A Sequent Calculus and a Theorem Prover for Standard Conditional Logics: Extended Version

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Abstract

In this paper we present a cut-free sequent calculus, called SeqS, for some standard conditional logics. The calculus uses labels and transition formulas and can be used to prove decidability and space complexity bounds for the respective logics. We also present CondLean, a theorem prover for these logics implementing SeqS calculi for weaker systems written in SICStus Prolog.

1 Introduction

Conditional logics have a long history. They have been studied first by Lewis ([35, 39, 6, 46]) in order to formalize a kind of hypothetical reasoning (if A were the case then B), that cannot be captured by classical logic with material implication.

In the last years, interesting applications of conditional logic to several domains of artificial intelligence such as knowledge representation, non-monotonic reasoning, belief revision, representation of counterfactual sentences, deductive databases have been proposed ([9]). For instance, in [28] knowledge and database update is formalized by some conditional logic. Conditional logics have also been used to modelize belief revision ([21, 36, 24, 25]). Conditional logics can provide an axiomatic foundation of non-monotonic reasoning ([31]), as it turns out that all forms of inference studied in the framework of non-monotonic (preferential) logics are particular cases of conditional axioms ([10]). Causal inference, which is very important for applications in action planning ([45]), has been modelled by conditional logics ([27]). Conditional Logics have been used to model hypothetical queries in deductive databases and logic programming;

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the conditional logic CK+ID is the basis of the logic programming language defined in [20]. In system diagnosis, conditional logics can be used to reason hypothetically about the expected functioning of system components with respect to the observed faults. [40] introduces a conditional logic, DL, suitable for diagnostic reasoning and which allows to represent and reason with assumptions in model-based diagnosis. Another interesting application of conditional logics is the formalization of *prototypical reasoning*, that is to say reasoning about typical properties and exceptions. Delgrande in [12] proposes a conditional logic for prototypical reasoning.

As we mentioned, conditional logics have been used to represent the so-called *counterfactual conditional sentences*, frequently used in artificial intelligence to let a reasoner to *learn from experiences that it has not done.* Counterfactuals are sentences like *p implies q* where the antecedent *p is false.* Intending *implies* with its classical meaning, all the counterfactuals are *true*; however, a reasoner could need to evaluate a counterfactual sentence, even as false. For this reason, conditional logics have been used to represent this kind of sentences, by using the conditional operator $\Rightarrow$ instead of the classical implication $\rightarrow$.
For a broader discussion about counterfactuals, see [7].

In knowledge representation, conditional logics are used to support the belief revision; in particular, Gardenfors introduced a relation between conditional logics and and the belief systems by the so-called *Ramsey test*: the conditional sentence *if A, then B* is acceptable in a *belief G* if and only if *B* is acceptable in *G revisited* with the information *A*. Formally:

$$B \in G * A$$
$$\text{if and only if}$$
$$A \Rightarrow B \in G$$

where $*$ is the belief operator of a knowledge base G given the information A. For details about the belief revision see [21], [36], [24] and [25].

Another interesting application of the conditional logics is the representation of the *prototypical reasoning*; for instance, in [12] a conditional system is described to formalize the default reasoning; in this system the following knowledge base is consistent, thanks to the usage of the conditional operator $\Rightarrow$:

$$(\forall x)(\mathit{Penguin(x)} \rightarrow \mathit{Bird(x)})$$
$$(\forall x)(\mathit{Penguin(x)} \rightarrow \neg \mathit{Fly(x)})$$
$$(\forall x)(\mathit{Bird(x)} \Rightarrow \mathit{Fly(x)})$$

Replacing $\Rightarrow$ with the classical implication $\rightarrow$, the above knowledge base is consistent only if there are no penguins.

Finally, an obvious application concerns natural language semantics where conditional logics are used in order to give a formal treatment of hypothetical and counterfactual sentences as presented in [39]. A broader discussion about counterfactuals can be found in [7].

In spite of their significance, very few proof systems have been proposed for conditional logics: we just mention [33, 14, 8, 1, 22, 11, 26]. One possible reason of the underdevelopment of proof-methods for conditional logics is the lack of a universally accepted semantics for them. This is in sharp contrast to modal and temporal logics which have a consolidated semantics based on a standard kind of Kripke structures.

Similarly to modal logics, the semantics of conditional logics can be defined in terms of possible world structures. In this respect, conditional logics can be seen as a generalization of modal logics (or a type of multi-modal logic) where the conditional operator is a sort of modality indexed by a formula of the same language.
The two most popular semantics for conditional logics are the so-called *sphere semantics* ([35]) and the *selection function semantics* ([39]). Both are possible-world semantics, but are based on different (though related) algebraic notions. Here we adopt the selection function semantics, which is more general than the sphere semantics.

Since we adopt the selection function semantics, **CK** is the fundamental system; it has the same role as the system K (from which it derives its name) in modal logic: CK-valid formulas are formulas that are valid in every selection function model.

In this work we present a sequent calculus for CK and for some standard extensions of it, namely CK+ID, CK+MP[1], CK+CS, CK+CEM and other combinations of these systems. This calculus makes use of labels, following the line of [47] and [19]. To the best of our knowledge, this is the first calculus for these systems. Some tableaux calculi were developed in [26] and in [42] for other more specific conditional systems.

Our goal is to obtain a decision procedure for the logics under consideration. For this reason, we undertake a proof theoretical analysis of our calculi. In order to get a terminating calculus, it is crucial to show that all the rules of the calculi are *analytic* and that rules having premises with a higher complexity than the conclusion can be applied in a controlled way. In this way, the calculus not only provides a decision procedure, but it can also be used to establish a complexity bound for these logics (the decidability for these logics has been shown in [39]). Roughly speaking, if the rules are analytic, the length of each branch is bounded essentially by the length of the initial sequent; therefore, we can easily obtain that the calculus give a polynomial space complexity.

In this work we also present a more detailed analysis of systems CK{+ID}{+MP}, which are characterized by other remarkable properties, such as the so-called *disjunction property* for conditional formulas: if $(A_1 \Rightarrow B_1) \vee (A_2 \Rightarrow B_2)$ is valid, then either $(A_1 \Rightarrow B_1)$ or $(A_2 \Rightarrow B_2)$ is valid too.

As a difference with modal logics, for which there are lots of efficient implementations ([3], [16], [2]), to the best of our knowledge very few theorem provers have been implemented for conditional logics ([33] and [1]). We present here a simple implementation of our sequent calculi, called **CondLean**; it is a Prolog program which follows the *lean* methodology ([3], [16]), in which every clause of a predicate `prove` implements an axiom or rule of the calculus and the proof

[1] This conditional system is related to modal logic T.

search is provided for free by the mere depth-first search mechanism of Prolog, without any ad hoc mechanism. We also present an alternative version of our theorem prover inspired by the tableau calculi for modal logics introduced in [2].

The plan of the paper is as follows: in section 2 we introduce the conditional systems we consider, in section 3 we present the sequent calculi for conditional systems above. In section 4 we analyze the calculi in order to obtain a decision procedure for the basic conditional system, CK, and for the mentioned extensions of it. In section 5 we give a more detailed analysis of CK and the extensions MP and ID. In section 6 we present the theorem prover CondLean. In section 7 we discuss about the possibility of extending our work to other conditional systems. In section 8 we discuss some related work.

2 Conditional Logics

Conditional logics are extensions of classical logic obtained by adding the conditional operator $\Rightarrow$. In this paper, we only consider propositional conditional logics.

A propositional conditional language $\mathcal{L}$ contains the following items:

- a set of propositional variables ATM;
- the symbol of *false* $\bot$;
- a set of connectives[2] $\rightarrow$, $\Rightarrow$.

We define formulas of $\mathcal{L}$ as follows:

- $\bot$ and the propositional variables of ATM are *atomic formulas*;
- if A and B are formulas, $A \rightarrow B$ and $A \Rightarrow B$ are *complex formulas.*

We adopt the *selection function semantics.* We consider a non-empty set of possible worlds $\mathcal{W}$. Intuitively, the selection function f selects, for a world w and a formula A, the set of worlds of $\mathcal{W}$ which are *closer* to w given the information A. A conditional formula $A \Rightarrow B$ holds in a world w if the formula B holds in *all the worlds selected by f for w and A.*

A model is a triple:

$$\mathcal{M} = \langle\, \mathcal{W},\, f,\, [\,]\, \rangle$$

where:

- $\mathcal{W}$ is a non empty set of items called *worlds*;
- f is the so-called *selection function* and has the following type:

$$f\colon \mathcal{W} \times 2^{\mathcal{W}} \longrightarrow 2^{\mathcal{W}}$$

[2] The usual connectives $\top$, $\wedge$, $\vee$ and $\neg$ can be defined in terms of $\bot$ and $\rightarrow$.

- [ ] is the *evaluation function*, which assigns to an atom $P \in ATM$ the set of worlds where P is true, and is extended to the other formulas as follows:

    * $[\bot] = \emptyset$
    * $[A \rightarrow B]$=($\mathcal{W}$ - $[A]$) $\cup$ $[B]$
    * $[A \Rightarrow B]$=$\{w \in \mathcal{W} \mid f(w, [A]) \subseteq [B]\}$

Observe that we have defined f taking $[A]$ rather than A (i.e. $f(w,[A])$ rather than $f(w,A)$) as an argument; this is equivalent to define f on formulas, i.e. $f(w,A)$ but imposing that if $[A]$=$[A']$ in the model, then $f(w, A)$=$f(w, A')$. This condition is called *normality*.

The semantics above characterizes the *basic conditional system*, called **CK**. An axiomatization of the CK system is given by:

- all tautologies of classical propositional logic.

- (Modus Ponens) $$\frac{A \quad A \rightarrow B}{B}$$

- (RCEA) $$\frac{A \leftrightarrow B}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow C)}$$

- (RCK) $$\frac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)}$$

Other conditional systems are obtained by assuming further properties on the selection function; we consider the following standard extensions of the basic system CK:

System	Axioms	Model condition
ID	$A \Rightarrow A$	$f(w, [A]) \subseteq [A]$
MP	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$	$w \in [A] \rightarrow w \in f(w, [A])$
CS	$(A \wedge B) \rightarrow (A \Rightarrow B)$	$w \in [A] \rightarrow f(w, [A]) \subseteq \{w\}$
CEM	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$	$\mid f(w, [A]) \mid \leq 1$

3 A Sequent Calculus for Conditional Logics

In this section we present **SeqS**, a sequent calculs for the conditional systems introduced above. S stands for {CK, ID, MP, CS, CEM, ID+MP, CS+ID, CEM+ID, CS+MP, CEM+MP, CEM+CS, CS+ID+MP, CEM+ID+MP, CEM+CS+ID}; notice that we have not to take care of combinations SeqCEM+CS{+ID}+MP, since condition (CS) is derivable in systems characterized by conditions (CEM) and (MP). Indeed, for (CEM) we have that $(*)$ $\mid f(w, [A]) \mid \leq 1$; if $w \in [A]$, then we have that $w \in f(w, [A])$ for (MP), but for $(*)$ we have that $f(w, [A]) = \{w\}$, which satisfies the (CS) condition. Therefore, SeqCEM+CS{+ID}+MP correspond to SeqCEM{+ID}+MP.

The calculi make use of labels to represent possible worlds.
We consider a conditional language $\mathcal{L}$ and a denumerable alphabet of labels $\mathcal{A}$, whose elements are denoted by $x, y, z, \ldots$.
There are two kinds of labelled formulas:

1. *world formulas*, denoted by x: A, where $x \in \mathcal{A}$ and $A \in \mathcal{L}$, used to represent that A holds in a world x;
2. *transition formulas*, denoted by $x \xrightarrow{A} y$, where $x, y \in \mathcal{A}$ and $A \in \mathcal{L}$. A transition formula $x \xrightarrow{A} y$ represents that $y \in f(x, [A])$.

A **sequent** is a pair $\langle \Gamma, \Delta \rangle$, usually denoted with $\Gamma \vdash \Delta$, where Γ and Δ are multisets of labelled formulas. The intuitive meaning of $\Gamma \vdash \Delta$ is: every model that satisfies all labelled formulas of Γ in the respective worlds (specified by the labels) satisfies at least one of the labelled formulas of Δ (in those worlds). This is made precise by the notion of *validity* of a sequent given in the next definition:

Definition 3.1 (Sequent validity): Given a model

$$\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$$

for $\mathcal{L}$, and a label alphabet $\mathcal{A}$, we consider any *mapping*

$$I : \mathcal{A} \to \mathcal{W}$$

Let F be a labelled formula, we define $\mathcal{M} \models_I F$ as follows:

- $\mathcal{M} \models_I x$: A *iff* $I(x) \in [A]$
- $\mathcal{M} \models_I x \xrightarrow{A} y$ *iff* $I(y) \in f(I(x), [A])$

We say that $\Gamma \vdash \Delta$ is *valid* in $\mathcal{M}$ if for every mapping $I : \mathcal{A} \to \mathcal{W}$, if $\mathcal{M} \models_I F$ for every $F \in \Gamma$, then $\mathcal{M} \models_I G$ for some $G \in \Delta$. We say that $\Gamma \vdash \Delta$ is valid in a system (CK or one of its extensions) if it is valid in every $\mathcal{M}$ satisfying the specific conditions for that system (if any).

In Figures 1 and 2 we present the calculi for CK and its mentioned extensions.

Example 3.2: We show a derivation of the (ID) axiom.

$$\dfrac{\dfrac{y : A \vdash y : A}{x \xrightarrow{A} y \vdash y : A}\,(ID)}{\vdash x : A \Rightarrow A}\,(\Rightarrow R)$$

$$(\mathbf{AX}) \quad \Gamma, x : P \vdash \Delta, x : P \quad (P \in ATM) \qquad\qquad (\mathbf{A}\bot) \quad \Gamma, x : \bot \vdash \Delta$$

$$(\rightarrow \mathbf{L}) \quad \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} \qquad\qquad (\rightarrow \mathbf{R}) \quad \frac{\Gamma, x : A \vdash x : B, \Delta}{\Gamma \vdash x : A \rightarrow B, \Delta}$$

$$(\Rightarrow \mathbf{L}) \quad \frac{\begin{array}{c}\Gamma, x : A \Rightarrow B \vdash x \xrightarrow{A} y, \Delta \\ \Gamma, x : A \Rightarrow B, y : B \vdash \Delta\end{array}}{\Gamma, x : A \Rightarrow B \vdash \Delta} \qquad\qquad (\Rightarrow \mathbf{R}) \quad \frac{\Gamma, x \xrightarrow{A} y \vdash y : B, \Delta}{\Gamma \vdash x : A \Rightarrow B, \Delta} \ (y \notin \Gamma, \Delta)$$

$$(\mathbf{EQ}) \quad \frac{u : A \vdash u : B \qquad u : B \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{B} y, \Delta}$$

Fig. 1: Sequent calculi SeqCK.

$$(\mathbf{ID}) \quad \frac{\Gamma, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta}$$

$$(\mathbf{MP}) \quad \frac{\Gamma \vdash x : A, \Delta}{\Gamma \vdash x \xrightarrow{A} x, \Delta}$$

$$(\mathbf{CS}) \quad \frac{\Gamma \vdash \Delta, x : A \qquad \Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \ (x \neq y, u \notin \Gamma, \Delta)$$

$$(\mathbf{CEM}) \quad \frac{\Gamma \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash \Delta[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \ (y \neq z, u \notin \Gamma, \Delta)$$

Fig. 2: SeqS's rules for systems extending CK.

$$(\wedge\mathbf{L})\ \frac{\Gamma, x : A, x : B \vdash \Delta}{\Gamma, x : A \wedge B \vdash \Delta} \qquad (\wedge\mathbf{R})\ \frac{\Gamma \vdash \Delta, x : A \qquad \Gamma \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : A \wedge B}$$

$$(\vee\mathbf{L})\ \frac{\Gamma, x : A \vdash \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \vee B \vdash \Delta} \qquad (\vee\mathbf{R})\ \frac{\Gamma \vdash \Delta, x : A, x : B}{\Gamma \vdash \Delta, x : A \vee B}$$

$$(\neg\mathbf{L})\ \frac{\Gamma \vdash \Delta, x : A}{\Gamma, x : \neg A \vdash \Delta} \qquad (\neg\mathbf{R})\ \frac{\Gamma, x : A \vdash \Delta}{\Gamma \vdash \Delta, x : \neg A}$$

$$(\mathbf{A}\top)\ \Gamma \vdash \Delta, x : \top$$

Fig. 3: Additional axioms and rules in SeqS for the other boolean operators, derived from the rules in Figure 1 by the usual equivalences.

Example 3.3: We show a derivation of the (MP) axiom.

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A \vdash x : A, x : B}{x : A \vdash x \xrightarrow{A} x, x : B}\,(MP) \qquad x : A, x : B \vdash x : B}{x : A \Rightarrow B, x : A \vdash x : B}\,(\Rightarrow L)}{x : A \Rightarrow B \vdash x : A \rightarrow B}\,(\rightarrow R)}{\vdash x : (A \Rightarrow B) \rightarrow (A \rightarrow B)}\,(\rightarrow R)$$

Example 3.4: We show a derivation of the (CS) axiom.

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A, x : B \vdash y : B, x : A \qquad u : A, u : B, u \xrightarrow{A} u \vdash u : B}{x : A, x : B, x \xrightarrow{A} y \vdash y : B}\,(CS)}{x : A \wedge B, x \xrightarrow{A} y \vdash y : B}\,(\wedge L)}{x : A \wedge B \vdash x : A \Rightarrow B}\,(\Rightarrow R)}{\vdash x : A \wedge B \rightarrow x : A \Rightarrow B}\,(\rightarrow R)$$

Example 3.5: We show a derivation of the (CEM) axiom.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \xrightarrow{A} y, z : B \vdash y : B, x \xrightarrow{A} y \qquad x \xrightarrow{A} u, x \xrightarrow{A} u, u : B \vdash u : B}{x \xrightarrow{A} y, x \xrightarrow{A} z, z : B \vdash y : B}\,(CEM)}{x \xrightarrow{A} y, x \xrightarrow{A} z \vdash y : B, z : \neg B}\,(\neg R)}{x \xrightarrow{A} y \vdash y : B, x : A \Rightarrow \neg B}\,(\Rightarrow R)}{\vdash x : A \Rightarrow B, x : A \Rightarrow \neg B}\,(\Rightarrow R)}{\vdash x : (A \Rightarrow B) \vee (A \Rightarrow \neg B)}\,(\vee R)
$$

3.1 Basic Structural Properties of SeqS

In order to prove that the sequent calculus SeqS is sound and complete with respect to the semantics, we introduce some remarkable structural properties holding in these calculi. In particular:

- label substitution is height-preserving admissible: if $\Gamma \vdash \Delta$ is derivable with a proof of height h, then the sequent obtained by replacing a label x with a label y wherever it occurs in $\Gamma \vdash \Delta$ is derivable with a proof of height $\leq h$;
- weakening is height-preserving admissible: if we have a proof of $\Gamma \vdash \Delta$, then we can find a proof of no greater height of $\Gamma \vdash \Delta, F$ $(\Gamma, F \vdash \Delta)$, where F is any labelled formula;
- the rules ($\rightarrow$ L), ($\rightarrow$ R), ($\Rightarrow$ L) and ($\Rightarrow$ R) are height-preserving invertible: roughly speaking, if $\Gamma \vdash \Delta$ is derivable, and it could be the conclusion of an application of one of these four rules, then any premise of that application is also derivable in our calculi, with a proof of no greater height. As an example, if $\Gamma \vdash \Delta^{'}, x : A \rightarrow B$ is derivable with a proof of height h, then $\Gamma, x : A \vdash \Delta^{'}, x : B$ is derivable with a proof of height $\leq h$;
- contraction is admissible: if $\Gamma \vdash \Delta, F, F$ is derivable, then there is a proof of $\Gamma \vdash \Delta, F$.

We discuss these properties in detail.

Lemma 3.6 (Height-preserving label substitution): If a sequent $\Gamma \vdash \Delta$ has a derivation of height h, then $\Gamma[x/y] \vdash \Delta[x/y]$ has a derivation of height $\leq h$, where $\Gamma[x/y] \vdash \Delta[x/y]$ is the sequent obtained from $\Gamma \vdash \Delta$ by replacing a label x by a label y wherever it occurs.

Proof. By a straightforward induction on the height of a derivation. We show the most interesting cases, the other cases are easy and left to the reader. We start with the case when (CS) is applied to $\Gamma \vdash \Delta$ with a derivation of height h of the form:

$$\frac{(1)\Gamma^{'} \vdash y : A, \Delta \qquad (2)\Gamma^{'}[y/u, x/u], u \xrightarrow{A} u \vdash \Delta[y/u, x/u]}{\Gamma^{'}, y \xrightarrow{A} x \vdash \Delta} \; (CS)$$

Our goal is to find a derivation of height $\leq h$ of $\Gamma^{'}[x/y], y \xrightarrow{A} y \vdash \Delta[x/y]$. Applying the inductive hypothesis to (2), we have a proof of height $\leq h-1$ of the sequent $(3)\Gamma[u/y], y \xrightarrow{A} y \vdash \Delta[u/y]$, but (3) is $\Gamma[x/y], y \xrightarrow{A} y \vdash \Delta[x/y]$, since labels x and y have both been replaced by a *new* label u in (2), and the proof is over.

