an example, $< (L^-), (\mid\!\sim^+), (\text{ANY}) >$ is used to represent the case in which the leftmost conclusion of (Weak-Cut) (1) is obtained by an application of (L^-), whereas the inner conclusion (2) is obtained by an application of ($\mid\!\sim^+$); the rightmost conclusion (3) can be either obtained by an application of (L^-) or by an application of ($\mid\!\sim^+$).

   We distinguish the following cases: (†) $< (\text{ANY})(\text{ANY})(L^-) >$, (††) $< (\text{ANY}), (L^-), (\mid\!\sim^+) >$, and (†††) $< (\text{ANY}), (\mid\!\sim^+), (\mid\!\sim^+) >$. The list is exhaustive.

   - (†) $< (\text{ANY})(\text{ANY})(L^-) >$: the tableau for (3) is started with an application of (L^-) to a formula $\neg LB \in \Gamma$ ($\Gamma = \Gamma', \neg LB$):

     $$\frac{(3)\Gamma', \neg LB, \Box\neg LA}{\Gamma'^{L^{\downarrow}}, \neg B}\ (L^-)$$

     We can conclude since $\Gamma'^{L^{\downarrow}}, \neg B$ can be obtained by applying (L^-) to $\Gamma = \Gamma'.\neg LB$ (the formula $\Box\neg LA$ is removed from the conclusion in an application of (L^-)). In case the tableau for (3) is started with an application of (L^-) on a formula $LB \in \Gamma$, and there is no $\neg LB_i \in \Gamma$, we can obviously conclude in the same manner;

- (††) $< (\text{ANY}), (L^-), (\mid\!\sim^+) >$: the proof of (3) is started with an application of $(\mid\!\sim^+)$ on a formula $C \mid\!\sim D \in \Gamma$; we have that the following sets of formulas have three closed tableaux without (Weak-Cut):
  - (3a) $\Gamma, \Box\neg LA, \neg LC$
  - (3b) $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LA, \Box\neg LC, LC$
  - (3c) $\Gamma, \Box\neg LA, \Box\neg LC, LC, LD$

  Moreover, since the first rule applied to (2) is (L^-), we have that $\{LA, \Gamma^{\Box^{\downarrow}}\}$ has a closed tableau without (Weak-Cut).

  We want to show that also the three following sets of formulas:
  - (I) $\Gamma, \neg LC$
  - (II) $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LC, LC$
  - (III) $\Gamma, \Box\neg LC, LC, LD$

  have a closed tableau without (Weak-Cut), from which we conclude by an application of $(\mid\!\sim^+)$. We thus want to show that:

  (I). By (3a) we know that $\Gamma, \neg LC, \Box\neg LA$ has a closed tableau without (Weak-Cut); by weakening from (1) we also know that $\Gamma, \neg LC, \neg LA$ has a closed tableau without (Weak-Cut); last, by (2) we know that $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LA, LA$ has a closed tableau without (Weak-Cut). Since the sum of the heights of (1), (2) and (3a) is smaller than h1 + h2 + h3 (because the height of (3a) is smaller than h3), we can apply the inductive hypothesis and conclude that (I) $\Gamma, \neg LC$ has a closed tableau without (Weak-Cut).

  (II) Since $\{LA, \Gamma^{\Box^{\downarrow}}\}$ has a closed tableau without (Weak-Cut), by weakening, also $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LC, LC$ has a closed tableau without (Weak-Cut), and from (3b) and Lemma 5.6, also (II) $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LC, LC$ has.

  (III). By weakening from (1), we know that $\Gamma, LC, \Box\neg LC, LD, \neg LA$ has a closed tableau without (Weak-Cut); by ($3c$) we know that $\Gamma, LC, \Box\neg LC, LD, \Box\neg LA$ has a closed tableau without (Weak-Cut), and by weakening from (2) we know that $\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LC, \Box\neg LA, LA$ has a closed tableau without (Weak-Cut). Furthermore, the sum of the heights of the tableaux for (1), (2) and (3c) is smaller than h1+h2+h3. We can then apply the inductive hypothesis to conclude that (III): $\Gamma, \Box\neg LC, LC, LD$ has a closed tableau without (Weak-Cut).

- (†††) $< (\text{ANY}), (\mid\!\sim^+), (\mid\!\sim^+) >$: as in the previous case, the proof of (3) is started with an application of $(\mid\!\sim^+)$ on a formula $C \mid\!\sim D \in \Gamma$;

we have that the following sets of formulas have three closed tableaux without (Weak-Cut):

- (3a) $\Gamma, \Box\neg LA, \neg LC$
- (3b) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LA, \Box\neg LC, LC$
- (3c) $\Gamma, \Box\neg LA, \Box\neg LC, LC, LD$

Moreover, we have that the first rule applied in (2) is $(\mid\!\sim^{+})$ applied to some conditional $C_1 \mid\!\sim D_1 \in \Gamma^{\mid\!\sim^{\pm}}$. We show that there is a conditional $C_i \mid\!\sim D_i \in \Gamma$ such that:

- (i) $\Gamma, \neg LC_i$
- (ii) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LC_i, LC_i$
- (iii) $\Gamma, \Box\neg LC_i, LC_i, LD_i$

have a closed tableau without (Weak-Cut), hence also Γ has, since (i), (ii), and (iii) can be obtained from Γ by applying $(\mid\!\sim^{+})$ to $C_i \mid\!\sim D_i$. The tableau for (2) can contain a sequence s of applications of $(\mid\!\sim^{+})$. By considering only the leftmost branch introduced by the applications of $(\mid\!\sim^{+})$ in s, consider the last application of $(\mid\!\sim^{+})$ to a conditional $C_i \mid\!\sim D_i$. We will have that (2a) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \ldots, \neg LC_i$ has a closed tableau; (2b) $\Gamma^{\mid\!\sim^{\pm}}, \neg LA, \Box\neg LC_i, LC_i$ has a closed tableau; (2c) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \ldots, \neg LC_{i-1}, \Box\neg LC_i, LC_i, LD_i$ has a closed tableau. The situation can be represented as follows:

$$\dfrac{\dfrac{\Gamma}{(1)\Gamma, \neg LA \qquad (2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad (3)\Gamma, \Box\neg LA}\ \text{(Weak-Cut)}}{\begin{array}{c}\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1 \quad \ldots \\ \vdots \\ (2a)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \ldots, \neg LC_i \quad \ldots (2b) \ldots (2c) \ldots\end{array}}\ (\mid\!\sim^{+})$$

Furthermore, since by hypothesis the tableau for (2) does not contain any application of (Weak-Cut), also the tableaux for (2a), (2b), and (2c) do not contain any application of (Weak-Cut).

Observe that (2a) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \ldots, \neg LC_i$ cannot be an instance of (AX), since axioms are restricted to atomic formulas only, and (2a) only contains conditionals, L-formulas, and boxed formulas. Therefore, we can observe that the rule applied to (2a) is (L^{-}): indeed, if the rule was $(\mid\!\sim^{-})$ applied to some $\neg(C_k \mid\!\sim C_j) \in \Gamma^{\mid\!\sim^{\pm}}$, then we could find a shorter derivation, obtained by applying $(\mid\!\sim^{-})$ to $(2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA$, against the minimality of the closed tableau we are considering. This situation would be as follows:

$$\dfrac{\Gamma}{(1)\Gamma, \neg LA \quad (2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \quad (3)\Gamma, \Box\neg LA}\,(\text{Weak-Cut})$$

$$\dfrac{(1)\Gamma, \neg LA \quad (2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \quad (3)\Gamma, \Box\neg LA}{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1 \quad \dots}\,(\mid\!\sim^{+})$$

$$\vdots$$

$$\dfrac{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \dots, \neg LC_i \quad \dots}{(\Gamma^{\mid\!\sim^{\pm}} - \{\neg(C_k \mid\!\sim C_j)\}), LC_k, \Box\neg LC_k, \neg LC_j}\,(\mid\!\sim^{-})$$

. . . .

$$\dfrac{\Gamma}{(1)\Gamma, \neg LA \qquad (2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \quad (3)\Gamma, \Box\neg LA}\,(\text{Weak-Cut})$$

$$\dfrac{(1)\Gamma, \neg LA \qquad (2)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \quad (3)\Gamma, \Box\neg LA}{\dots \qquad (\Gamma^{\mid\!\sim^{\pm}} - \{\neg(C_k \mid\!\sim C_j)\}), LC_k, \Box\neg LC_k, \neg LC_j \quad \dots}\,(\mid\!\sim^{-})$$

More precisely, we can observe that the rule applied to (2a) is (L^-) applied to $\neg LC_i$: indeed, if (L^-) was applied to a previously generated negated formula, there would be a shorter tableau obtained by immediately applying (L^-) to that formula, as represented by the following derivations:

$$\vdots$$

$$\dfrac{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \dots, \neg LC_h}{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \dots, \neg LC_h, \neg LC_{h+1} \quad \dots}\,(\mid\!\sim^{+})$$

$$\vdots$$

$$\dfrac{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \dots, \neg LC_h, \dots, \neg LC_i \qquad \dots}{A, \neg C_h}\,(L^{-})$$

. . . .

$$\vdots$$

$$\dfrac{\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA, \neg LC_1, \dots, \neg LC_h}{A, \neg C_h}\,(L^{-})$$

Hence, $A, \neg C_i$ has a closed tableau without (Weak-Cut), and hence also $(*)LA, \neg LC_i$ has a closed tableau without (Weak-Cut). From $(*)$

and (1), with opportune weakenings, by Lemma 5.6 we derive that $(i) : \Gamma, \neg LC_i$ has a closed tableau without (Weak-Cut). By weakening from $(2b)$, we have that $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LA, \Box\neg LC_i, LC_i$ has a closed tableau without (Weak-Cut). From this set of formulas, (2), weakening, and Lemma 5.6, we also know that: $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LA, \Box\neg LC_i, LC_i$ has a closed tableau without (Weak-Cut). From this set of formulas, (*) and Lemma 5.7, we conclude that also $(ii)\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \Box\neg LC_i, LC_i$ has a closed tableau without (Weak-Cut).