The other interesting case is when $(\Rightarrow$ R) is the rule ending the derivation (i.e. the rule applied to $\Gamma \vdash \Delta$); the situation is as follows:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta^{'}, y : B}{\Gamma \vdash \Delta^{'}, x : A \Rightarrow B} \; (\Rightarrow R)$$

We want to prove the existence of a proof of $\Gamma[x/y] \vdash \Delta^{'}[x/y], y : A \Rightarrow B$, of height less or equal than h. First, observe that the label y in the above derivation is a *new* label, not occurring in the conclusion of $(\Rightarrow$ R); therefore, we can rename it with another new label, for instance w:

$$\frac{\Gamma, x \xrightarrow{A} w \vdash \Delta^{'}, w : B}{\Gamma \vdash \Delta^{'}, x : A \Rightarrow B} \; (\Rightarrow R)$$

Applying the inductive hypothesis on the premise of $(\Rightarrow$ R), we obtain a proof of $\Gamma[x/y], y \xrightarrow{A} w \vdash \Delta^{'}[x/y], w : B$ of height no greater than $h-1$. We conclude by an application of $(\Rightarrow$ R), obtaining a proof (height $\leq h$) of $\Gamma[x/y] \vdash \Delta^{'}[x/y], y : A \Rightarrow B$.

□

Theorem 3.7 (Height-preserving admissibility of weakening): If a sequent $\Gamma \vdash \Delta$ has a derivation of height h, then $\Gamma \vdash \Delta, F$ and $\Gamma, F \vdash \Delta$ have a derivation of height $\leq h$.

Proof. By induction on the height of a derivation of $\Gamma \vdash \Delta$. The proof is easy and left to the reader.

□

Theorem 3.8 (Height-preserving invertibility of $(\rightarrow$ R), $(\rightarrow$ L), $(\Rightarrow$ R) and $(\Rightarrow$ L)): The rules $(\rightarrow$ R), $(\rightarrow$ L) , $(\Rightarrow$ R) and $(\Rightarrow$ L) are height-preserving invertible, i.e. if $\Gamma_2 \vdash \Delta_2$ has a derivation of height h and $\Gamma_2 \vdash \Delta_2$ could be the conclusion of an application of one of the above rules to $\Gamma_1 \vdash \Delta_1$, then there is a proof of $\Gamma_1 \vdash \Delta_1$ of height less or equal than h.

Proof. We consider each of the rules and proceed by an inductive argument on the height of a proof of $\Gamma_2 \vdash \Delta_2$. We only present the cases of $(\Rightarrow R)$ and $(\Rightarrow L)$; the cases of $(\rightarrow L)$ and $(\rightarrow R)$ are easy and left to the reader.

- $(\Rightarrow$ R): for any y, if $\Gamma \vdash \Delta, x : A \Rightarrow B$ is an axiom, $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$ is an axiom too, since axioms are restricted to atomic formulas. If $h > 0$ and the proof of $\Gamma \vdash \Delta, x : A \Rightarrow B$ is concluded (looking forward) by any rule other than $(\Rightarrow$ R), we apply the inductive hypothesis to the premise(s), then we conclude by applying the same rule. If the proof of $\Gamma \vdash \Delta, x : A \Rightarrow B$ is ended by $(\Rightarrow$ R) we have the following subcases:
  - ⋆ $x : A \Rightarrow B$ is the principal formula of $(\Rightarrow$ R): the proof is ended as follows:
  $$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B}\ (\Rightarrow R)$$
  We have a proof of $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$ of height $h - 1$ and the proof is over;
  - ⋆ $x : A \Rightarrow B$ is not the principal formula of $(\Rightarrow$ R): the proof is ended as follows:
  $$\frac{\Gamma, w \xrightarrow{C} z \vdash \Delta, z : D, x : A \Rightarrow B}{\Gamma \vdash \Delta, x : A \Rightarrow B, w : C \Rightarrow D}\ (\Rightarrow R)$$
  where z is a "new" label and then, without loss of generality, we can assume that z is not y, since we can apply the height-preserving label substitution. By inductive hypothesis on the premise we obtain a derivation with the same height ending with
  $$\frac{\Gamma, w \xrightarrow{C} z, x \xrightarrow{A} y \vdash \Delta, z : D, y : B}{\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B, w : C \Rightarrow D}\ (\Rightarrow R)$$
- $(\Rightarrow$ L): this rule is height-preserving invertible, since its premises are obtained by weakening from the conclusion, and weakening is height-preserving admissible. More in detail, given a proof (height h) of $\Gamma, x : A \Rightarrow B \vdash \Delta$, for the height-preserving admissibility of weakening we have proofs of height $\leq h$ of $\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y$ and $\Gamma, x : A \Rightarrow B, y : B \vdash \Delta$.

□

Now we prove the admissibility of contraction. Before stating this property, we need to mention some problems occurring with systems allowing (CEM), (CS) and (ID). Consider the case of SeqCEM; we can have the following proof (consider $\Sigma(u) = \Sigma[y/u, z/u]$ and $\Sigma(v) = \Sigma[y/v, w/v]$):

$$(\mathbf{ContrL}) \quad \frac{\Gamma, F, F \vdash \Delta}{\Gamma, F \vdash \Delta} \qquad\qquad (\mathbf{ContrR}) \quad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F}$$

Fig. 4: Contraction rules for SeqCEM{+CS}{+ID}{+MP} systems.

$$\frac{\Gamma, y : A \vdash \Delta, x : A \qquad \Gamma[x/u, y/u], u \xrightarrow{A} u, u : A \vdash \Delta[x/u, y/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} (\mathbf{CS}+\mathbf{ID}), x \neq y, u \notin \Gamma, \Delta$$

Fig. 5: Reformulated rule for Seq{CEM+}CS{+MP}+ID systems.

$$\cfrac{\cfrac{\Gamma \vdash \Delta, x \xrightarrow{A} z, x \xrightarrow{A} w \qquad \Gamma(v), x \xrightarrow{A} v \vdash \Delta(v), x \xrightarrow{A} w}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z} (CEM) \qquad \Gamma(u), x \xrightarrow{A} u, x \xrightarrow{A} u \vdash \Delta(u)}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta} (CEM)$$

The proof of the admissibility of contraction proceeds by induction on the height of the derivation of the sequent containing two instances of the same formula, in order to obtain a proof of the same sequent in which one of these instances has been discarded, eventually by applying the inductive hypothesis on a sequent obtained by a shorter derivation. In the above example, if $w \neq z$ we cannot obtain a proof of $\Gamma, x \xrightarrow{A} y \vdash \Delta$.
In systems SeqCEM{+CS}{+ID}{+MP} } we introduce two *explicit contraction rules*, presented in Figure 4.

We are also in trouble considering systems containing both (CS) and (ID) rules, since both these rules consider a transition formula in the left-hand side of a sequent as a principal formula; for instance, consider a proof in SeqCS+ID concluded as follows:

$$\cfrac{\cfrac{\Gamma, y : A \vdash \Delta, x : A \qquad \Gamma[x/u, y/u], u \xrightarrow{A} u, u : A \vdash \Delta[x/u, y/u]}{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta} (CS)}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta} (ID)$$

By an inductive argument on the height of the derivation we can conclude nothing, since we cannot apply the inductive hypothesis.

In systems Seq{CEM+}CS{+MP}+ID we reformulate the rules by *replacing rules* (ID) *and* (CS) *with the unified rule* presented in Figure 5.

Summarizing, we consider systems reformulated as follows:

Now we can state the admissibility of contractions by the following:

Tab. 1: Reformulations needed to prove the admissibility of contraction.

System	**Contractions**	**Replace R_1 and R_2 with R_3 ($R_1, R_2/R_3$)**
SeqCS+ID{+MP}	no	(CS),(ID)/(CS+ID)
SeqCEM{+ID}{+MP}	yes	nothing
SeqCEM+CS	yes	nothing
SeqCEM+CS+ID	yes	(CS),(ID)/(CS+ID)
SeqCEM{+ID}+MP	yes	nothing

Theorem 3.9 (Admissibility of contraction): The rules of contraction are admissible in SeqS, i.e. if a sequent $\Gamma \vdash \Delta, F, F$ is derivable in SeqS, then there is a proof of $\Gamma \vdash \Delta, F$, and if a sequent $\Gamma, F, F \vdash \Delta$ is derivable in SeqS, then there is a proof of $\Gamma, F \vdash \Delta$. Moreover, the proof of the contracted sequent does not add any rule application to the initial proof, except for explicit contractions in systems allowing (CEM).

Proof. By simultaneous induction on the height of derivation for left and right contraction. If $h = 0$, i.e. $\Gamma \vdash \Delta, F, F$ is an axiom, then we have to consider the following subcases:

- $w : \bot \in \Gamma$: in this case, obviously $\Gamma \vdash \Delta, F$ is an axiom too;
- $G \in \Gamma \cap \Delta$: we conclude, since $\Gamma \vdash \Delta, F$ is an axiom too;
- $F \in \Gamma$: the proof is over, observing that $\Gamma \vdash \Delta, F$ is an axiom too.

The proof of the case in which $\Gamma, F, F \vdash \Delta$ is an axiom is symmetric.

If $h > 0$, consider the last rule applied (looking forward) to derive the premise of contraction. We distinguish two cases:

- the contracted formula F is not principal in it: in this case, both occurrences of F are in the premise(s) of the rule, which have a smaller derivation height. By the inductive hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premise(s). As an example, consider a proof ended by an application of (CS) as follows:

$$\frac{\Gamma', F, F \vdash \Delta, x : A \qquad \Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u], F[x/u, y/u] \vdash \Delta[x/u, y/u]}{\Gamma', x \xrightarrow{A} y, F, F \vdash \Delta}\ (CS)$$

By the inductive hypothesis, we have a proof of the sequents $\Gamma', F \vdash \Delta, x : A$ and $\Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u] \vdash \Delta[x/u, y/u]$, from which we conclude as follows:

$$\frac{\Gamma', F \vdash \Delta, x : A \qquad \Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u] \vdash \Delta[x/u, y/u]}{\Gamma', x \xrightarrow{A} y, F \vdash \Delta}\ (CS)$$

- the contracted formula F is principal in it: we consider all the rules:

  ⋆ (→ R): we have a proof ending with:

$$\frac{\Gamma, x : A \vdash \Delta, x : B, x : A \rightarrow B}{\Gamma \vdash \Delta, x : A \rightarrow B, x : A \rightarrow B} (\rightarrow R)$$

  Applying the height-preserving invertibility of (→ R) (see Theorem 3.8 above), we obtain a proof of $\Gamma, x : A, x : A \vdash \Delta, x : B, x : B$. Applying the inductive hypothesis, we have a proof of $\Gamma, x : A \vdash \Delta, x : B$, from which we conclude applying (→ R) as follows:

$$\frac{\Gamma, x : A \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : A \rightarrow B} (\rightarrow R)$$

  ⋆ (→ L): the proof is ended as follows:

$$\frac{(1)\Gamma, x : A \rightarrow B \vdash \Delta, x : A \qquad (2)\Gamma, x : A \rightarrow B, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B, x : A \rightarrow B \vdash \Delta} (\rightarrow L)$$

  As in the previous case, we obtain proofs of the following sequents, by the application of height-preserving invertibility of (→ L):

  - $(1a)\Gamma \vdash \Delta, x : A, x : A$
  - $(1b)\Gamma, x : B \vdash \Delta, x : A$
  - $(2a)\Gamma, x : B \vdash \Delta, x : A$
  - $(2b)\Gamma, x : B, x : B \vdash \Delta$

  Applying the inductive hypothesis on $(1a)$ and $(2b)$ and applying (→ L) to the contracted sequents, we obtain a proof ending with (be $(1a^{'})$ and $(2b^{'})$ the contracted sequents):

$$\frac{(1a^{'})\Gamma \vdash \Delta, x : A \qquad (2b^{'})\Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} (\rightarrow L)$$

  ⋆ (⇒ L): we have a proof ending with:

$$\frac{\Gamma, x : A \Rightarrow B, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

  Applying the inductive hypothesis to both premises we can immediately conclude as follows:

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

⋆ (⇒ R): the proof is ended by:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \Rightarrow B, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B, x : A \Rightarrow B} (\Rightarrow R)$$

Applying the height-preserving invertibility (see Theorem 3.8) of (⇒ R), we have a proof of $(1)\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} z \vdash \Delta, y : B, z : B$. Applying the height-preserving label substitution (Lemma 3.6) to (1), replacing z with y, we obtain a proof of the sequent $(2)\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, y : B, y : B$, since y and z are new labels introduced (considering a backward proof search) by the applications of (⇒ R). We can then apply the inductive hypothesis on (2), obtaining a proof of $(3)\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$, from which we conclude by an application of (⇒ R):

$$\frac{(3)\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B} (\Rightarrow R)$$

⋆ (EQ): the proof is ended as follows:

$$\frac{u : A \vdash u : A^{'} \qquad u : A^{'} \vdash u : A}{\Gamma^{'}, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A^{'}} y} (EQ)$$

It is easy to observe that (EQ) does not require the second occurrence of $x \xrightarrow{A} y$; thus we obtain the following proof:

$$\frac{u : A \vdash u : A^{'} \qquad u : A^{'} \vdash u : A}{\Gamma^{'}, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A^{'}} y} (EQ)$$

⋆ (MP): the proof is ended as follows:

$$\frac{\Gamma \vdash \Delta, x \xrightarrow{A} x, x : A}{\Gamma \vdash \Delta, x \xrightarrow{A} x, x \xrightarrow{A} x} (MP)$$

The transition $x \xrightarrow{A} x$ occurring in the premise can be derived (looking forward) by an application of (EQ) or by (MP), since no other rules have a transition on the right-hand side of a sequent as a principal formula. If $x \xrightarrow{A} x$ derives from (EQ), then $x : A$ is removed by (EQ), since (EQ) is applied (looking backward) only on the transition formulas. Therefore, we can remove an occurrence of $x \xrightarrow{A} x$ from the conclusion, obtaining a proof of $\Gamma \vdash \Delta, x \xrightarrow{A} x$.
If $x \xrightarrow{A} x$ derives from (MP), then we can permute its application

over the other rules in the proof of $\Gamma \vdash \Delta, x : A, x \xrightarrow{A} x$, obtaining a proof of $\Gamma \vdash \Delta, x : A, x : A$, to which we apply the inductive hypothesis obtaining a proof of of $\Gamma \vdash \Delta, x : A$, and we can conclude by an application of (MP);

- ⋆ (CEM): given a proof of $\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta$, derived (looking forward) by (CEM), we obtain a proof of $\Gamma, x \xrightarrow{A} y \vdash \Delta$ by an application of (Contr L);
- ⋆ (ID): if we consider systems SeqID{+MP}, then we can conclude as in the case of (MP), since both the transitions in $\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta$ can be derived (forward) by (EQ) or by (ID), the only rules taking care of transitions in the left-hand side of a sequent; if both transitions have been derived by two applications of (ID), then we can permute these applications, obtaining a proof of $\Gamma, y : A, y : A \vdash \Delta$, to which apply the inductive hypothesis obtaining a proof of $\Gamma, y : A \vdash \Delta$, from which we can conclude by an application of (ID).
If we consider SeqCS+ID{+MP}, then rules (ID) and (CS) have been replaced by the unified rule (CS+ID); if both occurrences of $x \xrightarrow{A} y$ derives from applications of this rule, then we can permute them and the proof is ended as follows (we denote $\Sigma[x/u, y/u]$ with $\Sigma(u)$):

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma, y : A, y : A \vdash \Delta, x : A, x : A \quad \ldots\end{matrix}}{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta, x : A}\,(CS+ID) \qquad \begin{matrix}\Pi_2\\ \Gamma(u), u \xrightarrow{A} u, u \xrightarrow{A} u, u : A \vdash \Delta(u)\end{matrix}}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\,(CS+ID)$$

We can apply the inductive hypothesis on Π_1, obtaining a proof $\Pi_1^{'}$ of $\Gamma, y : A \vdash \Delta, x : A$, and on Π_2, obtaining a proof $\Pi_2^{'}$ of $\Gamma(u), u \xrightarrow{A} u, u : A \vdash \Delta(u)$, from which we can conclude by an application of (CS+ID).
If we consider SeqCEM+ID{+MP} we can immediately conclude that if $(1)\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta$ is derivable, then $\Gamma, x \xrightarrow{A} y \vdash \Delta$ is derivable too, since we can apply (Contr L) to (1);
- ⋆ (CS): in this case one can conclude as in the previous case of (ID).

It s easy to observe that, in each case, the proof is concluded by applying the same rule under consideration, with the exception of the explicit contractions; therefore, if $\Gamma, F, F \vdash \Delta$ is derivable, then we can find a proof of $\Gamma, F \vdash \Delta$ which does not add any application of SeqS rules to the initial proof. The same for $\Gamma \vdash \Delta, F, F$.

□

SeqS calculi have another remarkable property, the admissibility of the cut rule. By cut we mean the following rule:

$$\dfrac{\Gamma \vdash \Delta, F \quad F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\,(cut)$$

where F is any labelled formula. more in detail, cut is admissible in all SeqS systems, except in systems containing both (CEM) and (MP), i.e. seqCEM{+ID}+MP: for these system, we cannot give a counterexample in order to prove that cut is *not* admissible; however, we are not able to conclude the proof (as explained below), therefore we maintain the problem of giving a cut-free sequent calculus for CEM+MP systems as an open issue.

To prove the admissibility of cut we need to define the complexity of a labelled formula:

Definition 3.10 (Complexity of a labelled formula cp(F)): We define the complexity of a labelled formula F as follows:

1. cp $(x : A) = 2 * \mid A \mid$
2. cp $(x \xrightarrow{A} y) = 2 * \mid A \mid + 1$

where $\mid A \mid$ is the number of symbols occurring in the string representing the formula A.

Theorem 3.11 (Admissibility of cut): If $\Gamma \vdash \Delta, F$ and $F, \Gamma \vdash \Delta$ are derivable, so $\Gamma \vdash \Delta$.

Proof. As usual, the proof proceeds by a double induction over the complexity of the cut formula and the sum of the heights of the derivations of the two premises of the cut inference, in the sense that we replace one cut by one or several cuts on formulas of smaller complexity, or on sequents derived by shorter derivations. We have several cases: (*i*) one of the two premises is an axiom, (*ii*) the last step of *one* of the two premises is obtained by a rule in which F is *not* the principal formula, (*iii*) F is the principal formula in the last step of *both* derivations.

(i) If one of the two premises is an axiom then either $\Gamma \vdash \Delta$ is an axiom, or the premise which is not an axiom contains two copies of F and $\Gamma \vdash \Delta$ can be obtained by the admissibility of contraction (see Theorem 3.9 above).

(ii) We distinguish two cases:

1. the sequent where F is not principal is derived by any rule (R), except the (EQ) rule. This case is standard, we can permute (R) over the cut: i.e. we cut the premise(s) of (R) and then we apply (R) to the result of cut. As an example, consider the case when $F = x \xrightarrow{A} y$ and it is the principal formula of an application of (CS) in the right derivation, and (CS) is also the last rule in the left derivation; we are in SeqCS{+MP} and the situation is as follows (we denote the substitution $\Sigma[x/u, y/u]$ with $\Sigma(u)$):

$$\dfrac{\dfrac{\begin{array}{c}(1)\Gamma^{'} \vdash \Delta, x \xrightarrow{A} y, x : A^{'} \\ (2)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u\end{array}}{(5)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}\,(CS) \qquad \dfrac{\begin{array}{c}(3)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A \\ (4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)\end{array}}{(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta}\,(CS)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}\,(cut)$$

We can apply the inductive hypothesis on the height to replace the following cut:

$$\dfrac{(4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u) \qquad (2)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u}{(7)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u)}\,(cut)$$

Applying the height-preserving admissibility of weakening (Theorem 3.7), we have a derivation of $(7^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)$. Thus, we replace the cut as follows:

$$\dfrac{\dfrac{\begin{array}{l}(1^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}, x \xrightarrow{A} y \\ (6^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta, x : A^{'}\end{array}}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}}\,(cut) \qquad (7^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)}{(8)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta}\,(CS)$$

where $(1^{'})$ and $(6^{'})$ are also obtained by the height-preserving admissibility of weakening on (1) and (6), respectively. The proof is over applying the admissibility of contraction (see Theorem 3.9) to $(8)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta$, obtaining a derivation of $\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta$.