Consider now (iii): we can show that $\Gamma, \Box\neg LC_i, LC_i, LD_i$ has a closed tableau without (Weak-Cut). Indeed, we can repeat the same proofs of case (††) in order to show that (I) $\Gamma\neg LC$ and (III) $\Gamma, \Box\neg LC, LC, LD$ have a closed tableau without (Weak-Cut) (in (††) there was no assumption on the first rule applied to (2) to show that (I) and (III) have a closed tableau). In order to show that (iii) has a closed tableau without (Weak-Cut), we observe that: by weakening from (I), (I')$\Gamma, \Box\neg LC_i, LC_i,\ LD_i, \neg LC$ has a closed tableau without (Weak-Cut); by weakening from (III), (III')$\Gamma, \Box\neg LC_i, LC_i, LD_i, \Box\neg LC, LC, LD$ has a closed tableau without (Weak-Cut); since (3b) and (*) have closed tableaux without (Weak-Cut), by Lemma 5.6 we have that (II')$\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, \neg LC_i, \Box\neg LC, LC$ has a closed tableau without (Weak-Cut). We conclude that $(iii)\Gamma, LC_i, \Box\neg LC_i, LD_i$ has a closed tableau without (Weak-Cut), obtained by applying $(\mid\!\sim^{+})$ to (I'), (II'), and (III').

We have hence proven that $(i), (ii)$, and (iii) have a closed tableau without (Weak-Cut), then we can conclude by applying $(\mid\!\sim^{+})$ to them to obtain a closed tableau for Γ.

■

We prove the completeness of our calculus by modifying the procedure described in the proof of Theorem 3.7 above. The completeness is a consequence of the admissibility of the (Weak-Cut) rule. Hence, in the following completeness proof, we will make use of the (Weak-Cut) rule. We prove that given any finite $\Gamma_0 \subseteq \mathcal{L}_{L'}$, if it does not have any closed tableau, then it is satisfiable in a C-preferential model.

In order to build a *finite* model for Γ_0, we consider a restricted version of $\mathcal{L}_{L'}$, only containing the formulas in $\mathcal{L}_{L'}$ made out of propositional formulas appearing in Γ_0. We call this restricted language $\mathcal{L}_{\Gamma_0}$.

Theorem 5.9 (Completeness of $\mathcal{T}$C) *$\mathcal{T}$**C** is complete with respect to C-preferential models, i.e. if a set of formulas Γ is unsatisfiable, then it has a closed tableau in $\mathcal{T}$**C**.*

Proof. We assume that no tableau for Γ_0 is closed, and we construct a model for Γ_0.

Notice that the language $\mathcal{L}_{\Gamma_0}$ may contain infinitely many propositional formulas (obtained by boolean combinations of the atomic propositions in Γ_0). In order to keep the construction of the model finite, we define a notion of equivalence between formulas w.r.t. their propositional part (or *p-equivalence*) so to identify those formulas having the same structure and containing equivalent propositional components. Let $\equiv_{PC}$ be logical equivalence in the classical Propositional Calculus. We define the notion of *p-equivalence* between two formulas F and G (written $F \equiv_p G$) as an equivalence relation satisfying the following conditions:

- if F and G are propositional formulas, then $F \equiv_p G$ iff $F \equiv_{PC} G$;
- $LA \equiv_p LB$ iff $A \equiv_{PC} B$;
- $\neg LA \equiv_p \neg LB$ iff $A \equiv_{PC} B$;
- $\Box\neg LA \equiv_p \Box\neg LB$ iff $A \equiv_{PC} B$.

For instance, $LA \equiv_p L(A \wedge A)$, $\Box\neg LA \equiv_p \Box\neg L(A \wedge A)$.

We say that two sets of formulas Γ and Γ' are p-equivalent if for every formula in Γ there is a p-equivalent formula in Γ', and viceversa.

Observe that this notion of p-equivalence is very weak and, for instance, we do not recognize that the set $\{LA, LB\}$ is equivalent to the set $\{L(A \wedge B)\}$. The notion of p-equivalence has been introduced with the purpose of limiting the application of the rule (Weak-Cut) so that it does not generate infinitely many equivalent formulas. Moreover, as we will see in the construction of the model below, this notion of equivalence will be used to control the addition of a new set of formulas to the current set of worlds.

In our construction of the model below we will identify those p-equivalent sets of formulas Γ. Before adding a new set of formulas Γ to the current set of worlds X, we check that X does not already contain a set Γ' which is p-equivalent to Γ; for short, we will write $\Gamma \notin_P X$ in the case X does not already contain such a Γ'.

We define the procedure `SAT'` that for any $\Gamma \subseteq \mathcal{L}_{\Gamma_0}$ extends Γ by applying the transformations described below and, at the same time, builds a set of formulas Γ^S initially set to $\emptyset$. The transformations below are performed in sequence:

- applies to Γ the propositional rules, once to each formula, as far as possible. In case of branching, makes the choice that leads to an open tableau (this step saturates Γ with respect to the static rules in **C**);
- for each $A \mid\!\sim B \in \Gamma$, applies $(\mid\!\sim^+)$ to it. If the leftmost branch is open, then adds $\neg LA$ to Γ; otherwise, if the rightmost branch is open, then adds $\Box\neg LA, LA, LB$ to Γ; if the only open branch is the inner one, then adds the set $\{\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA\}$ to a support set Γ^S associated with Γ, that is initially set to $\emptyset$;
- for all $LA \in \mathcal{L}_{\Gamma_0}$, if there is no $A \mid\!\sim B \in \Gamma$, and there is no A' propositionally equivalent to A on which (Weak-Cut) has been previously applied (in

Γ), applies (Weak-Cut) to it. If the leftmost branch is open, then adds $\neg LA$ to Γ; otherwise, if the rightmost branch is open, adds $\Box\neg LA$ to Γ; if the only open branch is the inner one, then adds $\{\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA\}$ to the support set Γ^S.

Observe that SAT' terminates, extends Γ to a finite set of formulas, and produces a set Γ^S which a finite set of finite sets of formulas, since: (a) there are only a finite number of conditionals that can lead to create sets of formulas in Γ^S; (b) there are only a finite number of p-equivalent formulas that can lead to create a set in Γ^S by the third transformation above.
We build X, the set of worlds of the model, and $<$, as follows:

1. initialize $X = \{\Gamma_0\}$; mark Γ_0 as unresolved;
2. **while** X contains unresolved nodes **do**
    3. choose an unresolved Γ from X;
    4. **for** each $\{\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA\} \in \Gamma^S$, associated to Γ,
        let $\Gamma_{LA} =$ SAT'($\{\Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA\}$);
        4a. **for** all $\Gamma' \in X$ p-equivalent to Γ_{LA}
            add the relation $\Gamma' < \Gamma$;
        4b. **if** $\Gamma_{LA} \notin_P X$ **then** let $X = X \cup \{\Gamma_{LA}\}$
    5. **for** each formula $\neg LA \in \Gamma$, let $\Gamma_{\neg LA}$ =SAT'(APPLY($\Gamma, \neg LA$));
        5a. **for** all $\Gamma' \in X$ equivalent to $\Gamma_{\neg LA}$
            add the relation $\Gamma' R \Gamma$;
        5b. **if** $\Gamma_{\neg LA} \notin_P X$ **then** let $X = X \cup \{\Gamma_{\neg LA}\}$
    6. **for** each formula $\neg(A \mid\sim B) \in \Gamma$
        6a. let $\Gamma_{\neg(A\mid\sim B)}$ =SAT'(APPLY($\Gamma, \neg(A \mid\sim B)$));
        6b. **if** $\Gamma_{\neg(A\mid\sim B)} \notin_P X$ **then** $X = X \cup \{\Gamma_{\neg(A\mid\sim B)}\}$;
    7. mark Γ as resolved;

endWhile;

If Γ_0 is finite, the procedure terminates. Indeed, it can be easily seen that SAT' terminates, as there is only a finite number of propositional evaluations. Furthermore, the whole procedure terminates, since the number of possible different sets of formulas that can be added to X starting from a finite set Γ_0 is finite. Indeed, the number of non-p-equivalent sets of formulas that can be introduced in X is finite, as the number of p-equivalent classes is finite.

We construct the model $\mathcal{M} = \langle X, R_X, <_X, V \rangle$ as follows.

- X,V and R_X are defined as in the completeness proof for $\mathcal{T}$**CL** (Theorem 4.2);
- $<_X$ is defined as follows:

    (i) if $\Gamma' < \Gamma$, then $\Gamma' <_X \Gamma$;

    (ii) if $\Gamma' < \Gamma$, and $\Gamma R_X \Gamma''$, then $\Gamma' <_X \Gamma''$;

    As a difference from $<_X$ used in the completeness proof for $\mathcal{T}$**CL** (Theorem 4.2), $<_X$ is not transitive.

In order to show that $\mathcal{M}$ is a **C**-preferential model for Γ, we prove the following:

Fact 5.10 *The relation $<_X$ is irreflexive.*

Proof of Fact 5.10. By the procedure above, $\Gamma' <_X \Gamma$ only in two cases 1) $\Gamma' = \Gamma_{LA}$. In this case, it can be easily seen that $\Gamma \neq \Gamma'$. 2) $\Gamma' < \Gamma''$, i.e. $\Gamma' = \Gamma''_{LA}$, and $\Gamma'' R_X \Gamma$. Also in this case, it can be easily seen that $\Gamma \neq \Gamma'$, since Γ does not contain L-formulas, whereas Γ' does.

□ *Fact 5.10*

By reasoning analogously to what done for Facts 3.11 and 4.4 above, we show that:

Fact 5.11 *For all formulas F and for all sets $\Gamma \in X$ we have that: (i) if $F \in \Gamma$ then $\mathcal{M}, \Gamma \models F$; (ii) if $\neg F \in \Gamma$ then $\mathcal{M}, \Gamma \not\models F$.*

Proof of Fact 5.11. The proof is very similar to the one of Facts 3.11 and 4.4 above. Obviously, the case of negated boxed formulas disappears. Here we only consider the case of positive conditional formulas since the rule $(\mid\!\sim^+)$ in $\mathcal{TC}$ is slightly different than before. Let $\Delta \in Min_{<_X}(LA)$. We distinguish two cases:

- $A \mid\!\sim B \in \Delta$. By definition of `SAT'` and construction of the model above, either (1)$\neg LA \in \Delta$ or (2) there is $\Delta_{LA} \in X$ such that $LA \in \Delta_{LA}$ and $\Delta_{LA} < \Delta$, hence $\Delta_{LA} <_X \Delta$ or (3)$LB \in \Delta$. Similarly to what done in the proof for Fact 4.4, (1) and (2) cannot be the case, since they both contradict the fact that $\Delta \in Min_{<_X}(LA)$. Thus it must be that (3)$LB \in \Delta$, and by inductive hypothesis $\mathcal{M}, \Delta \models LB$.
- $A \mid\!\sim B \not\in \Delta$. We can reason in the same way than in the analogous case in the proof of Fact 4.4 above: Δ must have been generated by applying (L^-) to Δ' with $A \mid\!\sim B \in \Delta'$, hence $\Delta' R_X \Delta$. By definition of `SAT'` above, either (1)$\neg LA \in \Delta'$ or (2) there is $\Delta'_{LA} \in X$ such that $LA \in \Delta'_{LA}$ and $\Delta'_{LA} < \Delta'$, or (3) $LB \in \Delta'$. (1) is not possible: by inductive hypothesis, it would be $\mathcal{M}, \Delta' \not\models LA$, i.e. there is Δ'' such that $\Delta R_X \Delta''$ and $\mathcal{M}, \Delta'' \not\models A$. By definition of R_X (see point (ii) in the definition of R_X, proof of Theorem 4.2), also $\Delta R_X \Delta''$, hence also $\mathcal{M}, \Delta \not\models LA$, which contradicts $\Delta \in Min_{<_X}(LA)$. If (2), by definition of $<_X$, $\Delta'_{LA} <_X \Delta$, which contradicts $\Delta \in Min_{<_X}(LA)$. It follows that $LB \in \Delta'$, hence by inductive hypothesis $\mathcal{M}, \Delta' \models LB$, and also $\mathcal{M}, \Delta \models LB$ (indeed, since Δ does not contain any $L-$ formula, by construction of the model and by definition of R_X - see point (ii) in the definition of R_X, proof of Theorem 4.2 - $\Delta R_X \Delta''$ just in case $\Delta' R_X \Delta''$, from which the result follows).

□ *Fact 5.11*

Furthermore, we prove that:

Fact 5.12 *The relation $<$ satisfies the smoothness condition on L-formulas.*

Proof of Fact 5.12. Let $\mathcal{M}, \Gamma \models LA$. Then by Fact 5.11 above, $\neg LA \notin \Gamma$. By definition of `SAT'` and point 4 in the procedure above, either $\Box\neg LA \in \Gamma$ or there is $\Gamma' \in X$ s.t. $\Gamma' = \Gamma_{LA}$, and $\Gamma' <_X \Gamma$. In the first case, by Fact 5.11, $\mathcal{M}, \Gamma \models \Box\neg LA$, and it is minimal w.r.t. the set of LA-worlds. In the second case $\mathcal{M}, \Gamma^{'} \models \Box\neg LA$, it is minimal w.r.t. the set of LA-worlds, and $\Gamma^{'} <_X \Gamma$.

□ *Fact 5.12*

■

From the above Theorem 5.9, together with Proposition 2.29, it follows that for any boolean combination of conditionals Γ_0, if it does not have any closed tableau, then it is satisfiable in a cumulative model.

Similarly to what done for **P** and **CL**, we can show the following Corollary.

Corollary 5.13 (Finite model property) **C** *has the finite model property.*

As a difference from **P** and **CL** we cannot prove that $<$ does not have infinite descending chains. This is due to the fact that in the construction above we cannot prove that $<$ is acyclic.
In the case of logic **C**, non-termination can be caused by the generation of infinitely many worlds, producing infinite branches. By Theorem 5.8, only the formulas occurring in the initial set Γ_0 of formulas can occur on a branch. Hence, the number of possibly different sets of formulas Γ on the branch is finite (and they are exponentially many in the size of Γ_0). A loop checking procedure can be used in order to avoid that a given set of formulas is expanded again on a branch, so to ensure the termination of the procedure.

The satisfiability problem for a set of formulas Γ_0 can be solved by showing that all the tableaux for Γ_0 have an open branch. As there are exponentially many tableaux that have to be taken in consideration, each one of exponential size with respect to the size of the initial set of formulas, our tableau method provides an hyper exponential procedure to check the satisfiability.

In further investigations it might be considered if this bound can be improved. For this, a more accurate analysis of derivation structures (and, in particular, an analysis of permutability of the rules) might be required.

6 The Tableau Calculus for Rational Logic R

In this section we present $\mathcal{T}$**R**, a tableau calculus for rational logic **R**. We have already mentioned that, as a difference with the calculi presented for the other weaker logics, here we adopt a *labelled* tableau calculus, which seems to be a more natural approach. Indeed, in order to capture the modularity condition, intuitively we must keep all worlds generated by $(\Box^-)$ and we need to propagate formulas among them according to all possible modular orderings. In an unlabelled calculus, this might be achieved, for instance, by introducing ad hoc modal operators (that act as a marker) or by adding additional structures to tableau nodes similarly to hypersequents calculi (see for instance [3]). However, the resulting calculus would be unavoidably rather cumbersome. In contrast, by using world labels, we can easily keep track of multiple worlds and their relations. This provides a much simpler, intuitive and natural tableau calculus. On the other hand, even if we use labels, we do not run into problems with complexity and termination, and we are able to define an optimal decision procedure for **R**.

The calculus makes use of labels to represent possible worlds. We consider a language $\mathcal{L}_R$ and a denumerable alphabet of labels $\mathcal{A}$, whose elements are denoted by *x, y, z, ...* $\mathcal{L}_R$ extends $\mathcal{L}$ by formulas of the form $\Box\neg A$ as for the other logics.

Our tableau calculus includes two kinds of labelled formulas:

- *world formulas* $x : F$, whose meaning is that F holds in the possible world represented by x;
- *relation formulas* of the form $x < y$, where $x, y \in \mathcal{A}$, used to represent the relation $<$.

We denote by $\alpha, \beta \dots$ a world or a relation formula.
We define:

$$\Gamma^M_{x \to y} = \{y : \neg A, y : \Box\neg A \mid x : \Box\neg A \in \Gamma\}$$

The calculus $\mathcal{TR}$ is presented in Figure 11. As for **P**, the rules $(\mid\!\sim^-)$ and $(\Box^-)$ that introduce new labels in their conclusion are called *dynamic rules*; all the other rules are called *static rules.*

Definition 6.1 (Truth conditions of formulas of $\mathcal{T}$R) *Given a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and a labelled alphabet $\mathcal{A}$, we consider a mapping $I : \mathcal{A} \mapsto \mathcal{W}$. Given a formula α of the calculus $\mathcal{T}$**R**, we define $\mathcal{M} \models_I \alpha$ as follows:*

- $\mathcal{M} \models_I x : F$ *iff* $\mathcal{M}, I(x) \models F$
- $\mathcal{M} \models_I x < y$ *iff* $I(x) < I(y)$.

We say that a set of formulas Γ is satisfiable if, for all formulas $\alpha \in \Gamma$, we have that $\mathcal{M} \models_I \alpha$, for some model $\mathcal{M}$ and some mapping I.

In order to verify that a set of formulas Γ is unsatisfiable, we label all the formulas in Γ with a new label x_0, and verify that the resulting set of labelled formulas has a closed tableau. For instance, in order to verify that the set $\{adult \mid\!\sim worker, \neg(adult \mid\!\sim \neg married), \neg(adult \land married \mid\!\sim worker)\}$ is unsatisfiable (and thus $adult \land married \mid\!\sim worker$ is entailed by $\{adult \mid\!\sim worker, \neg(adult \mid\!\sim \neg married)\}$), we can build the closed tableau in Figure 12.

$$(\mathbf{AX})\ \Gamma, x : P, x : \neg P \quad \text{with } P \in ATM \qquad\qquad (\mathbf{AX})\ \Gamma, x < y, y < x$$

$$(\wedge^+)\ \frac{\Gamma, x : F \wedge G}{\Gamma, x : F, x : G} \qquad\qquad (\wedge^-)\ \frac{\Gamma, x : \neg(F \wedge G)}{\Gamma, x : \neg F \qquad \Gamma, x : \neg G}$$

$$(\neg)\ \frac{\Gamma, x : \neg\neg F}{\Gamma, x : F}$$

$$(\mid\!\sim^+)\ \frac{\Gamma, u : A \mid\!\sim B}{\Gamma, x : \neg A, u : A \mid\!\sim B \qquad \Gamma, x : \neg\Box\neg A, u : A \mid\!\sim B \qquad \Gamma, x : B, u : A \mid\!\sim B}$$

$$(\mid\!\sim^-)\ \frac{\Gamma, u : \neg(A \mid\!\sim B)}{\Gamma, x : A, x : \Box\neg A, x : \neg B}\ \begin{array}{c} x \text{ new} \\ \text{label} \end{array} \qquad (\Box^-)\ \frac{\Gamma, x : \neg\Box\neg A}{\Gamma, y < x, \Gamma^M_{x \to y}, y : A, y : \Box\neg A}\ \begin{array}{c} y \text{ new} \\ \text{label} \end{array}$$

$$(<)\ \frac{\Gamma, x < y}{\Gamma, x < y, z < y, \Gamma^M_{y \to z} \qquad \Gamma, x < y, x < z, \Gamma^M_{z \to x}}\ \begin{array}{c} z \text{ occurs in } \Gamma \text{ and} \\ \{x < z, z < y\} \cap \Gamma = \emptyset \end{array}$$

Figure 11: The calculus $\mathcal{TR}$. To save space, rules for $\rightarrow$ and $\vee$ are omitted.