We present another interesting example in SeqCEM+CS{+ID}{+MP}. $F = x \xrightarrow{A} y$ is the cutting formula, and it is principal only in the right derivation, introduced (forward) by an application of (CEM). In the left derivation (CS) is applied to another transition $x \xrightarrow{A^{'}} y$ (consider $\Sigma(u) = \Sigma[x/u, y/u]$ and $\Sigma(v) = \Sigma[y/v, z/v]$):

$$\dfrac{\dfrac{\begin{array}{c}(1)\Gamma^{'} \vdash \Delta, x \xrightarrow{A} y, x : A^{'} \\ (2)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u\end{array}}{(5)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}\,(CS) \qquad \dfrac{\begin{array}{c}(3)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} z \\ (4)\Gamma^{'}(v), x \xrightarrow{A^{'}} v, x \xrightarrow{A} v \vdash \Delta(v)\end{array}}{(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta}\,(CEM)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}\,(cut)$$

From (1) we obtain a proof of (at most) the same height of $(1^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y, x : A^{'}$, by Theorem 3.7; the same for $(6^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta, x : A^{'}$, obtained from (6). Applying Lemma 3.6 to (6) we obtain a proof of no greater height of $(6^{''})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)$.
For the inductive hypothesis on the height, we cut $(1^{'})$ with $(6^{'})$, obtaining $(7)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}$; then we cut (2) with $(6^{''})$, obtaining a proof of $(8)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u)$, from which we obtain a proof of (at most) the same height of $(8^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)$ by Theorem 3.7. The initial cut is replaced as follows:

$$\dfrac{\dfrac{\dfrac{(1^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y, x : A^{'} \qquad (6^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta, x : A^{'}}{(7)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}}\,(cut) \qquad (8^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta}\,(CS)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}\,(ContrL)$$

Notice the application of the (Contr L) rule to conclude the proof.

2. if one of the sequents, say $\Gamma \vdash \Delta, F$ is obtained by the (EQ) rule, where F is not principal, then also $\Gamma \vdash \Delta$ is derivable by the (EQ) rule and we are done.

(iii) F is the principal formula in both the inferences steps leading to the two cut premises. There are nine subcases: F is introduced (a) by $(\rightarrow$ L), $(\rightarrow$ R), (b) by $(\Rightarrow L)$, $(\Rightarrow R)$, (c) by (EQ), (d) by (ID) on the left and by (EQ) on the right, (e) by (EQ) on the left and by (MP) on the right, (f) by (ID) on the left and by (MP) on the right, (g) by (CEM) on the left and by (MP) on the right, (h) by (CS) on the left and by (EQ) on the right and (i) by (CEM) on the left and (EQ) on the right. The list is exhaustive[3].

(a) This case is standard and left to the reader.

(b) $F = x : A \Rightarrow B$ is introduced by $(\Rightarrow R)$ and $(\Rightarrow L)$. Then we have

$$\dfrac{\dfrac{(1)\Gamma, x \xrightarrow{A} z \vdash \Delta, z : B}{(2)\Gamma \vdash \Delta, x : A \Rightarrow B}\,(\Rightarrow R) \qquad \dfrac{(3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{(5)\Gamma, x : A \Rightarrow B \vdash \Delta}\,(\Rightarrow L)}{\Gamma \vdash \Delta}\,(cut)$$

[3] Notice that the case when $F = x \xrightarrow{A} x$ is introduced by (CS) on the left and (MP) on the right has not to be considered, since (CS) derives a transition $x \xrightarrow{A} y$ (looking forward) only if $x \neq y$.

where z does not occur in Γ, Δ and $z \neq x$; By Lemma 3.6, we obtain that $(1^{'})\Gamma, x \xrightarrow{A} y \vdash y : B, \Delta$ is derivable by a derivation of no greater height than (1); moreover, we can apply the height-preserving admissibility of weakening (Theorem 3.7) to (2) in order to obtain a proof of no greater height of $(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y$ and of $(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B$.
First, we can make the following cut, which uses the inductive hypothesis on the height:

$$\frac{(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y \qquad (3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y}{(6)\Gamma \vdash \Delta, x \xrightarrow{A} y}\,(cut)$$

Applying the height-preserving admissibility of weakening to (6), we have a proof of no greater height of $(6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B$. Thus we can replace the initial cut as follows:

$$\frac{\dfrac{\begin{array}{c}(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B \\ (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta\end{array}}{\Gamma, y : B \vdash \Delta}\,(cut) \qquad \dfrac{\begin{array}{c}(1^{'})\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B \\ (6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B\end{array}}{\Gamma \vdash \Delta, y : B}\,(cut)}{\Gamma \vdash \Delta}\,(cut)$$

The upper cut on the left uses the induction hypothesis on the height, the others the induction hypothesis on the complexity of the cutting formula.

(c) $F = x \xrightarrow{A} y$ is introduced by (EQ) in both premises, we have

$$\frac{\dfrac{(1)\, u : A^{'} \vdash u : A \quad (2)\, u : A \vdash u : A^{'}}{\Gamma', x \xrightarrow{A^{'}} y \vdash x \xrightarrow{A} y, \Delta}\,(EQ) \qquad \dfrac{(3)\, u : A \vdash u : A^{''} \quad (4)\, u : A^{''} \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{A^{''}} y, \Delta'}\,(EQ)}{\Gamma', x \xrightarrow{A^{'}} y \vdash x \xrightarrow{A^{''}} y, \Delta'}\,(cut)$$

where $\Gamma = \Gamma', x \xrightarrow{A^{'}} y$, $\Delta = x \xrightarrow{A^{''}} y, \Delta'$. (1)-(4) have been derived by a shorter derivation; thus we can replace the cut by cutting (1) and (3) on the one hand, and (4) and (2) on the other, which give respectively

$$(5)\ u : A^{'} \vdash u : A^{''} \text{ and } (6)\ u : A^{''} \vdash u : A^{'}.$$

Using (EQ) we obtain $\Gamma', x \xrightarrow{A^{'}} y \vdash \Delta', x \xrightarrow{A^{''}} y$.

(d) $F = x \xrightarrow{A} y$ is introduced on the left by (ID) rule, and it is introduced on the right by (EQ). Thus we have

$$\dfrac{\dfrac{u : A^{'} \vdash u : A \quad u : A \vdash u : A^{'}}{\Gamma', x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \quad \dfrac{(1)\Gamma', x \xrightarrow{A^{'}} y, y : A \vdash \Delta}{x \xrightarrow{A} y, \Gamma', x \xrightarrow{A^{'}} y \vdash \Delta}(\text{ID})}{\Gamma', x \xrightarrow{A^{'}} y \vdash \Delta}(\text{cut})$$

By Lemma 3.6 and Theorem 3.7, the sequent $(2)\Gamma', x \xrightarrow{A^{'}} y, y : A^{'} \vdash y : A, \Delta$ can be derived by a derivation of no greater height than $u : A^{'} \vdash u : A$'s; by Theorem 3.7 we have also a proof $(1^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y, y : A, y : A^{'} \vdash \Delta$. Thus, the cut is replaced as follows

$$\dfrac{\dfrac{(2)\Gamma', x \xrightarrow{A^{'}} y, y : A^{'} \vdash y : A, \Delta \quad (1^{'})\Gamma', x \xrightarrow{A^{'}} y, y : A, y : A' \vdash \Delta}{\Gamma', x \xrightarrow{A^{'}} y, y : A^{'} \vdash \Delta}(cut)}{\Gamma', x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta}(\text{ID})$$

Since contractions are admissible (see Theorem 3.9 above), we conclude obtaining a derivation of $\Gamma', x \xrightarrow{A^{'}} y \vdash \Delta$.

(e) $F = x \xrightarrow{A} x$ is introduced on the left by (EQ) rule, and it is introduced on the right by (MP). Thus we have

$$\dfrac{\dfrac{(1)\Gamma \vdash x : A, \Delta', x \xrightarrow{A'} x}{\Gamma \vdash \Delta', x \xrightarrow{A'} x, x \xrightarrow{A} x}(\text{MP}) \quad \dfrac{u : A \vdash u : A' \quad u : A' \vdash u : A}{\Gamma, x \xrightarrow{A} x \vdash \Delta', x \xrightarrow{A'} x}(EQ)}{\Gamma \vdash \Delta', x \xrightarrow{A'} x}(\text{cut})$$

By Lemma 3.6 and Theorem 3.7, the sequent $(2)\Gamma, x : A \vdash x : A', \Delta^{'}, x \xrightarrow{A'} x$ can be derived by a derivation of at most the same height as $u : A \vdash u : A'$. Furthermore, we can also apply the Theorem 3.7 to (1), obtaining a proof of no greater height of $(1')\Gamma \vdash x : A, \Delta', x \xrightarrow{A'} x, x : A'$. Thus the cut is replaced as follows:

$$\dfrac{\dfrac{(1')\Gamma \vdash x : A, \Delta', x \xrightarrow{A'} x, x : A' \quad (2)\Gamma, x : A \vdash x : A', \Delta^{'}, x \xrightarrow{A'} x}{\Gamma \vdash x : A', \Delta^{'}, x \xrightarrow{A'} x}(cut)}{\Gamma \vdash \Delta', x \xrightarrow{A'} x, x \xrightarrow{A'} x}(\text{MP})$$

from which we obtain a proof of $\Gamma \vdash \Delta', x \xrightarrow{A'} x$ by the admissibility of contraction (Theorem 3.9).

(f) $F = x \xrightarrow{A} x$ is introduced on the right by (MP) rule and on the left by (ID). Thus we have

$$\dfrac{\dfrac{\Gamma \vdash x : A, \Delta}{\Gamma \vdash \Delta, x \xrightarrow{A} x}(MP) \quad \dfrac{\Gamma, x : A \vdash \Delta}{\Gamma, x \xrightarrow{A} x \vdash \Delta}(ID)}{\Gamma \vdash \Delta}(\text{cut})$$

We replace this cut by the following:

$$\dfrac{\Gamma \vdash x : A, \Delta \quad \Gamma, x : A \vdash \Delta}{\Gamma \vdash \Delta}(cut)$$

(g) $F = x \xrightarrow{A} x$ is introduced on the right by (MP) and on the left by (CEM). This is the case in which we cannot conclude the proof, as we mentioned above.

(h) $F = x \xrightarrow{A} y$ is derived on the left by (CS) and on the right by (EQ). Thus we have (we denote with $\Sigma(u)$ the substitution $\Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{(1) u : A \vdash u : A^{'} \quad (2) u : A^{'} \vdash u : A}{(5)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \quad \dfrac{(3)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A \quad (4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta}(CS)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}(cut)$$

Applying the label substitution to (1) we have a proof of $x : A \vdash x : A^{'}$ with (at most) the same height, then we apply the height-preserving admissibility of weakening obtaining a proof of $(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x : A \vdash x : A^{'}, \Delta$. Applying weakening to (3) we also have a derivation of $(7)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A, x : A^{'}$.
Applying the height-preserving label substitution to (5), replacing every occurrence of x and y with u, we have a proof of $(8)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u$. We have (inductive hypothesis on the height):

$$\dfrac{(8)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u \qquad (4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)}{(9)\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u)}(cut)$$

from which we can derive $(9^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)$ by weakening. We replace the initial cut as follows:

$$\dfrac{\dfrac{\begin{array}{c}(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x : A \vdash x : A^{'}, \Delta \\ (7)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A, x : A^{'}\end{array}}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}}(cut) \qquad (9^{'})\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A^{'}} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta}(CS)$$

and we conclude $\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta$ thanks to the admissibility of contraction.

(i) $F = x \xrightarrow{A} y$ is derived on the left by (CEM) and on the right by (EQ). Thus we have (we denote with $\Sigma(u)$ the substitution $\Sigma[y/u, z/u]$):

$$\dfrac{\dfrac{(1) u : A \vdash u : A^{'} \quad (2) u : A^{'} \vdash u : A}{(5)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \quad \dfrac{(3)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} z \quad (4)\Gamma^{'}(u), x \xrightarrow{A^{'}} u, x \xrightarrow{A} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta}(CEM)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}(cut)$$

where $y \neq z$. Applying weakening to (3) we obtain a proof of (at most) the same height of $(3^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} z, x \xrightarrow{A^{'}} z$. We also apply the height-preserving label substitution (Lemma 3.6) to (5) obtaining a proof of $(5^{'})\Gamma^{'}(u), x \xrightarrow{A^{'}} u \vdash \Delta(u), x \xrightarrow{A} u$. Therefore, we replace the cut as follows:

$$\dfrac{(1) u : A \vdash u : A^{'} \qquad (2) u : A^{'} \vdash u : A}{(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} z \vdash \Delta, x \xrightarrow{A^{'}} z}(EQ)$$

$$\dfrac{\dfrac{\begin{array}{c}(6)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} z \vdash \Delta, x \xrightarrow{A^{'}} z \\ (3^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} z, x \xrightarrow{A^{'}} z\end{array}}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A^{'}} z}(cut) \qquad (7)\Gamma^{'}(u), x \xrightarrow{A^{'}} u, x \xrightarrow{A^{'}} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A^{'}} y \vdash \Delta}(CEM)$$

where (7) is obtained by cutting (4) and (5′) (inductive hypothesis on the height) and applying Theorem 3.7. We conclude applying the admissibility of contraction to obtain a proof of $\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta$.

□

3.2 Soundness and completeness of SeqS

SeqS calculi are sound and complete with respect to the semantics.

Theorem 3.12 (Soundness): If $\Gamma \vdash \Delta$ is derivable in SeqS then it is valid in the corresponding system.

Proof. By induction on the height of a derivation of $\Gamma \vdash \Delta$. As an example, we examine the cases of ($\Rightarrow$ R), (MP), (CS) and (CEM). The other cases are left to the reader.

- ($\Rightarrow$ R) Let $\Gamma \vdash \Delta, x : A \Rightarrow B$ be derived from (1) $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$, where y does not occur in Γ, Δ and it is different from x. By induction hypothesis we know that the latter sequent is valid. Suppose the former is not, and that it is not valid in a model $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$, via a mapping I, so that we have:

  $\mathcal{M} \models_I F$ for every $F \in \Gamma$, $\mathcal{M} \not\models_I F$ for any $F \in \Delta$ and $M \not\models_I x : A \Rightarrow B$.

  As $M \not\models_I x : A \Rightarrow B$ there exists $w \in f(I(x), [A]) - [B]$. We can define an interpretation $I'(z) = I(z)$ for $z \neq y$ and $I'(y) = w$. Since y does not occur in Γ, Δ and is different from x, we have that $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma$, $\mathcal{M} \not\models_{I'} F$ for any $F \in \Delta$, $\mathcal{M} \not\models_{I'} y : B$ and $\mathcal{M} \models_{I'} x \xrightarrow{A} y$, against the validity of (1).

- (MP) Let $\Gamma \vdash \Delta, x \xrightarrow{A} x$ be derived from (2) $\Gamma \vdash \Delta, x : A$. Let (2) be valid and let $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$ be a model satisfying the MP condition. Suppose that for one mapping I, $\mathcal{M} \models_I F$ for every $F \in \Gamma$, then by the validity of (2) either $\mathcal{M} \models_I G$ for some $G \in \Delta$, or $\mathcal{M} \models_I x : A$. In the latter case, we have $I(x) \in [A]$, thus $I(x) \in f(I(x), [A])$, by MP, this means that $\mathcal{M} \models_I x \xrightarrow{A} x$.

- (CS) Let (3)$\Gamma, x \xrightarrow{A} y \vdash \Delta$, with $x \neq y$, be derived from (4)$\Gamma \vdash \Delta, x : A$ and (5)$\Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]$, where u does not occur in Γ, Δ. Suppose that (4) and (5) are valid, whereas (3) is not, considering models $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$ satisfying the CS condition. Therefore, there is a mapping I such that $\mathcal{M} \models_I F$ for every $F \in \Gamma$, $\mathcal{M} \models_I x \xrightarrow{A} y$ (i.e. $I(y) \in f(I(x), [A])$) and $\mathcal{M} \not\models_I G$ for every $G \in \Delta$. We distinguish two cases:

  - ⋆ $I(x) \notin [A]$: in this case, we have that $\mathcal{M} \not\models_I x : A$, against the validity of (4);
  - ⋆ $I(x) \in [A]$: we observe that $f(I(x), [A]) \subseteq \{I(x)\}$, since $\mathcal{M}$ respects the CS condition. We have also that $I(y) \in f(I(x), [A])$, then it can be only $I(x) = I(y)$: say $w = I(x) = I(y)$. We introduce another mapping $I^{'}$ as follows: $I^{'}(u) = w$, $I^{'}(v) = I(v)$ for every label different from u. Obviously, $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma[x/u, y/u]$, and $\mathcal{M} \not\models_{I'} G$ for every $G \in \Delta[x/u, y/u]$, but $\mathcal{M} \models_{I'} u \xrightarrow{A} u$, since $I^{'}(u) = w \in f(w, [A])$, against the validity of (5).

- (CEM) Let (8)$\Gamma, x \xrightarrow{A} y \vdash \Delta$ be derived from (6)$\Gamma \vdash \Delta, x \xrightarrow{A} z$ and (7)$\Gamma[y/u, z/u], x \xrightarrow{A} u \vdash \Delta[y/u, z/u]$, with $z \neq y$. Suppose (6) and (7) are valid, whereas (8) is not. $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$ respects the CEM condition. Then, one can find a mapping I such that $\mathcal{M} \models_I F$ for every $F \in \Gamma$,

$\mathcal{M} \not\models_I G$ for every $G \in \Delta$ and $\mathcal{M} \models_I x \xrightarrow{A} y$, thus $I(y) \in f(I(x), [A])$. We distinguish two cases:

- ⋆ $I(y) \neq I(z)$: since $\mathcal{M}$ respects CEM, we have that $\mid f(I(x), [A]) \mid \leq 1$. In this case, $f(I(x), [A]) = \{I(y)\}$, then $I(z) \notin f(I(x), [A])$. We can conclude $\mathcal{M} \not\models_I x \xrightarrow{A} z$, against the validity of (6);
- ⋆ $I(z) = I(y) = w$, therefore $f(I(x), [A]) = \{w\}$. We introduce another mapping $I^{'}$ in this way: $I^{'}(u) = w$; $I^{'}(v) = I(v)$ for every label v different from u. Obviously, $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma[y/u, z/u]$, and $\mathcal{M} \not\models_{I'} G$ for every $G \in \Delta[y/u, z/u]$. However, $\mathcal{M} \models_{I'} x \xrightarrow{A} u$ since $I^{'}(u) = w \in f(I^{'}(x), [A])$, against the validity of (7).

□

Completeness is an easy consequence of the admissibility of cut (see Theorem 3.11 above).

Theorem 3.13 (Completeness): If A is valid in CK or in one of its cited extensions, then $\vdash x : A$ is derivable in the respective SeqS system.

Proof. We must show that the axioms are derivable and that the set of derivable formulas is closed under (Modus Ponens), (RCEA), and (RCK). A derivation of axioms (ID), (MP), (CS) and (CEM) is shown in examples 3.2, 3.3, 3.4 and 3.5 respectively.
Let us examine the other axioms.
For (Modus Ponens), suppose that $\vdash x : A \rightarrow B$ and $\vdash x : A$ are derivable. We easily have that $x : A \rightarrow B, x : A \vdash x : B$ is derivable too. Since cut is admissible, by two cuts we obtain $\vdash x : B$.
For (RCEA), we have to show that if $A \leftrightarrow B$ is derivable, then also $(A \Rightarrow C) \leftrightarrow (B \Rightarrow C)$ is so. The formula $A \leftrightarrow B$ is an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. Suppose that $\vdash x : A \rightarrow B$ and $\vdash x : B \rightarrow A$ are derivable, we can derive $x : B \Rightarrow C \vdash x : A \Rightarrow C$ as follows: (the other half is symmetrical).

$$\dfrac{\dfrac{\dfrac{x : A \vdash x : B \quad x : B \vdash x : A}{x : A \Rightarrow B, x \xrightarrow{B} y \vdash x \xrightarrow{A} y, y : C}\,(EQ) \qquad x : A \Rightarrow B, x \xrightarrow{B} y, y : C \vdash y : C}{x \xrightarrow{B} y, x : A \Rightarrow C \vdash y : C}\,(\Rightarrow L)}{x : A \Rightarrow C \vdash x : B \Rightarrow C}\,(\Rightarrow R)$$

For (RCK), suppose that (1) $\vdash x : B_1 \wedge B_2 \ldots \wedge B_n \rightarrow C$, it must be derivable also $y : B_1, \ldots, y : B_n \vdash y : C$. Then we have (we omit side formulas in $x \xrightarrow{A} y \vdash x \xrightarrow{A} y$):

$$\dfrac{\dfrac{\dfrac{x \xrightarrow{A} y \vdash x \xrightarrow{A} y \qquad x : A \Rightarrow B_1, x : A \Rightarrow B_2, \ldots, x : A \Rightarrow B_n, y : B_1, \ldots, y : B_n \vdash y : C}{x \xrightarrow{A} y, x : A \Rightarrow B_1, x : A \Rightarrow B_2, \ldots, x : A \Rightarrow B_n, y : B_1, \ldots, y : B_{n-1} \vdash y : C} (\Rightarrow L)}{\vdots}}{}$$

$$\dfrac{\dfrac{x \xrightarrow{A} y \vdash x \xrightarrow{A} y \qquad x \xrightarrow{A} y, x : A \Rightarrow B_1, x : A \Rightarrow B_2, \ldots, x : A \Rightarrow B_n, y : B_1 \vdash y : C}{x \xrightarrow{A} y, x : A \Rightarrow B_1, x : A \Rightarrow B_2, \ldots, x : A \Rightarrow B_n \vdash y : C} (\Rightarrow L)}{x : A \Rightarrow B_1, x : A \Rightarrow B_2, \ldots, x : A \Rightarrow B_n \vdash x : A \Rightarrow C} (\Rightarrow R)$$

□

4 Decidability and complexity

In this section we analyze SeqS calculi in order to obtain a decision procedure for all conditional systems under consideration[4]. In particular, we describe some common properties, then we analyze separately weaker systems CK{+ID}{+MP}, systems CEM{+ID} and systems CS{+ID}{+CEM,+MP}.