6.1 Soundness, Termination, and Completeness of $\mathcal{TR}$

In this section we prove that the calculus $\mathcal{TR}$ is sound and complete w.r.t. the semantics. Moreover, we prove that, with a restriction on $(\mid\!\sim^+)$ based on the same idea of the one adopted for the other calculi, the calculus guarantees termination.

First of all, we reformulate the calculus, obtaining a terminating calculus $\mathcal{TR}^{\mathbf{T}}$. We will show in Theorem 6.12 below that, as for the calculi for **P**, **CL**, and **C**, non-termination of the procedure due to the generation of infinitely-many worlds (thus creating infinite branches) cannot occur. As a consequence, we will observe that only a finite number of labels are introduced in a tableau (Corollary 6.13).

Similarly to the other cases, $\mathcal{TR}$ does not ensure a terminating proof search due to $(\mid\!\sim^+)$, which can be applied without any control. We ensure the termination by putting on $(\mid\!\sim^+)$ in $\mathcal{TR}$ the same constraint used in the other calculi. More precisely, it is easy to observe that it is useless to apply the rule on the same conditional formula more than once by using the same label x. By the invertibility of the rules (Theorem 6.4) we can assume, without loss of generality, that two applications of $(\mid\!\sim^+)$ on x are consecutive. We observe that the second application is useless, since each of the conclusions has already been obtained after the first application, and can be removed. We prevent redundant

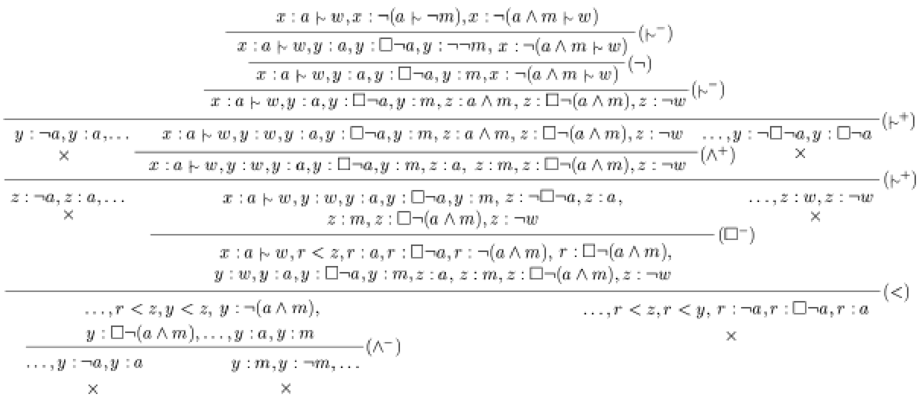

Figure 12: A derivation in $\mathcal{T}\mathbf{R}$ of $\{adult \mathrel{|\!\sim} worker, \neg(adult \mathrel{|\!\sim} \neg married), \neg(adult \land married \mathrel{|\!\sim} worker)\}$. To save space, we use a for $adult$, m for $married$, and w for $worker$.

applications of ($|\!\sim^+$) by keeping track of labels (worlds) in which a conditional $u : A \mathrel{|\!\sim} B$ has already been applied in the current branch. To this purpose, we add to each positive conditional a list of *used* labels; we then restrict the application of ($|\!\sim^+$) only to labels not occurring in the corresponding list.

Notice that also the rule ($<$) copies its principal formula $x < y$ in the conclusion; however, this rule will be applied only a finite number of times. This is a consequence of the side condition of the rule application and the fact that the number of labels in a tableau is finite.

The terminating calculus $\mathcal{T}\mathbf{R}^{\mathbf{T}}$ is obtained by replacing the ($|\!\sim^+$) rule in Figure 11 with the one presented in Figure 13.

$$\frac{\Gamma, u : A \mathrel{|\!\sim} B^{L}}{\Gamma, x : \neg A, u : A \mathrel{|\!\sim} B^{L,x} \qquad \Gamma, x : \neg\Box\neg A, u : A \mathrel{|\!\sim} B^{L,x} \qquad \Gamma, x : B, u : A \mathrel{|\!\sim} B^{L,x}} \ (|\!\sim^+)$$

with $x \notin L$

Figure 13: The rule ($|\!\sim^+$) in the tableau system $\mathcal{T}\mathbf{R}^{\mathbf{T}}$.

It is easy to prove the following structural properties of $\mathcal{T}\mathbf{R}^{\mathbf{T}}$:

Lemma 6.2 *For any set of formulas* Γ *and any world formula* $x : F$*, there is a closed tableau for* $\Gamma, x : F, x : \neg F$*.*

Proof. By induction on the complexity of the formula F.

Lemma 6.3 (Height-preserving admissibility of weakening) *Given any set of formulas Γ and any formula α, if Γ has a closed tableau of height h then Γ, α has a closed tableau whose height is no greater than h.*

Proof. By induction on the height of the closed tableau for Γ.

Moreover, one can easily prove that all the rules of $\mathcal{TR}^{\mathbf{T}}$ are height-preserving invertible, that is to say:

Theorem 6.4 (Height-preserving invertibility of the rules of $\mathcal{TR}^{\mathbf{T}}$) *Given any rule* $(\mathbf{R})$ *of $\mathcal{TR}^{\mathbf{T}}$, whose premise is Γ and whose conclusions are Γ_i, with $i \leq 3$, we have that if Γ has a closed tableau of height h, then there is a closed tableau, of height no greater than h, for each Γ_i, i.e. the rules of $\mathcal{TR}^{\mathbf{T}}$ are height-preserving invertible.*

Proof. We consider each rule of the calculus, then we proceed by induction on the height of the closed tableau for the premise.

- $(\mid\!\sim^+)$: given a closed tableau for $\Gamma, u : A \mid\!\sim B$, then we can immediately conclude that there is also a closed tableau for $\Gamma, u : A \mid\!\sim B, x : \neg A$, for $\Gamma, u : A \mid\!\sim B, x : \neg\Box\neg A$, and for $\Gamma, u : A \mid\!\sim B, x : B$, since weakening is height-preserving admissible (see Lemma 6.3);
- $(<)$: as in the previous case, a closed tableau for the two conclusions of the rule can be obtained by weakening from the premise;
- $(\Box^-)$: given a closed tableau for $(1)\Gamma, x : \neg\Box\neg A$ and a label y not occurring in Γ, we have to show that there is a closed tableau for $\Gamma, y < x, \Gamma^M_{x\rightarrow y}, y : A, y : \Box\neg A$. By induction on the height of the proof of (1), we distinguish the following cases:
  - the first rule applied is $(\Box^-)$ on $x : \neg\Box\neg A$: in this case, we are done, since we have a closed tableau for $\Gamma, y < x, \Gamma^M_{x\rightarrow y}, y : A, y : \Box\neg A$;
  - otherwise, i.e. another rule $(\mathbf{R})$ of $\mathcal{TR}^{\mathbf{T}}$ is applied to $\Gamma, x : \neg\Box\neg A$, we can apply the inductive hypothesis on the conclusion(s) of $(\mathbf{R})$, since no rule removes side formulas in a rule application. In detail, we have that $x : \neg\Box\neg A$ belongs to all the conclusions, then we can apply the inductive hypothesis and then conclude by re-applying $(\mathbf{R})$. As an example, suppose the derivation starts with an application of $(\mid\!\sim^-)$ as follows:

    $$\frac{(1)\Gamma', u : C \mid\!\sim D, x : \neg\Box\neg A}{(2)\Gamma', v : C, v : \Box\neg C, v : \neg D, x : \neg\Box\neg A}\ (\mid\!\sim^-)$$

    We can apply the inductive hypothesis on the closed tableau for (2), concluding that there is a closed tableau for $(2')\Gamma', v : C, v : \Box\neg C, v :$

$\neg D, y < x, \Gamma^M_{x \rightarrow y}, y : A, y : \Box\neg A$, from which we can conclude obtaining the following closed tableau:

$$\frac{\Gamma', u : C \mid\!\sim D, y < x, \Gamma^M_{x \rightarrow y}, y : A, y : \Box\neg A}{(2')\Gamma', v : C, v : \Box\neg C, v : \neg D, y < x, \Gamma^M_{x \rightarrow y}, y : A, y : \Box\neg A} \ (\mid\!\sim^-)$$

Notice that the proof has (at most) the same height of the closed tableau for (1).

- other rules: the proof is similar to the one for $(\Box^-)$ and then left to the reader.

■

Since all the rules are invertible, we have that in $\mathcal{TR}^{\mathbf{T}}$ the order of application of the rules is not relevant. Hence, no backtracking is required in the calculus, and we can assume without loss of generality that a given set of formulas Γ has a unique tableau.

Let us now prove that $\mathcal{TR}^{\mathbf{T}}$ is sound.

Theorem 6.5 (Soundness) *$\mathcal{TR}^{\mathbf{T}}$ is sound w.r.t. rational models, i.e. if there is a closed tableau for a set of formulas Γ, then Γ is unsatisfiable.*

Proof. By induction on the height of the closed tableau for Γ. If Γ is an axiom, then we distinguish two different cases. First case: $x : P \in \Gamma$ and $x : \neg P \in \Gamma$, therefore there is no I such that $I(x) \in \mathcal{W}$ and $\mathcal{M}, I(x) \models P$ and $\mathcal{M}, I(x) \not\models P$, and Γ is unsatisfiable. Second case: $x < y \in \Gamma$ and $y < x \in \Gamma$, therefore there is a cycle in the preference relation, and the smoothness condition cannot be satisfied. Indeed, since in any rational model $<$ is irreflexive an transitive, it can be easily shown that there cannot be an I such that $I(x) < I(y)$ and $I(y) < I(x)$, which makes Γ unsatisfiable.

For the inductive step, we have to show that, for each rule r, if all the conclusions of r are unsatisfiable, then the premise is unsatisfiable too. As usual, we prove the contrapositive, i.e. we prove for each rule that, if the premise is satisfiable, so is (at least) one of the conclusions. In order to save space, we only present the most interesting case of $(\Box^-)$. Since the premise is satisfiable, then there is a model $\mathcal{M}$ and a mapping I such that $\mathcal{M} \models_I \Gamma, x : \neg\Box\neg A$. Let $w \in \mathcal{W}$ such that $I(x) = w$; this means that $\mathcal{M}, w \not\models \Box\neg A$, hence there exists a world $w' < w$ such that $\mathcal{M}, w' \models A$. By the strong smoothness condition, we have that there exists a *minimal* such world, so we can assume that $w' \in Min_<(A)$, thus $\mathcal{M}, w' \models \Box\neg A$. In order to prove that the conclusion of the rule is satisfiable, we construct a mapping I' as follows: let y be a new label, not occurring in the current branch; we define (1) $I'(u) = I(u)$ for all $u \neq y$ and (2) $I'(y) = w'$. Since y does not occur in Γ, it follows that $\mathcal{M} \models_{I'} \Gamma$. By Definition 6.1, we have that $\mathcal{M} \models_{I'} y < x$ since $w' < w$. Moreover, since $I'(y) = w'$, we have that $\mathcal{M} \models_{I'} y : A$ and $\mathcal{M} \models_{I'} y : \Box\neg A$. Finally, $\mathcal{M} \models_{I'} \Gamma^M_{x \rightarrow y}$ follows from the fact that $I'(y) < I'(x)$ and from the transitivity of $<$. The only conclusion of the rule is then satisfiable in $\mathcal{M}$ via I'.

■

In order to prove the completeness of the calculus, we introduce the notion of saturated branch and we show that $\mathcal{TR}^{\mathbf{T}}$ ensures a terminating proof search. As a consequence, we will observe that the calculus introduces a finite number of labels in a tableau, and this result will be used to prove the completeness of the calculus.

Definition 6.6 (Saturated branch) *We say that a branch* $\mathbf{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ *of a tableau is* saturated *if the following conditions hold:*

1. *for the boolean connectives, the condition of saturation is defined in the usual way. For instance, if* $x : A \wedge B \in \Gamma_i$ *in* $\mathbf{B}$*, then there exists* Γ_j *in* $\mathbf{B}$ *such that* $x : A \in \Gamma_j$ *and* $x : B \in \Gamma_j$*;*
2. *if* $x : A \mid\!\sim B \in \Gamma_i$*, then for any label* y *in* $\mathbf{B}$*, there exists* Γ_j *in* $\mathbf{B}$ *such that either* $y : \neg A \in \Gamma_j$ *or* $y : \neg\Box\neg A \in \Gamma_j$ *or* $y : B \in \Gamma_j$*.*
3. *if* $x : \neg(A \mid\!\sim B) \in \Gamma_i$*, then there is a* Γ_j *in* $\mathbf{B}$ *such that, for some* y*,* $y : A \in \Gamma_j$*,* $y : \Box\neg A \in \Gamma_j$*, and* $y : \neg B \in \Gamma_j$*.*
4. *if* $x : \neg\Box\neg A \in \Gamma_i$*, then there exists* Γ_j *in* $\mathbf{B}$ *such that, for some* y*,* $y < x \in \Gamma_j$*,* $y : A \in \Gamma_j$ *and* $y : \Box\neg A \in \Gamma_j$*.*
5. *if* $x < y \in \Gamma_i$*, then for all labels* z *in* $\mathbf{B}$*, there exists* Γ_j *in* $\mathbf{B}$ *such that either* $z < y \in \Gamma_j$ *or* $x < z \in \Gamma_j$*.*

We say that a branch $\mathbf{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ is *open* if there is no Γ_i in $\mathbf{B}$ such that $x : F \in \Gamma_i$ and $x : \neg F \in \Gamma_i$. We can easily show the following Lemma:

Lemma 6.7 *Given a tableau starting with* $x_0 : F$*, for any open, saturated branch* $\mathbf{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$*, we have that:*

1. *if* $z < y \in \Gamma_i$ *in* $\mathbf{B}$ *and* $y < x \in \Gamma_j$ *in* $\mathbf{B}$*, then there exists* Γ_k *in* $\mathbf{B}$ *such that* $z < x \in \Gamma_k$*;*
2. *if* $x : \Box\neg A \in \Gamma_i$ *in* $\mathbf{B}$ *and* $y < x \in \Gamma_j$ *in* $\mathbf{B}$*, then there exists* Γ_k *in* $\mathbf{B}$ *such that* $y : \neg A \in \Gamma_k$ *and* $y : \Box\neg A \in \Gamma_k$*;*
3. *for no* Γ_i *in* $\mathbf{B}$*,* $x < x \in \Gamma_i$*.*

Proof. First, we prove the following Fact:

Fact 6.8 *Given a saturated branch* $\mathbf{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$*, if there are* Γ_i *and* Γ_j *in* $\mathbf{B}$ *such that* $x < y \in \Gamma_i$ *and* $y < x \in \Gamma_j$*, then* $\mathbf{B}$ *is closed.*

Proof of Fact 6.8. We distinguish two cases:

- both the two labels are different from the initial label x_0: by this fact, both x and y have been introduced in the tableau by an application of $(\Box^-)$ or $(\mid\!\sim^-)$, the only two rules introducing a new label, say u, in their conclusions. In both the cases of $(\Box^-)$ and $(\mid\!\sim^-)$, the rule introducing x also introduces $x : A, x : \Box\neg A$ for some A. The same for y: the rule introducing it also introduces $y : B, y : \Box\neg B$. Suppose that $y < x$ is introduced in the tableau before $x < y$: $y < x$ can only be introduced by $(\Box^-)$ or by $(<)$, and in both cases $\Gamma^M_{x \to y}$ is added to the current branch; therefore, by the presence of $x : \Box\neg A$, we have that $y : \neg A, y : \Box\neg A$ are introduced in the tableau. When $x < y$ is introduced in the tableau by $(\Box^-)$ or $(<)$, since positive box formulas are never removed from the tableau, we have that also $x : \neg A$ is introduced by $\Gamma^M_{y \to x}$, and the tableau is closed by the presence of $x : A$. The case when $y < x$ is introduced after $x < y$ is symmetric, and the tableau is closed by the presence of $y : B$ and $y : \neg B$;
- $x = x_0$ (resp. $y = x_0$): in this case, the tableau is closed since $x < y, y < x$ is an instance of (**AX**).

$\Box$ *Fact 6.8*

Let us consider an open, saturated branch $\mathbf{B} = \Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots$. We consider separately the three claims in the definition of the Lemma:

1. We are considering the case when $z < y \in \Gamma_i$ in $\mathbf{B}$ and $y < x \in \Gamma_j$ in $\mathbf{B}$. Since $\mathbf{B}$ is saturated, then there exists Γ_k in $\mathbf{B}$ such that either $z < x \in \Gamma_k$ or $x < y \in \Gamma_k$. If $x < y \in \Gamma_k$, then the branch is closed (Γ_k is an instance of **(AX)**). Thus, we conclude that $z < x \in \Gamma_k$.

2. A relation formula $y < x$ can only be introduced by an application of either $(\Box^-)$ or $(<)$; in both cases, $\Gamma^M_{x \to y}$ is added to the current branch of the tableau. Consider any $x : \Box\neg A \in \Gamma_i$; if $j > i$, i.e. $y < x$ is introduced in the branch *after* $x : \Box\neg A$, then we are done, since $y : \neg A \in \Gamma^M_{x \to y}$ and $y : \Box\neg A \in \Gamma^M_{x \to y}$. Otherwise, if $y < x$ is introduced in the branch *before* $x : \Box\neg A$, then we are considering the case such that $x : \Box\neg A$ is introduced by an application of $(<)$, i.e. $x : \Box\neg A \in \Gamma^M_{k \to x}$ (by the presence of some $k : \Box\neg A$ in the branch) for some k and $x < k$ is also introduced in the branch. Since the branch is saturated, then either (1) $x < y$ or (2) $y < k$ are introduced in the branch: (1) cannot be, otherwise the branch would be closed. If (2) is introduced after $k : \Box\neg A$, then we are done since $y : \neg A \in \Gamma^M_{k \to y}$ and $y : \Box\neg A \in \Gamma^M_{k \to y}$; otherwise, (2) has also been introduced by an application of $(<)$, and we can reason in the same way described here above. This process terminates: indeed, we can easily observe the following facts: - a boxed formula $u : \Box\neg A$ is initially introduced in the tableau by an application of either $(\Box^-)$ or $(\mid\!\sim^-)$, whereas $(<)$ and $(\Box^-)$ only "propagate" it to other worlds; - in

both cases ($\Box^-$) and ($\mid\!\sim^-$), u is a new label not occurring in the branch, therefore all the relation formulas $v < u$ will be introduced *later* in the branch, i.e. when $u : \Box\neg A$ already belongs to the branch. In conclusion of our proof, a relation formula $y < u$ will be introduced in the branch *not before* of $u : \Box\neg A$: by this fact and by the saturation of the branch, we conclude that also $y < u$ and, then, $y : \neg A$ and $y : \Box\neg A$ belong to the branch.