In general, cut-elimination alone does not ensure a terminating proof search in a given system of sequent calculi; the presence of labels and of the ($\Rightarrow$ L) rule, which increases the complexity of the sequent to prove in a backward proof search, are potential causes of a non-terminating proof search. In this section we show that SeqS's rules introduce only a finite number of labels in a backward proof search, and that ($\Rightarrow$ L) can be applied in a controlled way: these conditions allow to describe a decision procedure for the corresponding logics. We also give explicit complexity bounds for our systems.

As a first step, we show that it is useless to apply ($\Rightarrow$ L) on $x : A \Rightarrow B$ by introducing (looking backward) the same transition formula $x \xrightarrow{A} y$ *more than once in a proof search.* More in detail, we have the following:

Lemma 4.1 (Controlled application of ($\Rightarrow$ L)): If $\Gamma \vdash \Delta$ is derivable, then there is a proof of it which does not contain more than one application of ($\Rightarrow$ L) applied (looking backward) to $x : A \Rightarrow B$ with the same transition formula $x \xrightarrow{A} y$.

Proof. Consider a derivation of $\Gamma \vdash \Delta$ in which ($\Rightarrow$ L) is applied to $x : A \Rightarrow B$ with a transition $x \xrightarrow{A} y$ more than once; in particular, consider the two highest[5] applications. We have the following situation:

$$\dfrac{\dfrac{\overset{\Pi_a}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y} \qquad \overset{\Pi_b}{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1} (\Rightarrow L)}{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta\end{array}}$$

[4] Obviously, we restrict our concern to all cut-free systems, i.e. all SeqS systems except SeqCEM{+ID}+MP.

[5] The applications having the greatest distance from the root $\Gamma \vdash \Delta$.

and in Π_a or in Π_b the rule ($\Rightarrow$ L) is applied (looking backward) to $x : A \Rightarrow B$ by using $x \xrightarrow{A} y$. We distinguish these two cases:

- the highest application is in Π_a; this case is as follows:

$$\dfrac{\dfrac{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2, x \xrightarrow{A} y \qquad \Gamma_2, x : A \Rightarrow B, y : B \vdash \Delta_2}{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2}(\Rightarrow L) \quad \vdots \quad \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y \qquad \overset{\Pi_b}{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$

The application of ($\Rightarrow$ L) in the left can be permuted over the other rules in Π_a (remember that ($\Rightarrow$ L) is invertible, see Theorem 3.8); we have the following proof tree ($\Pi_a^{'}$ is Π_a after the permutation):

$$\dfrac{\dfrac{\overset{\Pi_a^{'}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y, x \xrightarrow{A} y} \quad \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1, x \xrightarrow{A} y}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y}(\Rightarrow L) \qquad \overset{\Pi_b}{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$

For the admissibility of contraction (Theorem 3.9), we have a proof $\Pi_a^{''}$ of $\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y$, which does not contain any application of ($\Rightarrow$ L) on $x : A \Rightarrow B$ with the same transition $x \xrightarrow{A} y$ and then we have the following proof:

$$\dfrac{\overset{\Pi_a^{''}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y} \quad \overset{\Pi_b}{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$
$$\vdots$$
$$\Gamma \vdash \Delta$$

in which we have removed an useless second application of ($\Rightarrow$ L) on $x : A \Rightarrow B$ by using $x \xrightarrow{A} y$;

- the highest application is in Π_b; we are in the following situation:

$$\dfrac{\overset{\Pi_a}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y} \qquad \dfrac{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2, x \xrightarrow{A} y \qquad \Gamma_2, x : A \Rightarrow B, y : B \vdash \Delta_2}{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2}(\Rightarrow L) \quad \vdots \quad \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$

As in the other case, we can permute the upper application of ($\Rightarrow$ L), obtaining the following derivation:

$$\dfrac{\begin{matrix}\Pi_a \\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y\end{matrix} \quad \dfrac{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1, x \xrightarrow{A} y \quad \begin{matrix}\Pi_b^{'} \\ \Gamma_1, x : A \Rightarrow B, y : B, y : B \vdash \Delta_1\end{matrix}}{\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1} (\Rightarrow L)}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1} (\Rightarrow L)$$

For the Theorem 3.9 we have a proof $\Pi_b^{''}$ of $\Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1$, therefore we can obtain the following proof:

$$\dfrac{\begin{matrix}\Pi_a \\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y\end{matrix} \quad \begin{matrix}\Pi_b^{''} \\ \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1\end{matrix}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1} (\Rightarrow L)$$
$$\vdots$$
$$\Gamma \vdash \Delta$$

□

From now on, we analyze separately the decidability of systems CK{+ID}{+MP}, CK+CEM{+ID} and CK+CS{+CEM,+MP}{+ID}.

4.1 Proof-theoretical analysis of CK{+ID}{+MP}

In this subsection we prove some properties characterizing calculi SeqS for weaker conditional logics CK{+ID}{+MP}, in order to give a decision procedure for these systems. From now on we refer to calculi for these systems with the name **SeqG**. In particular, we have to show that one can apply ($\Rightarrow$ L) in a controlled way.

First of all, we observe that the calculi for G systems are characterized by the following property:

Theorem 4.2 (Property of ($\Rightarrow$ L)): Let the sequent

$$\Gamma \vdash \Delta,\ x \xrightarrow{A} y$$

with $x \neq y$, be derivable in SeqG, then one of the following sequents:

1. $\Gamma \vdash \Delta$
2. $x \xrightarrow{F} y \vdash x \xrightarrow{A} y$, where $x \xrightarrow{F} y \in \Gamma$

is also derivable in SeqG .

Proof. (EQ) is the only SeqG's rule which operates, considering a backward proof search, on a transition formula on the right hand side (consequent) of a sequent. Thus, we have to consider three cases, analyzing the proof tree of $\Gamma \vdash \Delta,\ x \xrightarrow{A} y$:

1. $\Gamma \vdash \Delta, x \xrightarrow{A} y$ is an axiom: we have to consider two subcases:
   (a) a labelled formula F occurs in both Γ and Δ, therefore $\Gamma \vdash \Delta$ is derivable;
   (b) $w : \bot \in \Gamma$, then $\Gamma \vdash \Delta$ is derivable;
2. $x \xrightarrow{A} y$ descends in Π (looking forward) from an axiom $\Gamma_1 \vdash \Delta_1, x \xrightarrow{A} y$, where $\Gamma_1 \vdash \Delta_1$ is an axiom too. In this case, $\Gamma \vdash \Delta$ is derivable, since we can remove an occurrence of $x \xrightarrow{A} y$ from every sequent descending (looking forward) from $\Gamma_1 \vdash \Delta_1, x \xrightarrow{A} y$ in Π;
3. $x \xrightarrow{A} y$ is introduced (looking forward) by the (EQ) rule: in this case, another transition $x \xrightarrow{F} y$ *must* be in Γ, in order to apply (EQ). To see this, observe that the only rule that could introduce a transition formula (looking backward) in the antecedent of a sequent is ($\Rightarrow$ R), but it can only introduce a transition of the form $x \xrightarrow{F} z$, where z *does not occur in that sequent* (it is a *new* label), thus it cannot introduce the transition $x \xrightarrow{F} y$.
   The (EQ) rule is only applied to transition formulas:
   $$\frac{u : F \vdash u : A \qquad u : A \vdash u : F}{x \xrightarrow{F} y \vdash x \xrightarrow{A} y}\ (EQ)$$
   therefore we can say that $x \xrightarrow{F} y \vdash x \xrightarrow{A} y$ is derivable in SeqG.

□

Notice that this theorem holds for all the systems SeqG, but only if $x \neq y$. In systems with MP, considering a backward proof search, the (MP) rule operates on transitions in the consequent, although on transitions like $x \xrightarrow{A} x$. In this case the theorem does not hold, as shown by the following counterexample:

$$\frac{x : A \vdash x : A, x : B}{x : A \vdash x \xrightarrow{A} x, x : B}\ (MP)$$

for A and B arbitrary. The sequent $x : A \vdash x \xrightarrow{A} x, x : B$ is derivable in SeqMP, but $x : A \vdash x : B$ is not derivable in this system and the second condition is not applicable (no transition formula occurs in the antecedent).
The first hypothesis of the theorem ($x \neq y$) excludes this situation.

In order to control the application of ($\Rightarrow$ L) we need to show one more property; roughly speaking, if a sequent $\Gamma, x : A \Rightarrow B \vdash \Delta$ derives in a proof (looking forward) from $\Gamma_1, x : A \Rightarrow B \vdash \Delta_1$ and *all the transitions of the form* $x \xrightarrow{A_i} y_i$ *belong to* Γ_1, then the proof of $\Gamma_1, x : A \Rightarrow B \vdash \Delta_1$ does not contain any *proper application* of ($\Rightarrow$ L) by using a transition $x \xrightarrow{A} z$, with z different from x and different from each y_i. We say that an application of ($\Rightarrow$ L) is *proper* if it is needed to end the proof, i.e. the transition $x \xrightarrow{A} y$ in the left premise cannot be removed by the admissibility of weakening (Theorem 3.7); for instance, the following application of ($\Rightarrow$ L):

$$\dfrac{\dfrac{x : A, x : B, x : A \Rightarrow B \vdash x : A, x \xrightarrow{A} y}{x : A \wedge B, x : A \Rightarrow B \vdash x : A, x \xrightarrow{A} y}(\wedge L) \quad \dfrac{x : A, x : B, x : A \Rightarrow B, y : B \vdash x : A}{x : A \wedge B, x : A \Rightarrow B, y : B \vdash x : A}(\wedge L)}{x : A \wedge B, x : A \Rightarrow B \vdash x : A}(\Rightarrow L)$$

is obviously not proper, since $x \xrightarrow{A} y$ can be removed from the left premise[6]. The intuition is that it is useless to apply (backward) the ($\Rightarrow$ L) rule on $x : A \Rightarrow B$ by using a transition $x \xrightarrow{A} y$ if no $x \xrightarrow{A'} y$ belongs to the left-hand side of the sequent, since there will be no way to prove that transition. Indeed, a transition $x \xrightarrow{A} y$ in the right-hand side of a sequent can only be derived (looking forward) from (EQ) or (MP). In the case of (EQ), a transition $x \xrightarrow{A'} y$ must belong to the left-hand side of the sequent by Property 4.2. In the case of (MP) the same label x is used.

Let us state the above property in a formal manner:

Lemma 4.3 (Property of proper applications of ($\Rightarrow$ L)): Consider a proof of $\Gamma, x : A \Rightarrow B \vdash \Delta$ ending with:

$$\begin{array}{c} \Pi_A \\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1 \\ \Pi_B \\ \Gamma, x : A \Rightarrow B \vdash \Delta \end{array}$$

where all the transitions of the form $x \xrightarrow{A_i} y_i$, $i = 1, 2, ..., n$, $y_i \neq x$ and $y_i \neq y_j$ for all $j = 1, 2, ..., n$ with $j \neq i$, belong to Γ_1 and there are no negative conditionals with label x in $\Gamma_1, x : A \Rightarrow B \vdash \Delta_1$ (i.e. no negative conditional introduces in Π_A, looking backward, a transition $x \xrightarrow{A_k} k$ in the left-hand side of the sequents).

In this situation, Π_A does not contain any proper application of ($\Rightarrow$ L) of the form:

$$\dfrac{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2, x \xrightarrow{A} z \qquad \Gamma_2, x : A \Rightarrow B, z : B \vdash \Delta_2}{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2}(\Rightarrow L)$$

with $z \neq x$ and $z \neq y_i$ for each $i = 1, 2, ..., n$.

Proof. All negative conditionals introduce (looking backward) all the transitions $x \xrightarrow{A_i} y_i$ in Π_B by n applications of ($\Rightarrow$ R), therefore there is not another

[6] A proof tree of the sequent can be obtained by an application (backward) of ($\wedge$ L) on $x : A \wedge B$ as a first step in the derivation.

conditional $x : A_z \Rightarrow C$ introducing another transition $x \xrightarrow{A_z} z$ in the left-hand side of sequents. By this fact and by the Property 4.2, we can conclude that an application of ($\Rightarrow$ L) using a transition $x \xrightarrow{A} z$, where z is different from each y_i and from x, cannot be proper, since $x \xrightarrow{A} z$ cannot be proved.

□

Now we have all the elements to prove the decidability of G systems:

Theorem 4.4 (Terminating proof search for CK{+MP}{+ID}): Systems SeqCK, SeqID, SeqMP and SeqID+MP ensure a terminating proof search.

Proof. In all rules the premises have a smaller complexity than the conclusion, except for ($\Rightarrow$ L). However, Lemma 4.1 guarantees that ($\Rightarrow$ L) can be applied in a controlled way, i.e. one needs to apply ($\Rightarrow$ L) only once on a formula $x : A \Rightarrow B$ with the same transition $x \xrightarrow{A} y$. Therefore, considering a derivation of a sequent $\Gamma, x : A \Rightarrow B \vdash \Delta$, the number of proper applications of ($\Rightarrow$ L) on $x : A \Rightarrow B$ is bounded by the cardinality of $\{x \xrightarrow{A} y \mid \Gamma \vdash x \xrightarrow{A} y\}$. In systems allowing the (MP) rule this bound is incremented by 1, considering the opportunity of applying ($\Rightarrow$ L) by using $x \xrightarrow{A} x$.
The number of different y, excluded x, such that $\Gamma \vdash x \xrightarrow{A} y$ is finite. Indeed, we can assume, without loss of generality, that Γ contains all the transitions of the form $x \xrightarrow{A_i} y_i$ introduced by negative conditionals, by the permutability of the rules: the only non trivial case is when a negative conditional $x : A^{'} \Rightarrow C$, introducing (looking backward) a transition $x \xrightarrow{A^{'}} z$, is introduced (backward) by an application of ($\Rightarrow$ L) on $x : A \Rightarrow B$ itself, i.e. $x : A^{'} \Rightarrow C$ is a subformula of $x : A \Rightarrow B$; this happens only if ($\Rightarrow$ L) is applied by using $x \xrightarrow{A} x$, and $x : B$ introduces (backward) $x : A^{'} \Rightarrow C$: since x is used, all the rules introducing $x : A^{'} \Rightarrow C$ can be permuted until the application of ($\Rightarrow$ L). We can have this situation only in MP systems; in CK{+ID} no subformula of $x : A \Rightarrow B$ can introduce transitions $x \xrightarrow{A_i} y_i$ in a backward proof search, since they are all labelled with successors of x. We conclude by Lemma 4.3 that the number of different labels that can be used in a proper application of ($\Rightarrow$ L), and thus the number of application of this rule in a backward proof search, is finite.

Moreover, observe that the rules are analytic, so that the premises contains only (labelled) subformulas of the formulas in the conclusion. In the search of a proof of $\vdash x_0 : D$, with $\mid D \mid= n$, new labels are introduced only by (negative) conditional subformulas of D.

The number of different labels occurring in a proof is $O(n)$, and the length of each branch of a proof tree is bounded by $O(n^2)$.

□

This itself gives decidability:

Theorem 4.5 (CK{+ID}{+MP} decidability): Logic CK{+ID} is decidable.

Proof. We just observe that there is only a finite number of derivations to check of a given sequent $\vdash x_0 : D$, as both the length of a proof and the number of labelled formulas which may occur in it is finite.

□

We conclude this subsection by giving an explicit space complexity bound for CK{+ID}{+MP}:

Theorem 4.6 (Space complexity of CK{+ID}{+MP}): Provability in CK{+ID}{+MP} is decidable in $O(n^2 \ \log n)$ space.

Proof. As usual, a proof may have an exponential size since the branching introduced by the rules. However we can obtain a much sharper space complexity bound using a standard technique [29, 47], namely we do not need to store the whole proof, but only a sequent at a time plus additional information to carry on the proof search. In searching a proof there are two kinds of branching to consider: AND-branching caused by the rules with multiple premises and OR-branching (backtracking points in a depth first search) caused by the choice of the rule to apply.

We store only one sequent at a time and maintain a stack containing information sufficient to reconstruct the branching points of both types. Each stack entry contains the principal formula (either a world formula $x : B$, or a transition formula $x \xrightarrow{B} y$), the name of the rule applied and an index which allows to reconstruct the other branches on return to the branching points. The stack entries represent thus backtracking points and the index within the entry allows one to reconstruct both the AND branching and to check whether there are alternatives to explore (OR branching). The working sequent on a return point is recreated by replaying the stack entries from the bottom of the stack using the information in the index (for instance in the case of ($\Rightarrow$ L) applied to the principal formula $x : A \Rightarrow B$, the index will indicate which premise-first or second-we have to expand and the label y involved in the transition formula $x \xrightarrow{A} y$).

A proof begins with the end sequent $\vdash x_0 : D$ and the empty stack. Each rule application generates a new sequent and extends the stack. If the current sequent is an axiom we pop the stack until we find an AND branching point to be expanded. If there are not, the end sequent $\vdash x_0 : D$ is provable and we have finished. If the current sequent is not an axiom and no rule can be applied to it, we pop the stack entries and we continue at the first available entry with some alternative left (a backtracking point). If there are no such entries, the end sequent is not provable.

The entire process must terminate since: (i) the depth of the stack is bounded by the length of a branch proof, thus it is $O(n^2)$, where $\mid D \mid= n$, (ii) the branching is bounded by the number of rules, the number of premises of any

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash \Delta[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \ (\mathbf{CEM}), y \neq z$$

Fig. 6: Rule (CEM) reformulated for SeqCEM*{+ID} calculi.

rule and the number of labelled formulas occurring in one sequent, the last being $O(n^2)$.

To evaluate the space requirement, we have that each subformula of the initial labelled formula can be represented by a positional index into the initial labelled formula, which requires $O(\log n)$ bits. Moreover, also each label can be represented by $O(\log n)$ bits. Thus, to store the working sequent we need $O(n^2 \ \log n)$ space, since there may occur $O(n^2)$ labelled subformulas. Similarly, each stack entry requires $O(\log n)$ bits, as the name of the rule requires constant space and the index $O(\log n)$ bits. Having depth $O(n^2)$, to store the whole stack requires $O(n^2 \ \log n)$ space. Thus we obtain that provability in CK{+ID}{+MP} is decidable in $O(n^2 \ \log n)$ space.

□

4.2 Proof-theoretical analysis of CK+CEM{+ID}

In this subsection we analyze sequent calculi characterized by the CEM axiom in order to get a decision procedure for these calculi. However, some other properties holding in SeqG systems hold in these systems too, then we are able to present a terminating calculus for them.

In section 3 we have discussed about the introduction of *explicit contraction rules* in system SeqCEM (see Figure 4); these contraction rules are other potential causes of non termination, since their application increases the complexity of the current sequent in a backward proof search. By this fact, we reformulate the calculi for CEM in order to obtain a calculus with no explicit contraction rules: we call SeqCEM*{+ID} the resulting system.

First of all, we reformulate the (CEM) rule as shown in Figure 6.

It is easy to observe that in SeqCEM*{+ID} weakening and label substitution are height-preserving admissible (Theorem 3.7 and Lemma 3.6); moreover, we can prove the following:

Theorem 4.7 (Height-preserving invertibility of (CEM)): The rule (CEM) in SeqCEM*{+ID} is height-preserving invertible.

Proof. If $(1)\Gamma, x \xrightarrow{A} y \vdash \Delta$ is derivable, then $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ is derivable with a proof of no greater height since it is obtained from (1) by the height-preserving admissibility of weakening (Theorem 3.7). The sequent $(\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash \Delta[y/u, z/u]$ is also derivable with (at most) the same height since it is obtained from (1) by the height-preserving label substitution (Lemma 3.6).

□

We can also observe that the admissibility of contraction holds in SeqCEM*{+ID} system(s):

Theorem 4.8 (Admissibility of contraction in SeqCEM*{+ID}): The rules of contraction are admissible in SeqCEM*{+ID}, i.e. if a sequent $\Gamma \vdash \Delta, F, F$ is derivable in SeqCEM*{+ID}, then there is a proof of $\Gamma \vdash \Delta, F$, and if a sequent $\Gamma, F, F \vdash \Delta$ is derivable in SeqCEM*{+ID}, then there is a proof of $\Gamma, F \vdash \Delta$.

Proof. The proof is identical to the proof of Theorem 3.9 since we have the same rules. The only difference is when the contracted formula is a transition $x \xrightarrow{A} y$ in the left-hand side of the sequent: it can be introduced (looking forward) by (CEM) or by (ID); if it is introduced by (CEM) the situation is as follows:

$$\frac{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad \Gamma[y/u, z/u], x \xrightarrow{A} u, x \xrightarrow{A} u \vdash \Delta[y/u, z/u]}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\ (CEM)$$

By inductive hypothesis on the left premise, we obtain a proof of $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$, and by inductive hypothesis on the right premise we obtain a proof of $\Gamma[y/u, z/u], x \xrightarrow{A} u \vdash \Delta[y/u, z/u]$. Applying (CEM) to the last two sequents we are done.

If the contracted transition is introduced (forward) by an application of (ID), then we have:

$$\frac{(*)\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\ (ID)$$

The sequent $(*)$ derives (forward) from 1. another application of (ID) or 2. an application of (CEM)[7]. In case 1., since (ID) permutes over the other rules of the calculus, we can consider the following proof:

$$\frac{\dfrac{(**)\Gamma, y : A, y : A \vdash \Delta}{(*)\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}\ (ID)}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\ (ID)$$

Applying the inductive hypothesis on $(**)$ we obtain a proof of $\Gamma, y : A \vdash \Delta$, thus we conclude by an application of (ID). In case 2. we can permute the application of (CEM) over the application of (ID) by Theorem 4.7, thus we can conclude as we made for the case when (CEM) is the rule applied to $\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta$.

[7] As usual, if $x \xrightarrow{A} y$ derives from axioms, then $\Gamma, x \xrightarrow{A} y \vdash \Delta$ can be obviously obtained by removing that weakening.

□

Our goal is to prove that SeqCEM*{+ID} is sound and complete with respect to the semantics. The proof of soundness is obvious and left to the reader. To prove the completeness, we proceed in two steps:

1. we show that if a sequent $\Gamma \vdash \Delta$ is derivable in SeqCEM{+ID}, say $\Gamma \vdash_{old} \Delta$, then it is also derivable in SeqCEM*{+ID} *plus the explicit contraction rules*, say $\Gamma \vdash_{*Contr} \Delta$;
2. we show that $\Gamma \vdash \Delta$ is derivable in SeqCEM*{+ID} with explicit contractions ($\Gamma \vdash_{*Contr} \Delta$) if and only if $\Gamma \vdash \Delta$ is derivable in SeqCEM*{+ID} (without the contraction rules, say $\Gamma \vdash_{*} \Delta$).

Theorem 4.9: If $\Gamma \vdash \Delta$ is derivable in SeqCEM{+ID} ($\Gamma \vdash_{old} \Delta$), then it is derivable in SeqCEM*{+ID} + (Contr L) + (Contr R) ($\Gamma \vdash_{*Contr} \Delta$).

Proof. By induction on the proof of $\Gamma \vdash_{old} \Delta$. The only difference between the two calculi is the rule (CEM). Therefore, we have to show that if (CEM) is applied to $\Gamma \vdash_{old} \Delta$, then we can find a derivation of $\Gamma \vdash_{*Contr} \Delta$ with no applications of the "old version" of (CEM), eventually using the reformulated rule of Figure 6. We have the following situation:

$$\frac{\Gamma^{'} \vdash_{old} \Delta, x \xrightarrow{A} z \qquad (\Gamma^{'}, x \xrightarrow{A} y)[y/u, z/u] \vdash_{old} \Delta[y/u, z/u]}{\Gamma^{'}, x \xrightarrow{A} y \vdash_{old} \Delta} \ (CEM)$$

By the inductive hypothesis on the two premises, we can find derivations of $\Gamma^{'} \vdash_{*Contr} \Delta, x \xrightarrow{A} z$ and, by the admissibility of weakening (Theorem 3.7), of $\Gamma^{'}, x \xrightarrow{A} y \vdash_{*Contr} \Delta, x \xrightarrow{A} z$, and of $(\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash_{*Contr} \Delta[y/u, z/u]$, from which we conclude by an application of the invertible (CEM) as follows:

$$\frac{\Gamma^{'}, x \xrightarrow{A} y \vdash_{*Contr} \Delta, x \xrightarrow{A} z \qquad (\Gamma^{'}, x \xrightarrow{A} y)[y/u, z/u] \vdash_{*Contr} \Delta[y/u, z/u]}{\Gamma^{'}, x \xrightarrow{A} y \vdash_{*Contr} \Delta} \ (CEM)$$

□

Theorem 4.10: $\Gamma \vdash \Delta$ is derivable in SeqCEM*{+ID} + (Contr L) + (Contr R) ($\Gamma \vdash_{*Contr} \Delta$) if and only if it is derivable in SeqCEM*{+ID} ($\Gamma \vdash_{*} \Delta$).

Proof. First, we show that if $\Gamma \vdash_{*Contr} \Delta$, then $\Gamma \vdash_{*} \Delta$. By induction on the height of the derivation in SeqCEM*{+ID} + (Contr L) + (Contr R), we have that the only interesting case is when an explicit contraction is applied to the sequent. Suppose $\Gamma^{'}, F \vdash \Delta$ derives (looking forward) from $\Gamma^{'}, F, F \vdash \Delta$ by an application of (Contr L): by the inductive hypothesis, we can find a proof in SeqCEM*{+ID} of $\Gamma^{'}, F, F \vdash_{*} \Delta$, then we conclude with a proof of $\Gamma^{'}, F \vdash_{*} \Delta$

by the admissibility of contraction (Theorem 4.8 above). The same for the case when $\Gamma \vdash \Delta^{'}, F$ derives from $\Gamma \vdash \Delta^{'}, F, F$ by an application of (Contr R).

The other half is obvious, since SeqCEM*{+ID} offers a proper subset of rules of SeqCEM*{+ID} + (Contr L) + (Contr R).

□

The completeness of the reformulated calculus SeqCEM*{+ID} follows by these facts:

Corollary 4.11 (Completeness of SeqCEM*{+ID}): SeqCEM*{+ID} is complete.

Proof. This Corollary follows immediately from Theorems 4.9 and 4.10.

□

We conclude this subsection by proving that SeqCEM*{+ID} ensures a terminating proof search. We proceed in a similar manner as we made for SeqG; in particular, we have to show that both (CEM) and ($\Rightarrow$ L) rules can be applied in a controlled way. An application of these rules introduces the same principal formula in (at least) one of the premises; this fact is a potential cause of non termination in a backward proof search. However, we prove that the number of applications of both (CEM) and ($\Rightarrow$ L) is finite, and this gives the decidability.

In order to obtain a decision procedure for SeqCEM*{+ID} it is crucial to analyze how a sequent $\Gamma \vdash \Delta, x \xrightarrow{A} y$ can be derived: in particular, it is fundamental to discuss about the introduction of the transition formula $x \xrightarrow{A} y$. In these systems, (EQ) is the only rule having a transition on the right-hand side of a sequent as a principal formula; however, given a sequent $\Gamma^{'}, v \xrightarrow{A^{'}} w \vdash \Delta, x \xrightarrow{A} y$, the transition $x \xrightarrow{A} y$ could be derived (looking forward) by an application of (CEM) on $v \xrightarrow{A^{'}} w$, since it could play a fundamental role in the derivation of the right premise of (CEM), i.e. the premise where two labels are identified. For instance, consider the following derivation:

$$\dfrac{\Gamma^{'}, x \xrightarrow{A^{'}} z \vdash \Delta, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \qquad \Gamma^{'}[z/u, y/u], x \xrightarrow{A^{'}} u \vdash \Delta[z/u, y/u], x \xrightarrow{A} u}{\Gamma^{'}, x \xrightarrow{A^{'}} z \vdash \Delta, x \xrightarrow{A} y}\ (CEM)$$

The transition $x \xrightarrow{A} y$ in the right-hand side of the conclusion of the rule is *not* the principal formula in it, however it could play a key role in the derivation of the right premise, where it is transformed in $x \xrightarrow{A} u$ by the label substitution: for instance, $x \xrightarrow{A^{'}} u$ and $x \xrightarrow{A} u$ can be used to apply (backward) the (EQ) rule. In this kind of situation, we say that $x \xrightarrow{A} y$ is *derived indirectly* by (CEM).

By this fact, we can conclude that a transition $x \xrightarrow{A} y$ in the right-hand side of a sequent can be derived (looking forward) by (EQ) or indirectly by (CEM) with the above meaning or it can be removed, by the admissibility of weakening (see Theorem 3.7). However, when $x \xrightarrow{A} y$ derives indirectly from an application of (CEM), in the left premise of the rule *it still appears in the right-hand side of the sequent*[8], therefore it can be derived by (EQ) or indirectly by (CEM) in the derivation of the left premise, and so on. Obviously, given a proof tree of $\Gamma \vdash \Delta, x \xrightarrow{A} y$, we can repeat this reasoning on each left premise of an application of (CEM) introducing indirectly $x \xrightarrow{A} y$, until finding that $x \xrightarrow{A} y$ is derived by an application of (EQ) or by weakening, since the proof tree is finite, as shown below[9]:

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ \Gamma \vdash \Delta_n, x \xrightarrow{A} y \quad \ldots \\ \vdots \\ \Gamma \vdash \Delta_2, x \xrightarrow{A} y \quad \ldots\end{array}}{\Gamma \vdash \Delta_1, x \xrightarrow{A} y \quad \ldots}\ (CEM)}{\Gamma \vdash \Delta, x \xrightarrow{A} y}\ (CEM)$$

In Π the transition $x \xrightarrow{A} y$ can be only introduced by weakening or by an application of (EQ) with a transition $x \xrightarrow{A'} y$ in the left-hand side of a sequent. In the first case, all instances of $x \xrightarrow{A} y$ can be removed; in the second case, we can conclude that the transition $x \xrightarrow{A'} y$ belongs to Γ, since we can reason as we made to prove Theorem 4.2: ($\Rightarrow$ R) is the only rule introducing (looking backward) a transition $x \xrightarrow{A'} y$ in the left-hand side of a sequent; moreover, y is a new label, then it is not possible that $x \xrightarrow{A'} y$ is introduced in Π, since y is already in all sequents; thus, $x \xrightarrow{A'} y \in \Gamma$.

The above discussion on how a transition formula in the right-hand side of a sequent can be derived in SeqCEM*{+ID} suggests properties stated by the following lemmas:

Lemma 4.12 (Controlled use of (CEM) (1)): SeqCEM*{+ID} is complete even if the (CEM) rule is applied with the following restrictions:

$$\dfrac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash \Delta[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\ (CEM)$$

[8] In the example, $x \xrightarrow{A} y$ is still in the right-hand side of the left premise $\Gamma', x \xrightarrow{A'} z \vdash \Delta, x \xrightarrow{A'} y, x \xrightarrow{A} y$.

[9] We can assume, without loss of generality, that all the applications of (CEM) are consecutive, since (CEM) is invertible and then we can permute them over the other rules.

1. $y \neq z$;

2. there exists a transition $x \xrightarrow{A'} z \in \Gamma$.

Proof. Consider the transition $x \xrightarrow{A} z$ introduced (backward) by the (CEM) rule. As explained above, it derives (forward) from an application of (EQ) or indirectly from one or more applications of (CEM). In the first case, a transition $x \xrightarrow{A'} z$ must belong to Γ and the restriction 2. is satisfied. In the second case, we have observed that, sooner or later, the (EQ) rule must be applied to close the left branch of several applications of (CEM), hence a transition $x \xrightarrow{A'} z$ must be in Γ too. If $x \xrightarrow{A} z$ is introduced by weakening, the application of (CEM) is useless. Notice that the restriction 1. is the initial restriction on the application of (CEM) given in SeqCEM{+ID}.

□

Lemma 4.13 (Controlled use of (⇒ L)): SeqCEM*{+ID} is complete even if the (⇒ L) rule is applied as follows:

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

by choosing a label y such that there is a transition $x \xrightarrow{A'} y \in \Gamma$.

Proof. We conclude exactly as we made for Lemma 4.12 above: $x \xrightarrow{A} y$ in the left premise can be only derived (forward) by weakening, thus the application of (⇒ L) is useless, or by (EQ) or, indirectly, by one or more applications of (CEM). In the last two cases, a transition $x \xrightarrow{A'} y$ must be in Γ, since the proof can only be closed by an application of (EQ).

□

To prove that SeqCEM*{+ID} ensure a terminating proof search we also need the following:

Lemma 4.14 (Controlled use of (CEM) (2)): It is useless to apply (CEM) to $x \xrightarrow{A} y$ by introducing the same transition $x \xrightarrow{A} z$ in the left premise of the rule more than once in a backward proof search.

Proof. Consider a proof where (CEM) is applied to $x \xrightarrow{A} y$ more than once by introducing the same transition $x \xrightarrow{A} z$; consider the two highest applications: since (CEM) is invertible, it permutes over the other rules, then we can consider, without loss of generality, the following proof (as usual, we denote with $\Sigma(u)$ the substitution $\Sigma[y/u, z/u]$):

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z, x \xrightarrow{A} z\end{matrix} \quad (\Gamma, x \xrightarrow{A} y)(u) \vdash (\Delta, x \xrightarrow{A} z)(u)}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z}\,(CEM) \quad \begin{matrix}\Pi_2\\ (\Gamma, x \xrightarrow{A} y)(u) \vdash \Delta(u)\end{matrix}}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(CEM)$$

By the admissibility of contraction (see Theorem 4.8), one can find a proof $\Pi_1^{'}$ of the sequent $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$, then we can conclude as follows, obtaining a proof where the upper application of (CEM) has been removed:

$$\dfrac{\begin{matrix}\Pi_1^{'}\\ \Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z\end{matrix} \quad \begin{matrix}\Pi_2\\ (\Gamma, x \xrightarrow{A} y)(u) \vdash \Delta(u)\end{matrix}}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(CEM)$$

□

By the above Lemmas 4.12, 4.13 and 4.14 we prove the decidability of CK+CEM{+ID}:

Theorem 4.15 (Terminating proof search for CK+CEM{+ID}): Systems SeqCEM{+ID} ensure a terminating proof search.

Proof. As we made for SeqG systems, we observe that premises in all the rules have smaller complexity than the conclusion, with the exceptions of (CEM) and (⇒ L). However, by Lemmas 4.12 and 4.14 one can control the application of (CEM) on $\Gamma, x \xrightarrow{A} y \vdash \Delta$ by using only transitions $x \xrightarrow{A} z$ where z is such that $x \xrightarrow{A'} z \in \Gamma$, and only once with the same transition. By Lemmas 4.13 and 4.1 one can control the application of (⇒ L) on $\Gamma, x : A \Rightarrow B \vdash \Delta$ by applying it by using only transitions $x \xrightarrow{A} y$ where y is such that $x \xrightarrow{A'} y \in \Gamma$, and only once with the same transition. Moreover, all the rules are analytic (notice that an application of (CEM) identifies two labels with the same *new* label u in the right premise, therefore the number of different labels in that branch decreases by 1). We can conclude that SeqCEM*{+ID} ensure a terminating proof search since labels are introduced only by negative conditionals, hence their number is finite; thus, the number of possible applications of (CEM) and (⇒ L) is finite too.

□

As in the case of CK{+ID}{+MP}, this itself gives decidability:

Theorem 4.16 (Decidability of CK+CEM{+ID}): Logic CK+CEM{+ID} is decidable.

We can easily extend results for space complexity given in the previous subsection to systems allowing (CEM):

Theorem 4.17 (Space complexity of CK+CEM{+ID}): Provability in CK+CEM{+ID} is decidable in $O(n^2 \ \log n)$ space.

4.3 Proof-theoretical analysis of CK+CS{+ID}{+MP,+CEM}

We conclude this section by giving the decidability of remaining cut-free systems, i.e. systems allowing the (CS) rule. As we made in the previous subsections, we have to show that one can apply the ($\Rightarrow$ L) rule in a controlled way, since it is the only rule of these calculi such that premises have higher complexities than the conclusion, determining a potential cause of a non terminating backward proof search. In systems SeqCEM+CS+{ID} we have also to prove that the application of (CEM) can be controlled[10].

In these systems we have the following problem: transitions are *transitive*, i.e. sequents of the form $\Gamma, x_0 \xrightarrow{A_1} x_1, x_1 \xrightarrow{A_2} x_2, ..., x_{n-1} \xrightarrow{A_n} x_n \vdash \Delta, x_0 \xrightarrow{A} x_n$ can be derived by applying the (CS) rule. Indeed, an application of (CS) on $x_{i-1} \xrightarrow{A_i} x_i$ has the effect of identifying labels x_{i-1} and x_i, therefore several applications of this rule lead to prove a "typical" sequent of the form $\Gamma', u \xrightarrow{A_n} x_n \vdash \Delta', u \xrightarrow{A} x_n$.
By this fact, it is easy to find the following potential cause of a non terminating proof search: one could prove the validity of $\Gamma, x \xrightarrow{A} y, y \xrightarrow{A} z, x : A \Rightarrow B \vdash \Delta$ by proving $\Gamma, x \xrightarrow{A} y, y \xrightarrow{A} z, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} z$ and $\Gamma, x \xrightarrow{A} y, y \xrightarrow{A} z, x : A \Rightarrow B, z : B \vdash \Delta$. Consider the following example:

Example 4.18:

$$\dfrac{\dfrac{\dfrac{... \vdash z : A, x : \top \qquad u : \top \Rightarrow A, u \xrightarrow{\top} u, u \xrightarrow{\top} z \vdash z : A, u \xrightarrow{\top} z}{x : \top \Rightarrow A, x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash z : A, x \xrightarrow{\top} z}(CS) \qquad ..., z : A \vdash z : A}{x : \top \Rightarrow A, x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash z : A}(\Rightarrow L)}{\dfrac{x : \top \Rightarrow A, x \xrightarrow{\top} y \vdash y : \top \Rightarrow A}{x : \top \Rightarrow A \vdash x : \top \Rightarrow (\top \Rightarrow A)}(\Rightarrow R)}(\Rightarrow R)$$

Without any control on this kind of situation, systems allowing (CS) do not offer a terminating proof search, since the label z, used in the above example, can be introduced (looking backward) by a nested negative conditional in $x : A \Rightarrow B$, as shown in the following example:

[10] In these systems we obviously refer to the reformulated rule (CEM) presented in Figure 6. It is easy to observe that one can extend results of the previous subsection even in presence of (CS) or (CS+ID).

Example 4.19:

$$
\dfrac{x \xrightarrow{\top} y \vdash x \xrightarrow{\top} y \qquad \dfrac{\dfrac{\dfrac{...x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash ..., x \xrightarrow{\top} z \qquad \begin{matrix}\Pi \\ x : \top \Rightarrow (\neg(\top \Rightarrow A)), ..., z : \neg(\top \Rightarrow A) \vdash y : A, z : A\end{matrix}}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash y : A, z : A}\,(\Rightarrow L)}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, \vdash y : A, y : \top \Rightarrow A}\,(\Rightarrow R)}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, y : \neg(\top \Rightarrow A) \vdash y : A}\,(\neg L)}{\dfrac{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y \vdash y : A}{x : \top \Rightarrow (\neg(\top \Rightarrow A)) \vdash x : \top \Rightarrow A}\,(\Rightarrow R)}\,(\Rightarrow L)
$$

In the above example the proof search does not ensure termination, since we have a loop: z is used to apply ($\Rightarrow$ L) on $x : \top \Rightarrow (\neg(\top \Rightarrow A))$, and it has been introduced (looking backward) by an application of ($\Rightarrow$ R) on $y : \top \Rightarrow A$, a nested conditional of the same formula $x : \top \Rightarrow (\neg(\top \Rightarrow A))$. It is easy to observe that the machinery goes on in Π, where the negative conditional $z : \neg(\top \Rightarrow A)$ introduces (backward) a transition $z \xrightarrow{\top} w$ and ($\Rightarrow$ L) could be applied by using $x \xrightarrow{\top} w$.

As mentioned above, transitivity can be proved by the following fact: a transition in the right-hand side of a sequent can be derived indirectly (looking forward) by an application of (CS), since (CS) identifies two labels in its right premise. Indeed, to prove $x \xrightarrow{A} y, y \xrightarrow{A} z \vdash x \xrightarrow{A} z$, for instance one can apply (CS) on $x \xrightarrow{A} y$: the right premise $u \xrightarrow{A} u, u \xrightarrow{A} z \vdash u \xrightarrow{A} z$ is derivable *by the identification of labels x and y with u*. However, the transition $x \xrightarrow{A} z$ is maintained in the left premise, where it can only be introduced by an application of (EQ) or by weakening[11]. Therefore, one can restrict the application of ($\Rightarrow$ L) on $\Gamma, x : A \Rightarrow B \vdash \Delta$ only by using a transition $x \xrightarrow{A} y$ such that $x \xrightarrow{A'} y \in \Gamma$. The intuition is as follows: if one needs to propagate a conditional $x : A \Rightarrow B$ from x to y, and then to z by an application of (CS), where (CS) has the effect of identifying x and y, then one can *first* identify labels x and y with u by (CS), and *then* propagate the conditional $u : A \Rightarrow B$ from u to z by an application of ($\Rightarrow$ L).

Lemma 4.20 (Controlled use of ($\Rightarrow$ L)): SeqCS{+ID}{+MP,+CEM} are complete even if the ($\Rightarrow$ L) rule is applied as follows:

$$
\dfrac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta}\,(\Rightarrow L)
$$

[11] It could be the case when $x \xrightarrow{A} z$ also derives indirectly by (CS) in the left premise, for instance to derive $\Gamma', x \xrightarrow{B'} y, y \xrightarrow{B''} z \vdash \Delta', x \xrightarrow{A} z$ by applying (CS) on $x \xrightarrow{B'} y$. In this case, the application of (CS) on $x \xrightarrow{A'} y$ is useless, since one can replace this application by applying (CS) only on $x \xrightarrow{B'} y$.

by choosing a label y such that there is a transition $x \xrightarrow{A'} y \in \Gamma$. In systems allowing the (MP) rule, one can apply ($\Rightarrow$ L) by using the same label x.