3. A relation $x < x$ cannot be introduced by rule ($\Box^-$), since this rule establishes a relation between x in **B** and a label distinct from x. On the other hand, it cannot be introduced by modularity. Indeed, for rule ($<$) to introduce a relation $x < x$, there must be in **B** some relation $y < x$ (resp. $x < y$) for some y. But in this case the side condition of the rule would not be fulfilled, and the rule could not be applied.

■

Also in $\mathcal{TR}^{\mathbf{T}}$ we introduce the restriction on the order of application of the rules adopted for the other systems (see Definition 3.14), that is to say: the application of the ($\Box^-$) rule is postponed to the application of all propositional rules and to the test of whether Γ is an instance of (**AX**) or not.

Let us now show that $\mathcal{TR}^{\mathbf{T}}$ ensures a terminating proof search. In order to prove this result in a rigorous manner, we proceed as follows: first, we introduce the complexity measure on a set of formulas Γ of Definition 6.10, denoted by $m(\Gamma)$, which consists of five measures c_1, c_2, c_3, c_4 and c_5 in a lexicographic order, and the auxiliary Definition 6.9; then, we prove that each application of $\mathcal{TR}^{\mathbf{T}}$'s rules reduces this measure, until the rules are no longer applicable, or leads to a closed tableau.

The complexity of a formula $cp(F)$ is defined in Definition 3.18; moreover, we use square brackets $[\ldots]$ to denote *multisets*.

Definition 6.9 *Given an initial set of formulas Γ_0, we define:*

- *the set $\mathcal{L}^{\Gamma_0}_{\Box^+}$ of boxed formulas $\Box\neg A$ that can be generated in a tableau for Γ_0, i.e. $\mathcal{L}^{\Gamma_0}_{\Box^+} = \{\Box\neg A \mid A \mid\!\sim B \in_+ \Gamma_0\} \cup \{\Box\neg A \mid A \mid\!\sim B \in_- \Gamma_0\}$. We let $n_0 = \mid \mathcal{L}^{\Gamma_0}_{\Box^+} \mid$;*

- *the* multiset *$\mathcal{L}^{\Gamma_0}_{\Box^-}$ of negated boxed formulas that can be generated in a tableau for Γ_0, i.e. $\mathcal{L}^{\Gamma_0}_{\Box^-} = [\neg\Box\neg A \mid A \mid\!\sim B \in_+ \Gamma_0]$. We let $k_0 = \mid \mathcal{L}^{\Gamma_0}_{\Box^-} \mid$;*

Given a label x and a set of formulas Γ in the tableau for the initial set Γ_0, we define:

- *the number n_x of positive boxed formulas $\Box\neg A$ not labelled by x, i.e. $n_x = n_0 - \mid \{\Box\neg A \in \mathcal{L}^{\Gamma_0}_{\Box^+} \mid x : \Box\neg A \in \Gamma\} \mid$;*

- *the number k_x of negated boxed formulas $\neg\Box\neg A$ not yet expanded in a world x, i.e.* $k_x = k_0 - \mid [\neg\Box\neg A \in \mathcal{L}^{\Gamma_0}_{\Box^-} \mid y : \Box\neg A \in \Gamma$ *and* $y < x \in \Gamma] \mid$[6].

Definition 6.10 (Lexicographic order) *We define $m(\Gamma) = \langle c_1, c_2, c_3, c_4, c_5\rangle$ where:*

- $c_1 = \mid \{u : A \mid\!\sim B \in_- \Gamma\} \mid$
- *c_2 is the multiset given by $[c_2^{x_1}, c_2^{x_2}, \dots, c_2^{x_n}]$, where $x_1, x_2, \dots, x_n$ are the labels occurring in Γ and, given a label x, c_2^x is a pair (n_x, k_x) in a lexicographic order (n_x and k_x are defined as in Definition 6.9). We consider the integer multiset ordering given by c_2*
- $c_3 = \mid \{\langle x, A \mid\!\sim B\rangle \mid u : A \mid\!\sim B^L \in \Gamma$ *and* $x \notin L\} \mid$
- $c_4 = \sum_z c_4^z$, *where z occurs in Γ and* $c_4^z = \mid \{x < y \in \Gamma \mid \{x < z, z < y\} \cap \Gamma = \emptyset\} \mid$
- $c_5 = \sum_{x:F\in\Gamma} cp(F)$

We consider the lexicographic order given by $m(\Gamma)$.

Roughly speaking, c_1 is the number of negated conditionals that can still be expanded in the tableau. The application of $(\mid\!\sim^-)$ reduces c_1. c_2 keeps track of positive conditionals *which can still create a new world.* The application of $(\Box^-)$ reduces c_2. c_3 represents the number of positive conditionals not yet expanded in a world x: the application of $(\mid\!\sim^+)$ reduces this measure, since the rule is applied to $u : A \mid\!\sim B^L$ by using x *only if* x does not belong to L, i.e. $u : A \mid\!\sim B$ has not yet been expanded in x. c_4 represents the number of relations $x < y$ not yet added to the current branch: the application of $(<)$ reduces c_4, since it applies the modularity of $<$ in case $x < y$ and, given z, neither $z < y$ nor $x < z$ belong to the current set of formulas. c_5 is the sum of the complexities of the world formulas in Γ: an application of the rules for boolean connectives reduces c_5.

First of all, we prove that the application of any rule of $\mathcal{TR}^{\mathbf{T}}$ reduces $m(\Gamma)$ or leads to a closed tableau, as stated by the following Lemma:

Lemma 6.11 *Let Γ' be a set of formulas obtained as a conclusion of an application of a rule of $\mathcal{TR}^{\mathbf{T}}$ to a set of formulas Γ. We have that either the tableau for Γ' is closed or $m(\Gamma') < m(\Gamma)$.*

Proof. We consider each rule of the calculus:

[6]Notice that, in case there are two positive conditionals $A \mid\!\sim B$ and $A \mid\!\sim C$ with the same antecedent, then the multiset $\mathcal{L}^{\Gamma_0}_{\Box^-}$ contains two instances of $\neg\Box\neg A$. Therefore, if the rule $(\Box^-)$ is applied to $x : \neg\Box\neg A$ (for instance, generated by an application of $(\mid\!\sim^+)$ in x on $A \mid\!\sim B$), then k_x decreases only by 1 unit, whereas the second instance of $\neg\Box\neg A$, i.e. the one "associated" with $A \mid\!\sim C$, is still considered to be *not expanded* in x, thus it still "contributes" to k_x.

- ($\mid\!\sim^-$): an application of this rule reduces c_1, since it is applied to a negated conditional $u : \neg(A \mid\!\sim B)$ belonging to its premise which is removed from the conclusion;

- ($\Box^-$): first of all, observe that c_1 is not augmented in the conclusion, since no negated conditional is added by the rule. The application of ($\Box^-$) reduces c_2 or leads to a closed tableau. We are considering the following rule application:

$$\frac{\Gamma, x : \neg\Box\neg A}{\Gamma, y < x, \Gamma^M_{x\to y}, y : A, y : \Box\neg A}\ (\Box^-)$$

  where y is a new label. c_2 in the premise, say c_{2_p}, is a multiset $[\ldots, c^x_{2_p}, \ldots]$, whereas in the conclusion we have to consider a measure, called c_{2_c}, of the form $[\ldots, c^x_{2_c}, c^y_{2_c}, \ldots]$. By the standard definition of integer multiset ordering, we prove that either $c_{2_c} < c_{2_p}$ (by showing that $c^x_{2_c} < c^x_{2_p}$ and $c^y_{2_c} < c^x_{2_p}$, i.e. we replace an integer $c^x_{2_p}$ with two smaller integers, see [10] for details on integer multiset orderings) or that the procedure leads to a closed tableau. We conclude the proof as follows:

  – let us consider $c^y_{2_c}$ and $c^x_{2_p}$. By definition, $c^y_{2_c}$ is a pair (n_{y_c}, k_{y_c}) and $c^x_{2_p}$ is a pair (n_{x_p}, k_{x_p}). We distinguish two cases: if $x : \Box\neg A \notin \Gamma$, then we easily prove that $n_{y_c} < n_{x_p}$, since $y : \Box\neg A$ belongs to the conclusion: therefore, the number of boxed formulas $\Box\neg A$ not occurring with label y is smaller than the number of boxed formulas not occurring with label x, and we are done (remember that all the positive boxed formulas labelled by x are also labelled by y in the conclusion, by the presence of $\Gamma^M_{x\to y}$); if $x : \Box\neg A \in \Gamma$, then the application of ($\Box^-$) leads to a node containing both $y : A$ and $y : \neg A$ and, by the restriction on the order of application of the rules (see Definition 3.14), the procedure terminates building a closed tableau;

  – let us consider $c^x_{2_c}$ and $c^x_{2_p}$. It is easy to observe that $n_{x_p} = n_{x_c}$, since no formula $x : \Box\neg A$ is added nor removed in the conclusion (the positive boxed formulas labelled by x are the same in both the premise and the conclusion). We conclude since $k_{x_c} < k_{x_p}$, since $x : \neg\Box\neg A$ has been expanded in x, so it "contributes" to k_{x_p} whereas it does not to k_{x_c}.