Proof. Consider a proof tree where ($\Rightarrow$ L) is applied by proving the transitivity of transitions; the situation is as follows (as usual, we denote with $\Sigma(u)$ the substitution $\Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} z, x : A'} \quad \dfrac{\Pi_2}{\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B \vdash \Delta(u), u \xrightarrow{A} z}}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} z}\,(CS) \quad \dfrac{\Pi_3}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B, z : B \vdash \Delta}}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta}\,(\Rightarrow L)$$

We show that there exists a proof tree of $\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta$ where ($\Rightarrow$ L) is applied without proving transitivity of transitions. If $x \xrightarrow{A} z$ derives in Π_1 by an application of (EQ), then a transition $x \xrightarrow{C} z$ belongs to the left-hand side of the sequent, therefore ($\Rightarrow$ L) is applied respecting the restriction stated by this Lemma. Otherwise, $x \xrightarrow{A} z$ can be removed in Π_1, since it is introduced by weakening. Therefore, there is a proof $\Pi_1^{'}$ of the sequent $\Gamma, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x : A'$. Moreover, by applying the label substitution (Lemma 3.6) to the sequent derived by proof Π_3 one can obtain a proof $\Pi_3^{'}$ of $\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B, z : B \vdash \Delta(u)$. We can conclude by applying first the (CS) rule (to identify labels x and y with u) and then by propagating the conditional formula from u to z, as follows:

$$\dfrac{\dfrac{\Pi_1^{'}}{\Gamma, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x : A'} \quad \dfrac{\dfrac{\Pi_2}{\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B \vdash \Delta(u), u \xrightarrow{A} z} \quad \dfrac{\Pi_3^{'}}{\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B, z : B \vdash \Delta(u)}}{\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B \vdash \Delta(u)}\,(\Rightarrow L)}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta}\,(CS)$$

We proceed in the same manner if we have a proof where (CS) is applied on $y \xrightarrow{A''} z$ and also when the transition $x_0 \xrightarrow{A} x_n$ is used to apply ($\Rightarrow$ L) on $\Gamma, x_0 \xrightarrow{A_1} x_1, x_1 \xrightarrow{A_2} x_2, ..., x_{n-1} \xrightarrow{A_n} x_n$.

□

Now we have all the elements to prove the following:

Theorem 4.21 (Terminating proof search for CK+CS{+ID}{+MP,+CEM}): Systems SeqCS{+ID}{+MP,+CEM} ensure a terminating proof search.

Proof. One can control the application of ($\Rightarrow$ L) by Lemma 4.20 above. In the case of SeqCEM+CS{+ID} just observe that one can control the application of (CEM) in this way: one needs to apply (CEM) on $\Gamma, x \xrightarrow{A} y \vdash \Delta$ by using a transition $x \xrightarrow{A} z$ (i.e. premises are $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ and $(\Gamma, x \xrightarrow{A} y)[y/u, z/u] \vdash \Delta[y/u, z/u]$) such that $x \xrightarrow{A'} z \in \Gamma$, since Lemma 4.12 holds in these systems too. Moreover, one needs to apply (CEM) at most once by using the same transition $x \xrightarrow{A} z$, since we can easily observe that Lemma 4.14 holds in these systems[12]. The number of applications of ($\Rightarrow$ L) and (CEM) is finite, since the number of transitions introduced by negative conditionals in a backward proof search is finite.

□

As in the previous cases, this itself gives decidability:

Theorem 4.22 (Decidability of CK+CS{+ID}{+MP,+CEM}): Logics CK+CS{+ID}{+MP,+CEM} are decidable.

We conclude by giving an explicit space complexity bound:

Theorem 4.23 (Space complexity of CK+CS{+ID}{+MP,+CEM}): Provability in CK+CS{+ID}{+MP,+CEM} is decidable in $O(n^2 \ \log n)$ space.

5 Refinements and other properties for CK{+ID}{+MP}

In this section we analyze the sequent calculus SeqG for CK{+ID}{+MP} in order to obtain a decision procedure for these conditional systems. G systems have other remarkable properties, such as the so-called *disjunction property* for conditional formulas: if $(A_1 \Rightarrow B_1) \vee (A_2 \Rightarrow B_2)$ is valid, then either $(A_1 \Rightarrow B_1)$ or $(A_2 \Rightarrow B_2)$ is valid too. This property follows an important proposition, which does not hold for all sequents, but only for non-x-branching sequents, i.e. those sequents which do not create a branching in x or in a predecessor of x in a derivation of the sequent.

Let us introduce some essential definitions, first of all the notion of *regular sequent*. Intuitively, regular sequents are those sequents whose set of transitions in the antecedent forms a *forest*. As we show in Theorem 5.3 below, any sequent in a proof beginning with a sequent of the form $\vdash x_0 : D$, for an arbitrary formula D, is regular. For this reason, we will restrict our concern to regular sequents.

We define the multigraph $\mathcal{G}$ of the transition formulas in the antecedent of a sequent:

Definition 5.1 (Multigraph of transitions $\mathcal{G}$): Given a sequent $\Gamma \vdash \Delta$, where $\Gamma = \Gamma', T$ and T is the multiset of transition formulas and Γ' does not contain transition formulas, we define the multigraph $\mathcal{G} =< V, E >$ associated to $\Gamma \vdash \Delta$ with vertexes V and edges E. V is the set of labels occurring in $\Gamma \vdash \Delta$ and $< x, y > \in E$ whenever $x \xrightarrow{F} y \in T$.

[12] We can repeat the same proof of the theorem even if we have (CS), since (CEM) is invertible.

Definition 5.2 (Regular sequent): A sequent $\Gamma \vdash \Delta$ is called regular if its associated multigraph of transitions $\mathcal{G}$ is a forest. In particular, there is at most one link between two vertexes and there are no loops.

We can observe that we can always restrict our concern to regular sequents, since we have the following theorem:

Theorem 5.3 (Proofs with regular sequents): Every proof tree with a sequent $\vdash x_0 : D$ as root and obtained by applying backward SeqG's rules contains only regular sequents.

Proof. First, we show that $\mathcal{G}$ is a *graph*, i.e. there is at most one link between two vertexes. This can be seen by an easy inductive argument on the distance of a sequent in the proof tree from its root: $\vdash x_0 : D$, the root itself, obviously respects this condition. For the inductive step, we consider the case when a SeqG's rule is applied, in the typical bottom-up way, to an arbitrary $\Gamma \vdash \Delta$ which respects this condition, then we conclude observing that the theorem holds for the sequent(s) in the premise(s) of the rule. ($\Rightarrow$ R) is the only rule of the calculus which introduces, looking backward, a transition formula in the antecedent of the sequent to which it is applied. In particular, ($\Rightarrow$ R) with principal formula $x : A \Rightarrow B$ introduces a transition $x \xrightarrow{A} y$ in a premise where y is a "new label", then there cannot be another transition $x \xrightarrow{F} y$ in the antecedent.
To see that $\mathcal{G}$ is a forest, again we do a simple inductive argument on the distance of a sequent in the proof tree from the root: the graph associated to $\vdash x_0 : D$ is certainly a forest ($\mathcal{G}$=<$\{x_0\}$,$\emptyset$ >, which is a tree); consider a rule application which has $\Gamma_1 \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2$ as premises and $\Gamma \vdash \Delta$ as a conclusion, and assume by induction hypothesis that the graph $\mathcal{G}$ associated to $\Gamma \vdash \Delta$ is a forest. It is easy to observe that applying any rule of SeqG, the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, associated to $\Gamma_1 \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2$ respectively, are forests. ($\Rightarrow$ L), ($\rightarrow$ L), ($\rightarrow$ R) and (MP) do not modify the graph $\mathcal{G}$; ($\Rightarrow$ R) adds a transition $x \xrightarrow{F} y$ in the initial forest, but y is a "new" label as discussed above, thus $\mathcal{G}_1$ is still a forest obtained by adding a new vertex and a new edge; consider the (ID) rule:

$$\frac{\Gamma, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\ (ID)$$

The application of (ID) deletes the edge $< x, y >$ from $\mathcal{G}$, and $\mathcal{G}_1$ is still a forest. When (EQ) is applied, the calculus tries to find two derivations starting with only one label u and no transitions; therefore, $\mathcal{G}_1$ and $\mathcal{G}_2$ are trees $<\{u\}, \emptyset >$.

□

As mentioned above, we restrict from now on our attention to regular sequents. Notice that the above Theorem 5.3 does not hold in systems with CEM or CS: indeed, if (CEM) or (CS) is applied (looking backward) to $\Gamma, x \xrightarrow{A} y \vdash \Delta$, then

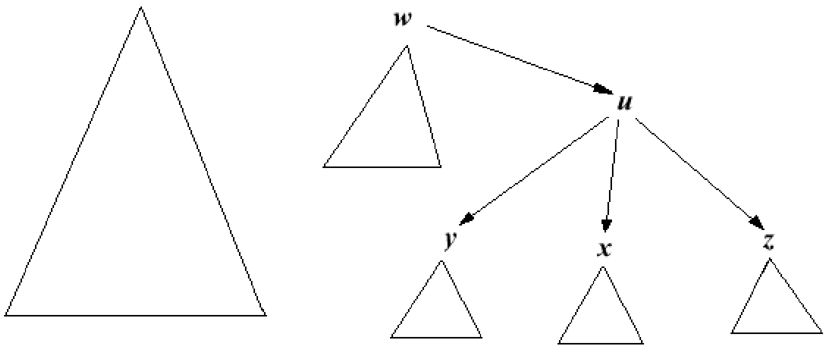

Fig. 7: The forest $\mathcal{G}$ of a sequent; w is predecessor of x, x is successor of w, u is father of x and x is son of u.

the labels y and z or y and x are identified by a new label u in the right premise, and the resulting graph of transitions is not a forest. In the following, we prove some elementary properties of regular sequents.

We complete the description of the forest of transitions with the following elementary definitions:

Definition 5.4 (Predecessor and successor, father and son): Given a sequent $\Gamma^{'}, T \vdash \Delta$, where all the transitions in the antecedent are in T, we say that a world w is a predecessor of a world x if there is a path from w to x in the graph of transitions $\mathcal{G} =< V, E >$ of the sequent. In this case, we also say that x is a successor of w. If $< w, x >\in E$, we say that w is the father of x and that x is a son of w.

As we mentioned above, the graph of transitions of regular sequents forms a forest, as shown in Figure 7.

We need some more definitions:

Definition 5.5 (Positive and negative occurrences of a world formula): Given a world formula $x : A$, we say that:

- $x : A$ occurs positively in $x : A$;
- if a world formula $x : B \rightarrow C$ occurs positively (negatively) in $x : A$, then $x : C$ occurs positively (negatively) in $x : A$ and $x : B$ occurs negatively (positively) in $x : A$;
- if a formula $x : B \Rightarrow C$ occurs positively (negatively) in $x : A$, then $x : C$ occurs positively (negatively) in $x : A$.

A world formula $x : A$ occurs positively (negatively) in a multiset Γ if $x : A$ occurs positively (negatively) in some world formula $x : G \in \Gamma$.

Now we introduce the definition of x-branching formula. Intuitively, $\mathcal{B}(x, T)$ contains formulas that create a branching in x or in a predecessor of x according to T in a derivation of a sequent. For instance, consider the sequent $x : A, x : A \rightarrow B \vdash x : B$, obviously valid in CK. $x : A \rightarrow B$ is an x-branching formula, since it creates a branching in x in a derivation of the sequent, as shown in the following proof tree:

$$\dfrac{x : A \vdash x : A, x : B \qquad x : A, x : B \vdash x : B}{x : A, x : A \rightarrow B \vdash x : B}\ (\rightarrow L)$$

$\mathcal{B}(x, T)$ also contains the conditionals $u : A \Rightarrow B$ such that $T \vdash u \xrightarrow{A} v$ and B creates a branching in x (i.e. $v = x$) or in a predecessor v of x.

Definition 5.6 (x-branching formulas): Given a multiset of transition formulas T, we define the set of x-branching formulas, denoted with $\mathcal{B}(x,\ T)$, as follows:

- $x : A \rightarrow B \in \mathcal{B}(x, T)$;
- $u : A \rightarrow B \in \mathcal{B}(x, T)$ if $T \vdash u \xrightarrow{F} x$ for some formula F;
- $u : A \Rightarrow B \in \mathcal{B}(x, T)$ if $T \vdash u \xrightarrow{A} v$ and $v : B \in \mathcal{B}(x, T)$.

We also introduce the notion of x-branching sequent. Intuitively, we say that $\Gamma \vdash \Delta$ is x-branching if it contains an x-branching formula occurring positively in Γ or if it contains an x-branching formula occurring negatively in Δ.

Consider the following proof of $\vdash x : (A \wedge B) \rightarrow (B \Rightarrow ((B \rightarrow \bot) \rightarrow \bot))$, valid in CK+ID:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : A, x : B, y : B \vdash y : B, y : \bot}{x : A, x : B, x \xrightarrow{B} y \vdash y : B, y : \bot}\ (ID) \qquad x : A, x : B, x \xrightarrow{B} y, y : \bot \vdash y : \bot}{x : A, x : B, x \xrightarrow{B} y, y : B \rightarrow \bot \vdash y : \bot}\ (\rightarrow L)}{x : A \wedge B, x \xrightarrow{B} y, y : B \rightarrow \bot \vdash y : \bot}\ (\wedge L)}{x : A \wedge B, x \xrightarrow{B} y \vdash y : (B \rightarrow \bot) \rightarrow \bot}\ (\rightarrow R)}{x : A \wedge B \vdash x : B \Rightarrow ((B \rightarrow \bot) \rightarrow \bot)}\ (\Rightarrow R)}{\vdash x : (A \wedge B) \rightarrow (B \Rightarrow ((B \rightarrow \bot) \rightarrow \bot))}\ (\rightarrow R)$$

The sequent $x : A \wedge B, x \xrightarrow{B} y, y : B \rightarrow \bot \vdash y : \bot$ is *not* x-branching, since no formula creates branching in the backward proof search on a path to x or in x; however, it is y-branching by the formula $y : B \rightarrow \bot$, which creates a branch on y in the derivation.

Since in systems containing (ID) a transition $u \xrightarrow{F} v$ in the antecedent can be derived, looking forward, from $v : F$ and $v : F$ can be x-branching, we impose that a sequent $\Gamma', u \xrightarrow{F} v \vdash \Delta$ is x-branching if $\Gamma', v : F \vdash \Delta$ is

x-branching; for the same reason, in systems containing (MP) we impose that a sequent $\Gamma \vdash \Delta^{'}, u \xrightarrow{F} u$ is x-branching if $\Gamma \vdash \Delta^{'}, u : F$ is x-branching.
In systems containing (MP) we also impose that a sequent $\Gamma^{'}, w : A \Rightarrow B \vdash \Delta$ is x-branching if the sequent $\Gamma^{'} \vdash \Delta, w \xrightarrow{A} w$ is derivable and w is a predecessor of x (or $w = x$), since $w : A$ can introduce x-branching formula(s) in the sequent.

Definition 5.7 (x-branching sequents): Given a sequent $\Gamma \vdash \Delta$, we denote by $\Gamma^{'}$ the world formulas in Γ and by T the transition formulas in Γ, so that Γ=$\Gamma^{'}$, T. To define when a sequent $\Gamma \vdash \Delta$ is x-branching according to each system, we consider the following conditions:

1. a world formula $u : F \in \mathcal{B}(x, T)$ occurs positively in Γ;
2. a world formula $u : F \in \mathcal{B}(x, T)$ occurs negatively in Δ.
3. T=$T^{'}, u \xrightarrow{F} v$ and the sequent $\Gamma^{'}, T^{'}, v : F \vdash \Delta$ is x-branching;
4. $u \xrightarrow{F} u \in \Delta$ and the sequent $\Gamma \vdash \Delta^{'}, u : F$ is x-branching ($\Delta = \Delta^{'}, u \xrightarrow{F} u$);
5. a formula $w : A \Rightarrow B \in \Gamma$, w is a predecessor of x in the graph $\mathcal{G}$ of transitions or $w = x$ and $\Gamma^{''} \vdash \Delta, w \xrightarrow{A} w$ is derivable, where $\Gamma = \Gamma^{''}, w : A \Rightarrow B$.

We say that $\Gamma \vdash \Delta$ is x-branching for each system if the following combinations of the previous conditions hold:

- CK: 1, 2
- CK+ID: 1, 2, 3
- CK+MP: 1, 2, 4, 5
- CK+MP+ID: 1, 2, 3, 4, 5

As anticipated at the beginning of this section, the disjunction property is *only* applicable to sequents that are not x-branching. The reason is twofold: on the one hand, only the formulas on the path from x going backwards through the transition formulas (i.e. on the worlds $u_1 \xrightarrow{A_1} u_2 \xrightarrow{A_2} \ldots \xrightarrow{A_n} x$) can contribute to a proof of a formula with label x. This is proved by Proposition 5.12 below. On the other hand, no formula on that path can create a branching in the derivation. As an example, consider the x-branching sequent $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B, x : C \Rightarrow D$; it is valid in CK, but neither $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B$ nor $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : C \Rightarrow D$ are valid.

To prove the disjunction property, we need to consider a more general setting; namely, we shall consider a sequent of the form $\Gamma \vdash \Delta, y : A, z : B$, whose forest of transitions has the form represented in Figure 8, i.e. it has one subtree with

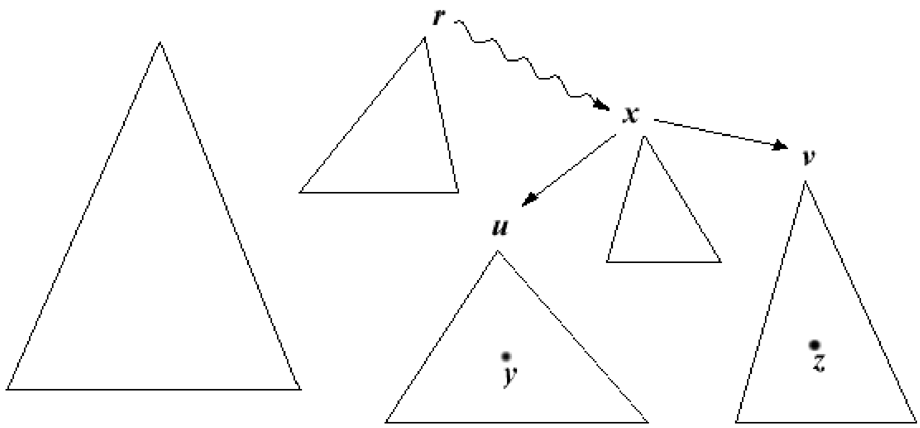

Fig. 8: The forest $\mathcal{G}$ of transitions used to prove the disjunction property.

root u and another subtree with root v, with $u \neq v$; y is a member of the tree with root u and z is a member of the tree with root v; x is the father of u and v and the tree containing x has root r.

We need some more definitions. In particular, given a sequent $\Gamma \vdash \Delta$ with its associated forest $\mathcal{G} =< V, E >$, consider a label k contained in the tree with root r. We define the set T°_k of the labels contained in the tree of $\mathcal{G}$ with root k and the sets Γ°_k and Δ°_k, containing all the labelled formulas of Γ and Δ whose labels are in the tree of $\mathcal{G}$ with root k; from now on, we denote with $\mathcal{G}(k)$ the tree of $\mathcal{G}$ with root k. We also define the set T^{*}_k of the labels contained in $\mathcal{G}(k)$ or in a path from r to k, and the sets Γ^{*}_k and Δ^{*}_k, containing all the labelled formulas of Γ and Δ whose labels are in $\mathcal{G}(k)$ or on the path from r to k.

Definition 5.8 (T°_k): T°_k is the set of labels in $\mathcal{G}(k)$; more precisely:

- $k \in T^{\circ}_k$
- if $< u, w > \in E$ and $u \in T^{\circ}_k$, then $w \in T^{\circ}_k$.

Definition 5.9 (T^{*}_k): T^{*}_k is the set of labels in $\mathcal{G}(k)$ or on a path from r[13] to k; more precisely:

$$T^{*}_k = T^{\circ}_k \cup P_k$$

where P_k it the set of labels on a path from r to k:

- $k \in P_k$
- if $< u, w > \in E$ and $w \in P_k$, then $u \in P_k$.

For a multiset of labelled formulae Σ we define Σ°_k and Σ^{*}_k.

[13] The label r is the root of the tree containing k, e.g. $k \in \mathcal{G}(r)$. It could be r=k.