- ($\mid\!\sim^+$): first of all, notice that c_1 and c_2 cannot be higher in the conclusions than in the premise. Notice that the addiction of $x : \neg\Box\neg A$ in the inner conclusion does not increase c_2, since $\neg\Box\neg A$ is a negated boxed formula still to be considered in both the premise and the conclusions. The application of ($\mid\!\sim^+$) reduces c_3. Suppose that this rule is applied to $u : A \mid\!\sim B^L$ by using label x in the conclusions; by the restriction in Figure 13, this means that $x \notin L$, so $\langle x, A \mid\!\sim B\rangle$ belongs to the set whose cardinality determines c_3 in the premise. Obviously, since x is added to L for $u : A \mid\!\sim B$ in the three conclusions of the rule, we can easily observe that $\langle x, A \mid\!\sim B\rangle$

does no longer belong to the set in the definition of c_3 in the conclusions: c_3 is then smaller in the conclusions than in the premise, and we are done;

- $(<)$: the application of $(<)$ reduces c_4, whereas c_1, c_2, and c_3 cannot be augmented (at most formulas $z : \Box\neg A$ are added in the conclusions by $\Gamma^M_{z\rightarrow x}$ and $\Gamma^M_{y\rightarrow z}$, reducing c_2). To conclude the proof, just observe that, given a label z and a formula $x < y$, $(<)$ is applied if $\{x < z, z < y\} \cap \Gamma = \emptyset$, i.e. $x < y$ belongs to the set used to define c_4^z in the premise, say $c_{4_p}^z$. When the rule is applied, in the left premise $z < y$ is added, and $x < y$ does no longer belong to the set used to define c_4^z in the conclusion, say $c_{4_c}^z$. Therefore, $c_{4_c}^z < c_{4_p}^z$. The same for the right premise, and we are done;
- rules for the boolean connectives: these rules do not increase values of c_1, c_2, c_3, and c_4. Their application reduces c_5, since the (sum of) complexity of the subformula(s) introduced in the conclusion(s) is lower then the complexity of the principal formula to which the rule is applied.

■

Now we can prove that $\mathcal{TR}^{\mathbf{T}}$ ensures a terminating proof search:

Theorem 6.12 (Termination of $\mathcal{TR}^{\mathbf{T}}$) *Let Γ be a finite set of formulas, then any tableau generated by $\mathcal{TR}^{\mathbf{T}}$ is finite.*

Proof. Let Γ' be obtained by an application of a rule of $\mathcal{TR}^{\mathbf{T}}$ to a premise Γ. By Lemma 6.11 we have that either the procedure leads to a closed tableau (and in this case we are done) or we have that $m(\Gamma') < m(\Gamma)$. This means that, similarly to the case of $\mathbf{P}$, a finite number of applications of the rules leads either to close the branch or to a node whose measure is as follows: $\langle 0, [(0,0),(0,0),\ldots,(0,0)], 0, 0, c_{5_{min}}\rangle$, where $c_{5_{min}}$ is the minimal value that c_5 can assume for Γ. This means that no rule of $\mathcal{TR}^{\mathbf{T}}$ is further applicable. This is a consequence of the following facts:

- $(\mid\!\sim^-)$ is no longer applicable, since $c_1 = 0$;
- since $c_2 = [(0,0),(0,0),\ldots,(0,0)]$, given any label x, we have that $c_2^x = (0,0)$. Therefore, the $(\Box^-)$ rule is no longer applicable, since we can observe that there is no $x : \neg\Box\neg A \in \Gamma$ (by the fact that $k_x = 0$);
- the rule $(\mid\!\sim^+)$ is not further applicable since $c_3 = 0$; indeed, $c_3 = 0$ means that all formulas $A \mid\!\sim B$ have already been expanded in each world x;
- $c_4 = 0$, i.e. for all formulas $x < y$, given any label z, we have that either $z < y \in \Gamma$ or $x < z \in \Gamma$, thus the $(<)$ rule is not further applicable;
- since c_5 assumes its *minimal* value $c_{5_{min}}$, no rule for a boolean connective is further applicable. If a boolean rule is applicable, then its application reduces the value of c_5 in its conclusion(s) by Lemma 6.11, against the minimality of $c_{5_{min}}$in the premise.

■

As a consequence of Theorem 6.12, we can observe that the tableau for a given set of formulas Γ_0 contains a finite number of labels, since all the branches in a tableau generated by $\mathcal{TR}^{\mathbf{T}}$ are finite.

Corollary 6.13 *Given a set of formulas* Γ*, the tableau generated by* $\mathcal{TR}^{\mathbf{T}}$ *for* Γ *only contains a finite number of labels.*

Let us now show that $\mathcal{TR}^{\mathbf{T}}$ is complete with respect to the semantics:

Theorem 6.14 (Completeness) $\mathcal{TR}^{\mathbf{T}}$ *is complete w.r.t. rational models, i.e. if a set of formulas* Γ *is unsatisfiable, then it has a closed tableau in* $\mathcal{TR}^{\mathbf{T}}$*.*

Proof. We show the contrapositive, i.e. if there is no closed tableau for Γ, then Γ is satisfiable. Consider the tableau starting with the set of formulas $\{x : F$ such that $F \in \Gamma\}$ and any open, saturated branch $\mathbf{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ in it. Starting from $\mathbf{B}$, we build a canonical model $\mathcal{M} = \langle \mathcal{W}_B, <, V \rangle$ satisfying Γ, where:

- $\mathcal{W}_B$ is the set of labels that appear in the branch $\mathbf{B}$;
- for each $x, y \in \mathcal{W}_B$, $x < y$ iff there exists Γ_i in $\mathbf{B}$ such that $x < y \in \Gamma_i$;
- for each $x \in \mathcal{W}_B$, $V(x) = \{P \in ATM \mid$ there is Γ_i in $\mathbf{B}$ such that $x : P \in \Gamma_i\}$.

We can easily prove that:
(i) by Corollary 6.13, we have that $\mathcal{W}_B$ is finite;
(ii) $<$ is an irreflexive, transitive and modular relation on $\mathcal{W}_B$ satisfying the smoothness condition. Irreflexivity, transitivity and modularity are obvious, given Definition 6.6 and Lemma 6.7 above. Since $<$ is irreflexive and transitive, it can be easily shown that it is also acyclic. This property together with the finiteness of $\mathcal{W}_B$ entails that $<$ cannot have infinite descending chains. In turn this last property together with the transitivity of $<$ entails the smoothness condition.
(iii) We show that, for all formulas F and for all Γ_i in $\mathbf{B}$, (i) if $x : F \in \Gamma_i$ then $\mathcal{M}, x \models F$ and (ii) if $x : \neg F \in \Gamma_i$ then $\mathcal{M}, x \not\models F$. The proof is by induction on the complexity of the formulas. If $F \in ATM$ this immediately follows from definition of V. For the inductive step, we only present the case of $F = A \mid\!\sim B$. The other cases are similar and then left to the reader. Let $x : A \mid\!\sim B \in \Gamma_i$. By Definition 6.6, we have that, for all y, there is Γ_j in $\mathbf{B}$ such that either $y : \neg A \in \Gamma_j$ or $y : B \in \Gamma_j$ or $y : \neg\Box\neg A \in \Gamma_j$. We show that for all $y \in Min_<(A)$, $\mathcal{M}, y \models B$. Let $y \in Min_<(A)$. This entails that $\mathcal{M}, y \models A$, hence $y : \neg A \not\in \Gamma_j$. Similarly, we can show that $y : \neg\Box\neg A \not\in \Gamma_j$. It follows that $y : B \in \Gamma_j$, and by inductive hypothesis $\mathcal{M}, y \models B$. (ii) If $x : \neg(A \mid\!\sim B) \in \Gamma_i$, since $\mathbf{B}$ is saturated, there is a label y in some Γ_j such that $y : A \in \Gamma_j$, $y : \Box\neg A \in \Gamma_j$, and $y : \neg B \in \Gamma_j$. By inductive hypothesis we can easily show that $\mathcal{M}, y \models A$, $\mathcal{M}, y \models \Box\neg A$, hence $y \in Min_<(A)$, and $\mathcal{M}, y \not\models B$, hence $\mathcal{M}, x \not\models A \mid\!\sim B$.

Since $\mathcal{TR}^{\mathbf{T}}$ makes use of the restriction in Figure 13, we have to show that this restriction preserves the completeness. We have only to show that if $(\mid\!\sim^+)$ is applied twice on the same conditional $A \mid\!\sim B$, in the same branch, by using the same label x, then the second application is useless. Since all the rules are invertible (Theorem 6.4), we can assume, without loss of generality, that the two applications of $(\mid\!\sim^+)$ are consecutive. We conclude that the second application is useless, since each of the conclusions has already been obtained after the first application, and can be removed.

■

By Theorem 6.5 above and by the construction of the model done in the proof of Theorem 6.14 just above, we can show the following Corollary.

Corollary 6.15 (Finite model property) **R** *has the finite model property.*

6.2 Decision Procedure and Optimal Proof Search for R

In this section we define a systematic procedure which allows the satisfiability problem for **R** to be decided in nondeterministically polynomial time, in accordance with the known complexity results for this logic.

Let n be the size of the starting set Γ of which we want to verify the satisfiability. The number of applications of the rules is proportional to the number of labels introduced in the tableau. In turn, this is $O(2^n)$ due to the interplay between the rules $(\mid\!\sim^+)$ and $(\Box^-)$. Hence, the complexity of the calculus $\mathcal{TR}^{\mathbf{T}}$ is exponential in n.
In order to obtain a better complexity bound for validity in **R** we provide the following procedure. Intuitively, we do not apply $(\Box^-)$ to all negated boxed formulas, but only to formulas $y : \neg\Box\neg A$ not already expanded, i.e. such that $z : A, z : \Box\neg A$ do not belong to the current branch. As a result, we build a *small* model for the initial set of formulas in accordance with Theorem 2.2. This is made possible by the modularity of $<$ in **R**.