Definition 5.10 (Σ_k°): Σ_k° is the multiset of labelled formulas of Σ contained in $\mathcal{G}(k)$; if $w \xrightarrow{F} k \in \Sigma$, then $w \xrightarrow{F} k \in \Sigma_k^\circ$; more precisely:

$$\Sigma_k^\circ = \{u : F \in \Sigma \mid u \in T_k^\circ\} \cup \{w \xrightarrow{F} u \in \Sigma \mid w \in T_k^\circ \text{ or } u \in T_k^\circ\}$$

Definition 5.11 (Σ_k^*): Σ_k^* is the multiset of labelled formulas of Σ contained in $\mathcal{G}(k)$ or on a path from r to k; more precisely:

$$\Sigma_k^* = \{w : F \in \Sigma \mid w \in T_k^*\} \cup \{v \xrightarrow{F} w \in \Sigma \mid w \in T_k^*\}$$

Now we have all the elements to prove the following proposition; intuitively, it says that a proof of a sequent $\Gamma \vdash \Delta, y : A, z : B$ whose forest $\mathcal{G}$ has form as in Figure 8 can be divided into three parts: one part that involves only formulae from $\mathcal{G}(u)$, another part that involves only formulae from $\mathcal{G}(v)$ and the other one that involves only formulae from the rest of $\mathcal{G}$; that is, the derivation can be related to the location of the labels (in the distinct subtrees $\mathcal{G}(u)$ and $\mathcal{G}(v)$ or in the rest of $\mathcal{G}$). This proposition is also crucial to prove the Lemma 5.15 below, which lets us to present a better space complexity bound for CK{+ID}.

Proposition 5.12: Given a sequent $\Gamma \vdash \Delta, y : A, z : B$ and its forest of transitions $\mathcal{G}$, if it is derivable in SeqG and has the following features:

1. $\mathcal{G}$ is a forest of the form as shown in Figure 8 (thus y is a member of $\mathcal{G}(u)$ and z is a member of $\mathcal{G}(v)$, with $u \neq v$; u and v are sons of x);
2. $\Gamma \vdash \Delta, y : A, z : B$ is not x-branching

then one of the following sequents is derivable in SeqG:

1. $\Gamma_u^* \vdash \Delta_u^*, y : A$
2. $\Gamma_v^* \vdash \Delta_v^*, z : B$
3. Γ - $(\Gamma_u^\circ \cup \Gamma_v^\circ) \vdash \Delta$ - $(\Delta_u^\circ \cup \Delta_v^\circ)$

Moreover, the proofs of 1, 2, and 3 do not add any application of ($\Rightarrow$ L) to the proof of $\Gamma \vdash \Delta, y : A, z : B$.

Proof. By induction on the height of the proof tree of the sequent $\Gamma \vdash \Delta, y : A, z : B$. We present two examples, the other cases are left to the reader.

1. Consider the case where $y : A$ is a conditional formula $y : C \Rightarrow D$ and is the principal formula of an application of the ($\Rightarrow$ R) rule. The proof tree of the sequent is ended by:

$$\frac{\Gamma, y \xrightarrow{C} k \vdash \Delta, k : D, z : B}{\Gamma \vdash \Delta, y : C \Rightarrow D, z : B} (\Rightarrow R)$$

We can apply the inductive hypothesis on the only premise of the ($\Rightarrow$ R) rule; indeed, $\Gamma, y \xrightarrow{C} k \vdash \Delta, k : D, z : B$ is not x-branching. It could become x-branching as an effect of the introduction of $k : D$ and $y \xrightarrow{C} k$, but this is impossible since k is a "new" label, then it is in the same tree of y and not on a path to x. Applying the inductive hypothesis, we must consider the three possible situations:

(a) $(\Gamma,\ y \xrightarrow{C} k)^*_u \vdash \Delta^*_u,\ k\!:\ D$ is derivable: it is easy to see that $y \xrightarrow{C} k \in (\Gamma,\ y \xrightarrow{C} k)^*_u$, since u is a predecessor of y and thus of k; then we obtain the following derivation:

$$\frac{\Gamma^*_u, y \xrightarrow{C} k \vdash \Delta^*_u, k : D}{\Gamma^*_u \vdash \Delta^*_u, y : C \Rightarrow D}\ (\Rightarrow R)$$

(b) $(\Gamma,\ y \xrightarrow{C} k)^*_v \vdash \Delta^*_v,\ z\!:\ B$ is derivable: k is in $\mathcal{G}(u)$, thus $y \xrightarrow{C} k \notin (\Gamma,\ y \xrightarrow{C} k)^*_v$: we obtain that

$$\Gamma^*_v \vdash \Delta^*_v, z : B$$

is derivable;

(c) $(\Gamma,\ y \xrightarrow{C} k)$ - $((\Gamma,\ y \xrightarrow{C} k)^\circ_u \cup (\Gamma,\ y \xrightarrow{C} k)^\circ_v) \vdash \Delta$ - $(\Delta^\circ_u \cup \Delta^\circ_v)$ is derivable: k is in $\mathcal{G}(u)$, thus $y \xrightarrow{C} k \in (\Gamma,\ y \xrightarrow{C} k)^\circ_u$ and then $y \xrightarrow{C} k \notin (\Gamma,\ y \xrightarrow{C} k)$ - $((\Gamma,\ y \xrightarrow{C} k)^\circ_u \cup (\Gamma,\ y \xrightarrow{C} k)^\circ_v)$, from which we obtain that

$$\Gamma - (\Gamma^\circ_u \cup \Gamma^\circ_v) \vdash \Delta - (\Delta^\circ_u \cup \Delta^\circ_v)$$

is derivable.

2. Let us now analyze the case where the principal formula of the sequent is a conditional formula $x : C \Rightarrow D \in \Gamma$; the ($\Rightarrow$ L) rule is applied to that formula by using the transition $x \xrightarrow{C} u$, as shown below:

$$\frac{\Gamma', x : C \Rightarrow D \vdash \Delta, y : A, z : B, x \xrightarrow{C} u \qquad \Gamma', x : C \Rightarrow D, u : D \vdash \Delta, y : A, z : B}{\Gamma', x : C \Rightarrow D \vdash \Delta, y : A, z : B}\ (\Rightarrow L)$$

We can obviously apply the inductive hypothesis on the premises. For the left premise we have the following alternatives:

- $(1a)$ $\Gamma'^*_u, x : C \Rightarrow D \vdash \Delta^*_u, y : A, x \xrightarrow{C} u$ is derivable (notice that $x : C \Rightarrow D$ and $x \xrightarrow{C} u$ belongs to $(\Gamma', x : C \Rightarrow D)^*_u$ and $(\Delta, x \xrightarrow{C} u)^*_u$, respectively);
- $(1b)$ $\Gamma'^*_v, x : C \Rightarrow D \vdash \Delta^*_v, z : B$ is derivable;
- $(1c)$ $\Gamma' - (\Gamma'^\circ_u \cup \Gamma'^\circ_v), x : C \Rightarrow D \vdash \Delta - (\Delta^\circ_u \cup \Delta^\circ_v)$ is derivable.

For the right premise we have the following alternatives:

- $(2a)$ $\Gamma_u^{'*}, x : C \Rightarrow D, u : D \vdash \Delta_u^*, y : A$ is derivable;
- $(2b)$ $\Gamma_v^{'*}, x : C \Rightarrow D \vdash \Delta_v^*, z : B$ is derivable;
- $(2c)$ $\Gamma' - (\Gamma_u^{'\circ} \cup \Gamma_v^{'\circ}), x : C \Rightarrow D \vdash \Delta - (\Delta_u^\circ \cup \Delta_v^\circ)$ is derivable.

We have to prove that one of the following sequents:

- (i) $\Gamma_u^{'*}, x : C \Rightarrow D \vdash \Delta_u^*, y : A$;
- (ii) $\Gamma_v^{'*}, x : C \Rightarrow D \vdash \Delta_v^*, z : B$;
- (iii) $\Gamma' - (\Gamma_u^{'\circ} \cup \Gamma_v^{'\circ}), x : C \Rightarrow D \vdash \Delta - (\Delta_u^\circ \cup \Delta_v^\circ)$.

is derivable in all possible situations (i.e. $(1a)-(2a), (1a)-(2b), ..., (1c)-(2c)$)). If we have $(1b)$, then the proof is over, since we have that (ii) is derivable. The same in the case of $(2b)$.

The proof is also over if we have $(1c)$ or $(2c)$, since (iii) is derivable.

We have only to analyze the case $(1a)-(2a)$; we conclude with the following proof:

$$\frac{(1a)\Gamma_u^{'*}, x : C \Rightarrow D \vdash \Delta_u^*, y : A, x \xrightarrow{C} u \qquad (2a)\Gamma_u^{'*}, x : C \Rightarrow D, u : D \vdash \Delta_u^*, y : A}{(i)\Gamma_u^{'*}, x : C \Rightarrow D \vdash \Delta_u^*, y : A}\,(\Rightarrow L)$$

In this case, we introduce a $(\Rightarrow \mathrm{L})$ to prove the Proposition; however, this application is already in the proof tree of the initial sequent, thus we do not *add* application of $(\Rightarrow \mathrm{L})$ on it (we use the same application on $x : C \Rightarrow D$).

□

Theorem 5.13 (Disjunction property): Given a non x-branching sequent

$$\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1, x : A_2 \Rightarrow B_2$$

derivable in SeqG with a derivation Π, one of the following sequents:

1. $\Gamma \vdash \Delta,\ x : A_1 \Rightarrow B_1$
2. $\Gamma \vdash \Delta,\ x : A_2 \Rightarrow B_2$

is derivable in SeqG.

Proof. The sequent $\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1, x : A_2 \Rightarrow B_2$ is derivable in SeqG, then we can find a derivation Π of it; the two conditional formulas can be introduced (looking forward) in two ways:

- in an axiom $\Gamma_1 \vdash \Delta_1, x : A_i \Rightarrow B_i$, where $\Gamma_1 \vdash \Delta_1$ is an axiom too;
- by the application of the $(\Rightarrow \mathrm{R})$ rule.

In the first case suppose that x: $A_1 \Rightarrow B_1$ descends (looking forward) from $\Gamma_1 \vdash \Delta_1, x : A_1 \Rightarrow B_1$, and $\Gamma_1 \vdash \Delta_1$ is an axiom: the proof is ended by erasing all the instances of x: $A_1 \Rightarrow B_1$ descending from $\Gamma_1 \vdash \Delta_1, x : A_1 \Rightarrow B_1$ in Π, obtaining a proof of $\Gamma \vdash \Delta$, x: $A_2 \Rightarrow B_2$.
In the second case both the conditional formulas are introduced, again looking forward, by an application of the ($\Rightarrow$ R) rule; by the permutability of this rule over all the others, we can consider a proof tree ending as follows:

$$\cfrac{\cfrac{\Gamma, x \xrightarrow{A_1} y, x \xrightarrow{A_2} z \vdash \Delta, y : B_1, z : B_2}{\Gamma, x \xrightarrow{A_1} y \vdash \Delta, y : B_1, x : A_2 \Rightarrow B_2}\ (\Rightarrow R)}{\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1, x : A_2 \Rightarrow B_2}\ (\Rightarrow R)$$

The sequent $\Gamma, x \xrightarrow{A_1} y, x \xrightarrow{A_2} z \vdash \Delta, y : B_1, z : B_2$ respects all the conditions to apply the Proposition 5.12 (y and z are "new" labels), then we have that one of the following sequents:

1. $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^*_y \vdash \Delta^*_y,\ y\text{: } B_1$
2. $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^*_z \vdash \Delta^*_z,\ z\text{: } B_2$
3. $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)$ - $((\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_y \cup (\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_z)$ $\vdash \Delta$ - $(\Delta^\circ_y \cup \Delta^\circ_z)$

is derivable in SeqG.
In all these cases we can prove the disjunction property:

1. z is not in $\mathcal{G}(y)$, and is not on a path towards y, then the transition formula $x \xrightarrow{A_2} z$ is not member of the multiset $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^*_y$; the sequent $\Gamma^*_y,\ x \xrightarrow{A_1} y \vdash \Delta^*_y,\ y$: B_1 is then derivable, from what we have a proof:

$$\cfrac{\Gamma^*_y, x \xrightarrow{A_1} y \vdash \Delta^*_y, y : B_1}{\Gamma^*_y \vdash \Delta^*_y, x : A_1 \Rightarrow B_1}\ (\Rightarrow R)$$

   and, since $\Gamma^*_y \subseteq \Gamma$ and $\Delta^*_y \subseteq \Delta$, we obtain a proof of $\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1$ simply by adding all formulas of $\Gamma - \Gamma^*_y$ and $\Delta - \Delta^*_y$ by the admissibility of weakening (Theorem 3.7);

2. symmetric to the previous case;
3. we can observe that $x \xrightarrow{A_1} y \in (\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_y$, and that $x \xrightarrow{A_2} z \in (\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_z$; both the transition formulas are members of $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_y \cup (\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^\circ_z$ and then they are *not*

members of $(\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)$ - $((\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^{\circ}_{y} \cup (\Gamma,\ x \xrightarrow{A_1} y,\ x \xrightarrow{A_2} z)^{\circ}_{z})$. Therefore, the sequent Γ - $(\Gamma^{\circ}_{y} \cup \Gamma^{\circ}_{z}) \vdash \Delta$ - $(\Delta^{\circ}_{y} \cup \Delta^{\circ}_{z})$ is derivable and, observing that Γ - $(\Gamma^{\circ}_{y} \cup \Gamma^{\circ}_{z}) \subseteq \Gamma$ and that Δ - $(\Delta^{\circ}_{y} \cup \Delta^{\circ}_{z}) \subseteq \Delta$, we can add all formulas of $(\Gamma^{\circ}_{y} \cup \Gamma^{\circ}_{z})$, of $(\Delta^{\circ}_{y} \cup \Delta^{\circ}_{z})$ and an instance of $x : A_1 \Rightarrow B_1$ or $x : A_2 \Rightarrow B_2$ by Theorem 3.7, obtaining a proof of $\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1$ or $\Gamma \vdash \Delta, x : A_2 \Rightarrow B_2$, respectively.

□

By the correctness and completeness of SeqG, it is easy to prove the following corollary of the disjunction property:

Corollary 5.14: If $(A \Rightarrow B) \vee (C \Rightarrow D)$ is valid in CK{+MP}{+ID}, then either $A \Rightarrow B$ or $C \Rightarrow D$ is valid in CK{+MP}{+ID}.

5.1 Refinements for CK{+ID}

In this subsection we give a further refinement for the basic conditional logic CK and its extension CK+ID. The Proposition 5.12 suggests the following fact: in SeqCK and SeqID systems, it is useless to apply ($\Rightarrow$ L) on the same formula $x : A \Rightarrow B$ *by using more than one transition* $x \xrightarrow{A} y$, with different y. Intuitively, if ($\Rightarrow$ L) is applied to $x : A \Rightarrow B$ by using two (or more) transitions $x \xrightarrow{A} y$ and $x \xrightarrow{A} z$, then the proof can be directed either on the subtree with root y (i.e. $\mathcal{G}(y)$) or on $\mathcal{G}(z)$. Therefore, to prove the validity of $\Gamma, x \xrightarrow{A} y_1, x \xrightarrow{A} y_2, ..., x \xrightarrow{A} y_n, x : A \Rightarrow B \vdash \Delta$ one needs *only one application of* ($\Rightarrow$ L) *in each branch*, then only one transition $x \xrightarrow{A} y_i$ is used on $x : A \Rightarrow B$; notice that in the other systems we potentially need to apply ($\Rightarrow$ L) on $x : A \Rightarrow B$ by using *all* the transitions $x \xrightarrow{A} y_i$ in the left-hand side of the sequent.

This fact is formalized as follows:

Lemma 5.15 (Controlled used of ($\Rightarrow$ L) in SeqCK and SeqID): If $\Gamma, x : A \Rightarrow B \vdash \Delta$ is derivable in SeqCK (SeqID), then it has a derivation with at most one application of ($\Rightarrow$ L) with $x : A \Rightarrow B$ as a principal formula in each branch.

Proof. Consider a proof of $\Gamma, x : A \Rightarrow B \vdash \Delta$ with more than one application of ($\Rightarrow$ L) introducing (backward) transitions $x \xrightarrow{A} y_1, x \xrightarrow{A} y_2, ..., x \xrightarrow{A} y_n$. Consider the two highest applications[14] of ($\Rightarrow$ L); we have the following situation:

$$\dfrac{...\vdash x \xrightarrow{A} y... \qquad \begin{array}{c}\dfrac{...\vdash x \xrightarrow{A} z... \qquad \Gamma_1, x : A \Rightarrow B, z : B \vdash \Delta_1}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}\,(\Rightarrow L)\\ \Pi_1 \\ \Gamma, x : A \Rightarrow B, y : B \vdash \Delta\end{array}}{\Gamma, x : A \Rightarrow B \vdash \Delta}\,(\Rightarrow L)$$

[14] The applications having the highest distance from the root of the tree, i.e. the initial sequent.

Without loss of generality, we can assume that the above proof can be transformed in the following one by the permutability of the rules. In particular, transitions $x \xrightarrow{A} y$ and $x \xrightarrow{A} z$ in the right-and side necessarily derives (looking forward) from applications of (EQ)[15] by Theorem 4.2, thus the two transitions $x \xrightarrow{A'} y$ and $x \xrightarrow{A''} z$ are introduced (looking backward) in the left-hand side of sequents by two applications of ($\Rightarrow$ R) on $x : A' \Rightarrow B'$ and $x : A'' \Rightarrow B''$, respectively. These negative conditionals are not subformulas of $x : A \Rightarrow B$, since all conditional subformulas of $x : A \Rightarrow B$ are labelled by descendant of x. If any ($\Rightarrow$ R) introducing the transitions is in Π_1, then we can permute them (even with the other rules introducing the negative conditional(s) to which is (are) applied, in a backward search); in each case, we can permute the two applications of ($\Rightarrow$ L) over the other rules, until the upper application of ($\Rightarrow$ R):

$$\dfrac{\dfrac{\dfrac{(\dagger) x \xrightarrow{A'} y \vdash x \xrightarrow{A} y \qquad \dfrac{(\ddagger) x \xrightarrow{A''} z \vdash x \xrightarrow{A} z \qquad \Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''}{\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B \vdash \Delta', y : B', z : B''}(\Rightarrow L)}{\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta', y : B', z : B''}(\Rightarrow L)}{\Gamma', x \xrightarrow{A'} y, x : A \Rightarrow B \vdash \Delta', y : B', x : A'' \Rightarrow B''}(\Rightarrow R)}{\Gamma', x : A \Rightarrow B \vdash \Delta', x : A' \Rightarrow B', x : A'' \Rightarrow B''}(\Rightarrow R)$$

We distinguish two cases:

- $\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$ is not x-branching: in this case, we can apply the Proposition 5.12, obtaining that one of the following sequents is derivable with no addicted applications of ($\Rightarrow$ L):

  1. $\Gamma'^{*}_{y}, x \xrightarrow{A'} y, x : A \Rightarrow B, y : B \vdash \Delta'^{*}_{y}, y : B'$;
  2. $\Gamma'^{*}_{z}, x \xrightarrow{A''} z, x : A \Rightarrow B, z : B \vdash \Delta'^{*}_{z}, z : B''$;
  3. Γ' - $(\Gamma'^{\circ}_{y} \cup \Gamma'^{\circ}_{z}), x : A \Rightarrow B \vdash \Delta' - (\Delta'^{\circ}_{y} \cup \Delta'^{\circ}_{z})$.

  In each case, we conclude the proof:

  1.

$$\dfrac{(\dagger) x \xrightarrow{A'} y \vdash x \xrightarrow{A} y \qquad \Gamma'^{*}_{y}, x \xrightarrow{A'} y, x : A \Rightarrow B, y : B \vdash \Delta'^{*}_{y}, y : B'}{\Gamma'^{*}_{y}, x \xrightarrow{A'} y, x : A \Rightarrow B \vdash \Delta'^{*}_{y}, y : B'}$$

[15] If one of the transitions derives from an axiom, i.e. it is introduced by weakening, then we can obviously remove the corresponding application of ($\Rightarrow$ L).

and the proof is over by applying the admissibility of weakening (Theorem 3.7) to derive $\Gamma, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, y : B', z : B''$ (remember that $\Sigma_k^* \subseteq \Sigma$);

2. symmetric to the previous case;
3. since Γ' - $(\Gamma_y'^{\circ} \cup \Gamma_z'^{\circ}) \subseteq \Gamma$ and $\Delta' - (\Delta_y'^{\circ} \cup \Delta_z'^{\circ}) \subseteq \Delta'$, by Theorem 3.7 on Γ' - $(\Gamma_y'^{\circ} \cup \Gamma_z'^{\circ}), x : A \Rightarrow B \vdash \Delta' - (\Delta_y'^{\circ} \cup \Delta_z'^{\circ})$ we can derive a proof of $\Gamma, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta', y : B', z : B''$.