Let us define a nondeterministic procedure CHECK(Γ) to decide whether a given set of formulas Γ is satisfiable. Let EXPAND(Γ) be a procedure that returns one saturated expansion of Γ w.r.t. all static rules. In case of a branching rule, EXPAND nondeterministically selects (guesses) one conclusion of the rule.

CHECK(Γ)
1. $\Gamma \longleftarrow$ EXPAND(Γ);
2. **if** Γ contains an axiom **then return** UNSAT;
3. $\Gamma \longleftarrow$ result of applying $(\mid\!\sim^-)$ to each negated conditional in Γ;
4. $\Gamma \longleftarrow$ EXPAND(Γ);
5. **if** Γ contains an axiom **then return** UNSAT;

while Γ contains a $y : \neg\Box\neg A$ not marked as CONSIDERED **do**
  6. select $y : \neg\Box\neg A \in \Gamma$ not already marked as CONSIDERED;
    6a. **if** there is z in Γ such that $z : A \in \Gamma$ and $z : \Box\neg A \in \Gamma$
      then 6a'. add $z < y$ and $\Gamma^M_{y \rightarrow z}$ to Γ;
      else 6a". $\Gamma \longleftarrow$ result of applying $(\Box^-)$ to $y : \neg\Box\neg A$;

6b. mark $y : \neg\Box\neg A$ as CONSIDERED;
7. $\Gamma \longleftarrow$ EXPAND(Γ);
8. **if** Γ contains an axiom **then return** UNSAT;
endWhile
9. **return** SAT;

Observe that the addition of the set of formulas $z < y, \Gamma^M_{y \rightarrow z}$ in step 6a' could be omitted and it has been added mostly to enhance the understanding of the procedure. Indeed, the rule $(<)$, which is applied at each iteration to assure modularity, already takes care of adding such formulas. The procedure CHECK nondeterministically builds an open branch for Γ.

Theorem 6.16 (Soundness and completeness of the procedure) *The above procedure is sound and complete w.r.t. the semantics.*

Proof. (*Soundness*). We prove that if the initial set of formulas Γ is satisfiable, then the above procedure returns SAT. More precisely, we prove that each step of the procedure preserves the satisfiability of Γ. As far as EXPAND is concerned, notice that it only applies the static rules of $\mathcal{T}\mathbf{R}^{\mathbf{T}}$ and the soundness follows from the fact that these rules preserve satisfiability (see Theorem 6.5). Consider now step 6. Let $y : \neg\Box\neg A$ the formula selected in this step. If $(\Box^-)$ is applied to $y : \neg\Box\neg A$ (step 6a") we are done, since $(\Box^-)$ preserves satisfiability (see Theorem 6.5). If Γ already contains $z : A, z : \Box\neg A$, then step 6a' is executed, and the relation $z < y$ is added. In this case we reason as follows. Since Γ is satisfiable, we have that there is a model $\mathcal{M}$ and a mapping I such that (1) $\mathcal{M}, I(y) \models \neg\Box\neg A$ and (2) $\mathcal{M}, I(z) \models A$ and $\mathcal{M}, I(z) \models \Box\neg A$. We can observe that $I(z) < I(y)$ in $\mathcal{M}$. Indeed, by the truth condition of $\neg\Box\neg A$ and by the strong smoothness condition, we have that there exists w such that $w < I(y)$ and $\mathcal{M}, w \models A, \Box\neg A$. By modularity of $<$, either 1. $w < I(z)$ or 2. $I(z) < I(y)$. 1 is impossible, since otherwise we would have $\mathcal{M}, w \models \neg A$, which contradicts $\mathcal{M}, w \models A$. Hence, 2 holds. Therefore, we can conclude that step 6a' preserves satisfiability.
(*Completeness*). It can be easily shown that in case the procedure above returns SAT, then the branch built is saturated (see Definition 6.6). Therefore, we can build a canonical model for the initial Γ, as done in the proof of Theorem 6.14.

■

Theorem 6.17 (Complexity of the CHECK procedure) *By means of the procedure CHECK the satisfiability of a set of formulas of logic* $\mathbf{R}$ *can be decided in nondeterministic polynomial time.*

Proof. Observe that the procedure generates at most *O(n)* labels by applying the rule $(\mid\!\sim^-)$ (step 3) and that the while loop generates at most one new label for each $\neg\Box\neg A$ formula. Indeed, the rule $(\Box^-)$ is applied to a labelled formula $y : \neg\Box\neg A$ to generate a new world only if there is not a label z such that $z : A \in \Gamma$ and $z : \Box\neg A \in \Gamma$ are already on the branch. In essence, the procedure does not add a new minimal A-world on the branch if there is already one. As the number of different $\neg\Box\neg A$ formulas is at most $O(n)$, then the while loop can add at most $O(n)$ new labels on the branch. Moreover, for each different label x, the expansion step can add at most $O(n)$ formulas $x : \neg\Box\neg A$ on the branch, one for each positive conditional $A \mid\!\sim B$ occurring in the set Γ. We can therefore conclude that the while loop can be executed at most $O(n^2)$ times.

As the number of generated labels is at most $O(n)$, by the subformula property, the number of labelled formulas on the branch is at most $O(n^2)$. Hence, the execution of step 6a has complexity $O(n^2)$. The execution of the nondeterministic procedure `EXPAND` has complexity $O(n^2)$, including a guess of size $O(n^2)$, whereas to verify if Γ contains an axiom has complexity $O(n^4)$ (since it requires to check whether, for each labelled formula $x : P \in \Gamma$, the formula $x : \neg P$ is also in Γ, and Γ contains at most $O(n^2)$ labelled formulas). We can therefore conclude that the execution of the `CHECK` procedure requires at most $O(n^6)$ steps.

■

By Theorem 6.17, the validity problem for **R** is in **coNP**. **coNP**-hardness is immediate, since **R** includes classical propositional logic. Thus, we can conclude that:

Theorem 6.18 (Complexity of R) *The problem of deciding the validity for rational logic* **R** *is* **coNP***-complete.*

7 Conclusions

In this paper, we have presented tableau calculi for all of the KLM logical systems for default reasoning. Some preliminary results have been presented in [18] and [19]. We have given a tableau calculus for rational logic **R**, preferential logic **P**, loop-cumulative logic **CL**, and cumulative logic **C**. The calculi presented give a decision procedure for the respective logics. Moreover, for **R**, **P** and **CL** we have shown that we can obtain **coNP** decision procedures by refining the rules of the respective calculi. In case of **C**, we obtain a decision procedure by adding a suitable loop-checking mechanism. Our procedure gives an hyper exponential upper bound. Further investigation is needed to get a more efficient procedure. On the other hand, we are not aware of any tighter complexity bound for this logic.

All the calculi presented in this paper have been implemented by a theorem prover called KLMLean. KLMLean (not presented here) is a SICStus Prolog implementation of the tableau calculi introduced in this paper, and it is inspired to the "lean" methodology [4, 13, 5], whose basic idea is to write short programs and exploit the power of Prolog's engine as much as possible. To the best of our knowledge, KLMLean is the first theorem prover for KLM logics.

Artosi, Governatori, and Rotolo [2] develop a labelled tableau calculus for **C**. Their calculus is based on the interpretation of **C** as a conditional logic with a selection function semantics. As a major difference from our approach, their calculus makes use of labelled formulas, where the labels represent possible worlds or sets of possible worlds. World labels in turn are annotated by formulas to express minimality assumptions, e.g. they represent by a label w^A the fact that w is a minimal A-world, or in terms of the selection function, belongs to $f(A, u)$. They use then a sophisticated unification mechanism on the labels to match two annotated worlds, e.g. w^A, w^B; observe that by CSO (which is equivalent to CUT+CM), the equivalence of A and B might also be enforced by the conditionals contained in a tableau branch. Even if they do not discuss decidability and complexity issues, their tableau calculus should give a decision procedure for **C**.

In [20] and [21] it is defined a labelled tableau calculus for the logic **CE** and some of its extensions. The flat fragment of **CE** corresponds to the system **P**. The similarity between the two calculi lies in the fact that both approaches use a modal interpretation of conditionals. The major difference is that the calculus presented here does not use labels, whereas the one proposed in [20] does. A further difference is that in [20] the termination is obtained by means of a loop-checking machinery, and it is not clear if it matches complexity bounds and if it can be adapted in a simpler way to **CL** and to **C**.

Lehmann and Magidor [27] propose a non-deterministic algorithm that, given a finite set K of conditional assertions $C_i \mid\sim D_i$ and a conditional assertion $A \mid\sim B$, checks if $A \mid\sim B$ is not entailed by K in the logic **P**. This is an abstract algorithm useful for theoretical analysis, but practically unfeasible, as it requires to guess sets of indexes and propositional evaluations. They conclude that entailment in **P** is **coNP**, thus obtaining a complexity result similar to ours. However, it is not easy to compare their algorithm with our calculus, since the two approaches are radically different. As far as the complexity result is concerned, notice that our result is more general than theirs, since our language is richer: we consider boolean combinations of conditional assertions (and also combinations with propositional formulas), whereas they do not. As remarked by Boutilier [8], this more general result is not an obvious consequence of the more restricted one. Moreover, we prove the **coNP** result also for the system **CL**. At the best of our knowledge, this result was unknown up to now.

We plan to extend our calculi to first order case. The starting point will be the analysis of first order preferential and rational logics by Friedman, Halpern and Koller in [15].

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