- $\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$ is x-branching: by permutation properties, a proof Π ending with $\Gamma, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$ can be transformed into a proof Π', where all the rules introducing x-branching formulas are permuted over the other rules (i.e. they are applied at the bottom of the tree). As an example, let the end sequent of Π have the form

$$\Gamma', x : C \rightarrow D, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$$

We have that the lower sequent is x-branching, (at least) since $x : C \rightarrow D$. We can permute Π so that the last step is the introduction of the x-branching formula $x : C \rightarrow D$ from the two sequents $\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B'', x : C$ and $\Gamma', x : D, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$. We have decomposed the x-branching formula, if the two sequents are still x-branching we perform a similar permutation upwards, so that at the end every branch of Π' will contain a sequent $\Gamma_i', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta_i, y : B', z : B''$, such that Γ_i', Δ_i' are no longer x-branching. Notice that if a sequent $\Gamma', w : C \Rightarrow D, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta_i', y : B', z : B''$ is x-branching because of $w : C \Rightarrow D$, we can permute ($\Rightarrow$ L) over the other rules, since w is a predecessor of x in the tree of transitions (see the definition 5.7 above): the label used to decompose the conditional formula is already in the sequent, then the permutation is possible[16]; $x : A \Rightarrow B$ is not x-branching too: a conditional formula is x-branching if it introduces (looking backward) a transition $x \xrightarrow{A} v$ and a world formula $v : B$, and either $v : A$ (for SeqID only) or $v : B$ is x-branching, but this is not the case since v is a successor of x. For the same reason, $x \xrightarrow{A'} y$, $x \xrightarrow{A''} z$, $y : B$, $z : B$, $y : B'$ and $z : B''$ are not x-branching. Then we can conclude by applying the Proposition 5.12, obtaining that one of the following sequents:

[16] As explained, ($\Rightarrow$ L) does not permute over the application of ($\Rightarrow$ R) which introduces the label used by ($\Rightarrow$ L).

$$\frac{x \xrightarrow{A'} y \vdash x \xrightarrow{A} y \qquad \Gamma, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} \ (\Rightarrow \mathbf{L}), x \xrightarrow{A'} y \in \Gamma$$

Fig. 9: $(\Rightarrow \text{L})$ rule for SeqCK and SeqID systems.

1. $\Gamma'^{*}_{iy}, x \xrightarrow{A'} y, x : A \Rightarrow B, y : B \vdash \Delta'^{*}_{iy}, y : B'$;
2. $\Gamma'^{*}_{iz}, x \xrightarrow{A''} z, x : A \Rightarrow B, z : B \vdash \Delta'^{*}_{iz}, z : B''$;
3. $\Gamma'_i - (\Gamma'^{\circ}_{iy} \cup \Gamma'^{\circ}_{iz}), x : A \Rightarrow B \vdash \Delta'_i - (\Delta'^{\circ}_{iy} \cup \Delta'^{\circ}_{iz})$

is derivable. Proceeding as in the previous case (i.e. when the initial sequent is not x-branching), we obtain a proof of $\Gamma'_i, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta'_i, y : B', z : B''$ with at most an application of $(\Rightarrow \text{L})$; the proof is concluded reapplying the rules of Π' to sequents $\Gamma'_i, x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$, obtaining a proof of $\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A''} z, x : A \Rightarrow B, y : B, z : B \vdash \Delta', y : B', z : B''$.

□

By the above Lemma 5.15 and Theorem 4.2 we can reformulate the $(\Rightarrow \text{L})$ rule as shown in Figure 9.

As mentioned above, thanks to the reformulation shown in Figure 9, it is possible to give a better space complexity bound for CK{+ID}:

Theorem 5.16 (Space complexity of CK{+ID}): Provability in CK{+ID} is decidable in $O(n \ \log n)$ space.

Proof. We proceed in the same way as we made for Theorem 4.6. In searching a proof there are two kinds of branching to consider: AND-branching caused by the rules with multiple premises and OR-branching (backtracking points in a depth first search) caused by the choice of the rule to apply, and how to apply it in the case of $(\Rightarrow \text{L})$.

We store only one sequent at a time and maintain a stack containing information sufficient to reconstruct the branching points of both types. Each stack entry contains the principal formula (either a world formula $x : B$, or a transition formula $x \xrightarrow{B} y$), the name of the rule applied and an index which allows to reconstruct the other branches on return to the branching points. The stack entries represent thus backtracking points and the index within the entry allows one to reconstruct both the AND branching and to check whether there are alternatives to explore (OR branching). The working sequent on a return point is recreated by replaying the stack entries from the bottom of the stack using the information in the index (for instance in the case of ($\Rightarrow$ L) applied to the principal formula $x : A \Rightarrow B$, the index will indicate which premise-first or second-we have to expand and the label y involved in the transition formula $x \xrightarrow{A} y$).

A proof begins with the end sequent $\vdash x_0 : D$ and the empty stack. Each rule application generates a new sequent and extends the stack. If the current sequent is an axiom we pop the stack until we find an AND branching point to be expanded. If there are not, the end sequent $\vdash x_0 : D$ is provable and we have finished. If the current sequent is not an axiom and no rule can be applied to it, we pop the stack entries and we continue at the first available entry with some alternative left (a backtracking point). If there are no such entries, the end sequent is not provable.

The entire process must terminate since: (i) the depth of the stack is bounded by the length of a branch proof, thus it is $O(n)$, where $\mid D \mid = n$, (ii) the branching is bounded by the number of rules, the number of premises of any rule and the number of labelled formulas occurring in one sequent, the last being $O(n)$.

To evaluate the space requirement, we have that each subformula of the initial labelled formula can be represented by a positional index into the initial labelled formula, which requires $O(\log n)$ bits. Moreover, also each label can be represented by $O(\log n)$ bits. Thus, to store the working sequent we need $O(n \ \log n)$ space, since there may occur $O(n)$ labelled subformulas. Similarly, each stack entry requires $O(\log n)$ bits, as the name of the rule requires constant space and the index $O(\log n)$ bits. Having depth $O(n)$, to store the whole stack requires $O(n \ \log n)$ space. Thus we obtain that provability in CK{+ID} is decidable in $O(n \ \log n)$ space.

□

6 CondLean: A Theorem Prover for Conditional Logics

In this section we present CondLean, a theorem prover implementing the sequent calculi SeqS for systems CK, CK+ID, CK+MP and CK+ID+MP; it is a SICStus Prolog program inspired by leanTAP [3]. The program comprises a set of clauses, each one of them represents a sequent rule or axiom. The proof search is provided for free by the mere depth-first search mechanism of Prolog, without any additional ad hoc mechanism. CondLean is available for free download at `http://www.di.unito.it/∼olivetti/CONDLEAN`.

We represent each component of a sequent (antecedent and consequent) by a *list* of labelled formulas, partitioned into three sub-lists: atomic formulas, transitions and complex formulas. Atomic and complex formulas (i.e. the labelled formulas) are represented by a list like `[x,a]`, where `x` is a Prolog constant and `a` is a formula. A transition $x \xrightarrow{A} y$ is represented by `[x,a,y]`. For example, the sequent $x : A \Rightarrow B, x : A \Rightarrow C, x \xrightarrow{A} y \vdash y : B, x : C, x : A \rightarrow B$ is represented by the following lists (the upper one represents the antecedent, the lower one represents the consequent):

```
[[],[[x,a,y]],[[x,a=>b],[x,a=>c]]]
    [[y,b],[x,c]],[],[[x,a->b]]
```

We present three different implementations:

1. a *constant labels* version;
2. a *free-variables* version;
3. an heuristic version.

The *constant labels* version makes use of Prolog constants to represent SeqS's labels. The sequent calculi are implemented by the predicate

prove(Sigma, Delta, Labels).

This predicate succeeds if and only if $\Sigma \vdash \Delta$ is derivable in SeqS, where `Sigma` and `Delta` are the lists representing the multisets Σ and Δ, respectively and `Labels` is the list of labels introduced in that branch. For example, to prove

$$x\text{: } A \Rightarrow (B \wedge C)^{17} \vdash x\text{: } A \Rightarrow B,\ x\text{: } C$$

in CK, one queries CondLean with the goal:

```
prove([[],[],[[x, a=>(b and c)]]], [[[x,c],[],[[x, a=>b]]], [x]).
```

Each clause of `prove` implements one axiom or rule of SeqS; for example, the clause implementing ($\Rightarrow$ L) is:

```
prove([LitSigma,TransSigma,ComplexSigma],[LitDelta,TransDelta,
      ComplexDelta], Labels):-
   select([X,A=>B],ComplexSigma,ResComplexSigma), member(Y,Labels),
   put([Y,B],LitSigma,ResComplexSigma,NewLitSigma,NewComplexSigma),
   prove([LitSigma,TransSigma,ResComplexSigma],
      [LitDelta,[[X,A,Y]|TransDelta],ComplexDelta],Labels),
   prove([NewLitSigma,TransSigma,NewComplexSigma],
      [LitDelta,TransDelta,ComplexDelta],Labels).
```

The predicate `select` removes `[X,A=>B]` from `ComplexSigma` returning `ResComplexSigma` as a result. The predicate `put` is used to put `[Y,B]` in the proper sub-list of the antecedent.
To search a derivation of a sequent $\Sigma \vdash \Delta$, CondLean proceeds as follows. First of all, if $\Sigma \vdash \Delta$ is an axiom, the goal will succeed immediately by using the clauses for the axioms. If it is not, then the first applicable rule will be chosen, e.g. if `ComplexSigma` contains a formula `[X,A and B]`, then the clause for ($\wedge$ L) rule will be used, invoking `prove` on the unique premise of ($\wedge$ L). CondLean

[17] CondLean extends the sequent calculi to formulas containing also $\neg$, $\wedge$, $\vee$ and $\top$.

proceeds in a similar way for the other rules. The ordering of the clauses is such that the application of the branching rules is postponed as much as possible. When the ($\Rightarrow$ L) clause is used to prove $\Sigma \vdash \Delta$, a backtracking point is introduced by the choice of a label `Y` occurring in the two premises of the rule; in case of failure, Prolog's backtracking tries every instance of the rule with every available label (if more than one). Choosing, sooner or later, the right label to apply ($\Rightarrow$ L) may strongly affect the theorem prover's efficiency: if there are n labels to choose for an application of ($\Rightarrow$ L) the computation might succeed only after *n-1* backtracking steps, with a significant loss of efficiency.

Our second implementation, called *free-variables*, makes use of *Prolog variables* to represent all the labels that can be used in a single application of the ($\Rightarrow$ L) rule. This version represents labels by integers starting from 1; by using integers we can easily express constraints on the range of the variable-labels. To this regard the library `clpfd` is used to manage free-variable domains (see [37] and [30] for details about the constraints satisfaction problems and the constraint logic programming). As an example, in order to prove $\Sigma^{'}$, *1:* $A \Rightarrow B \vdash \Delta$ the theorem prover will call `prove` on the following premises: $\Sigma^{'} \vdash \Delta$, $1 \xrightarrow{A} V$ and V*:* B, $\Sigma^{'} \vdash \Delta$, where V is a Prolog variable. This variable will be then instantiated by Prolog's pattern matching to apply either the (EQ) rule, or to *close a branch with an axiom.* Here below is the clause implementing the ($\Rightarrow$ L) rule:

```
prove([LitSigma,TransSigma,ComplexSigma],[LitDelta,
    TransDelta,ComplexDelta],Max):-
  select([X,A => B],ComplexSigma,ResComplexSigma),
  domain([Y],1,Max), Y#>X,
  put([Y,B],LitSigma,ResComplexSigma,NewLitSigma,NewComplexSigma),
  prove([NewLitSigma,TransSigma,NewComplexSigma],
      [LitDelta,TransDelta,ComplexDelta],Max),
  prove([LitSigma,TransSigma,ResComplexSigma],
      [LitDelta,[[X,A,Y]|TransDelta],ComplexDelta],Max).
```

The atom `Y#>X` adds the constraint `Y>X` to the constraint store: the constraints solver will verify the consistency of it during the computation. In SeqCK and SeqID we can only use labels introduced *after* the label `X`, thus we introduce the previous constraint. In SeqMP and SeqID+MP we can also use `X` itself, thus we shall add the constraint `Y#>=X`.
The third argument of predicate `prove` is `Max` and is used to define variables domains.
On a sequent with 65 labels on the antecedent this version succeeds in 460 mseconds, whereas the constant labels version takes 4326 mseconds.

We have also developed a third version, called *heuristic version*, that performs a "two-phase" computation: in "Phase 1" an *incomplete* theorem prover searches a derivation exploring a *reduced search space*; in case of failure, the free-variables version is called ("Phase 2"). Intuitively, the reduction of the search space in Phase 1 is obtained by committing the choice of the label to instantiate a free variable, whereby blocking the backtracking.

For SeqMP and SeqID+MP, we have developed a theorem prover which simplifies the reformulations given by BSeqMP and BSeqID+MP. In particular, the reformulation given in the previous section uses two different auxiliaries sets, namely K and Ψ, whereas the theorem prover implements only Ψ, by introducing another argument `CondContr` to the predicate `prove`; therefore, only $(\Rightarrow \mathrm{L})_1$ and $(\Rightarrow \mathrm{L})_3$ are implemented. The `prove` predicate now becomes:

prove(Sigma, Delta, Labels, CondContr).

The list `CondContr` stores the conditional formulas of the antecedent that have been duplicated so far. When $(\Rightarrow \mathrm{L})$ is applied to a formula x*:* $A \Rightarrow B$ in the antecedent, the formula is duplicated at the same time into the `CondContr` list; when $(\Rightarrow \mathrm{L})$ is applied to a formula in `CondContr`, in contrast, the formula is *no longer* duplicated. Thus the $(\Rightarrow \mathrm{L})$ rule is split in two rules, one taking care of "unused" conditionals of the antecedent, the other taking care of "used" (or duplicated) conditionals. Observe that this ensures termination. To understand the difference between the calculus and CondLean implementation, we can observe that the calculus BSeqMP (BSeqID+MP) ensures that every conditional formula $x : A \Rightarrow B$ is duplicated only once, no matter how many times it occurs in a branch. As a difference, CondLean ensures that every *occurrence* of $x : A \Rightarrow B$ is duplicated at most once. However, we have chosen of implementing this simplified version since it is easier and, at the current state, it is not clear if the exact implementation of BSeqMP (BSeqID+MP) would perform significantly better.

6.1 Performances of the Theorem Prover

The performances of the three versions of the theorem prover are promising even on a small machine. To test our program we used samples generated by modifying the samples from [2] and from [47].

We have tested CondLean, SeqCK system, obtaining the following results[18]:

1. the constant labels version succeeds in 79 tests over 90 in less than 2 seconds (78 in less than one second);

2. the free-variables version succeeds in 73 tests over 90 in less than 2 seconds (but 67 in less than 10 mseconds);

3. the heuristic version succeeds in 78 tests over 90 in less than 2 seconds (70 in less than 500 mseconds).

Considering the sequent-degree (defined as the maximum level of nesting of the $\Rightarrow$ operator) as a parameter, we have the following results, obtained by testing the SeqCK free-variables version:

[18] These results are obtained running SICStus Prolog 3.10.0 on an Intel Pentium 166 MMX, 96 MB RAM machine.

Sequent degree	2	6	9	11	15
Time to succeed (mseconds)	5	500	650	1000	2000

We have also tested CondLean, SeqMP system, on some sequents that require duplications of conditional formulas; in particular, we have obtained the following results running the heuristic version on an AMD Athlon XP 2400+ (2.0 GHz), 512 MB RAM machine, using SICStus Prolog 3.11.1:

Sequent	1	2	3	4	5	6	7	8	9	10
Number of applications of $(\Rightarrow \mathrm{L})_3$	1	1	2	2	2	1	2	3	4	5
Time to succeed (mseconds)	1	2500	1	1	1	1	1	1	1	1

As expected, in (MP) systems the free-variables version offers better performances than the constant label version; in the following table we show how many sequents have been derived by each implementation in less than 1 ms, 1 second and 2 seconds over 97 valid sequents:

Time to succeed	$1ms$	$1s$	$2s$
Constant labels	61	66	67
Free-variables	75	82	82

7 Extension of SeqS to other systems

There are a number of extensions of CK{+MP}{+ID} which have been considered in the literature. Each one of them is characterized by a set of axioms/semantic conditions. We list a few of them, without being exhaustive:

(AC) $(A \Rightarrow B) \land (A \Rightarrow C) \rightarrow (A \land C \rightarrow B)$
If $f(w, [A]) \subseteq [B]$ then $f(w, [A \land B]) \subseteq f(w, [A])$

(RT) $(A \land B \Rightarrow C) \land (A \Rightarrow B) \rightarrow (A \Rightarrow C)$
If $f(w, [A]) \subseteq [B]$ then $f(w, [A]) \subseteq f(w, [A \land B]$

(CV) $(A \Rightarrow B) \land \neg(A \Rightarrow \neg C) \rightarrow (A \land C \Rightarrow B)$
If $f(w, [A]) \subseteq [B]$ and $f(w, [A]) \cap [C] \neq \emptyset$ then $f(w, [A \land C]) \subseteq [B]$

(CA) $(A \Rightarrow C) \land (B \Rightarrow C) \rightarrow (A \lor B \Rightarrow C)$
$f(w, [A \lor B]) \subseteq f(w, [A]) \cup f(w, [B])$

These axioms/conditions are part of well-known conditional logics ([39]). Some of these conditions can be used to formalize non-monotonic inferences. For instance AC corresponds to the property of *cumulativity* and CV to the property of *rational monotony*. Preferential entailment corresponds to the first degree fragment (i.e. without nested conditionals) of CK+ID+AC+CUT+CA ([31]).

We can think of extending our sequent calculi to these logics. One would like to have a modular proof system in the form of a sequent calculus where each semantic condition/axioms corresponds to well-defined group of rules. To this regard, it is not difficult to devise rules capturing these semantic conditions. For instance, a possible rule for AC is the following:

$$(\mathbf{AC}) \quad \frac{\Gamma, x \xrightarrow{A} y \vdash \Delta \qquad \Gamma, x \xrightarrow{A} z \vdash z : B, \Delta}{\Gamma, x \xrightarrow{A \wedge B} y \vdash \Delta} \ (z \notin \Gamma, \Delta)$$

However, the addition of (AC) rule to SeqCK does not give a complete system unless *we allow cut*, or in other words, in the system SeqAC cut is not eliminable. To see this consider the formula

$$(A \Rightarrow (B \wedge C)) \rightarrow ((A \wedge B) \Rightarrow C)$$

This formula is provable by SeqAC without cut. But the equivalent formula

$$(A \Rightarrow (B \wedge C)) \rightarrow (\neg(\neg A \vee \neg B)) \Rightarrow C)$$

is not. In order to apply the rule for AC, we must be able to show the equivalence of $\neg(\neg A \vee \neg B)$ and $A \wedge B$. We cannot do this unless we allow cut on transition formulas, or we incorporate the (EQ)-test within the AC rule itself.

The same problem seems to arise with the other axiom/semantic conditions where we need to identify a subformula which is characterized by its syntactic structure (such as $A \wedge B$) above. We can conclude that a straightforward encoding of the semantic conditions we have exemplified results in a non-analytic calculus where cut cannot be eliminated. This is somewhat expected as the selection function cannot be assumed to satisfy any *compositionality* principle: i.e. the value of $f(w, [A\#B])$ for any connective $\#$ is not a function of $f(w, [A])$ and $f(w, [B])$, at most f satisfies some constraints as the above ones. However, further research is needed to see how and whether we can capture the above semantic conditions and alike within the labelled calculus by analytic rules.

8 Conclusions, Comparison with Other Works and Future Work

In this work we have provided a labelled calculus for minimal conditional logic CK, and its standard extensions with conditions ID, MP, CS and CEM. The calculus is cut-free and analytic. By a proof-theoretical analysis, we have shown that these logics are decidable. To the best of our knowledge, sequent calculi for these logics have not been previously studied.

We have also developed CondLean, a theorem prover implementing the calculus written in SICStus Prolog.

We briefly remark on some related works. Most of the works have concentrated on extensions of CK.

De Swart [11] and Gent [22] give sequent/tableaux calculi for the strong conditional logics VC and VCS. Their proof systems are based on the entrenchment connective $\leq$, from which the conditional operator can be defined. Their systems are analytic and comprise an infinite set of rules $\leq F(n, m)$, with a uniform pattern, to decompose each sequent with m negative and n positive entrenchment formulas.

Crocco and Fariñas [8] present sequent calculi for some conditional logics including CK, CEM, CO and others. Their calculi comprise two levels of sequents: principal sequents with $\vdash_P$ correspond to the basic deduction relation, whereas auxiliary sequents with $\vdash_a$ correspond to the conditional operator: thus the constituents of $\Gamma \vdash_P \Delta$ are sequents of the form $X \vdash_a Y$, where X, Y are sets of formulas.

Artosi, Governatori, and Rotolo [1] develop labelled tableau for the *first-degree* fragment (i.e. without nested conditionals) of the conditional logic CU that corresponds to cumulative non-monotonic logics. In their work they use labels similarly to ours. Formulas are labelled by path of worlds containing also variable worlds (see also our free-variable implementation). Differently from us, they do not use a specific rule to deal with equivalent antecedents of conditionals. They use instead a unification procedure to propagate positive conditionals. The unification process itself provides to check the equivalence of antecedents. Their tableau system is not analytic, since it contains a cut-rule, called PB, which is not eliminable. Moreover it is not clear how to extend it to nested conditionals.

Lamarre [33] presents tableaux systems for the conditional logics V, VN, VC, and VW. Lamarre's method is a consistency-checking procedure which tries to build a system of sphere falsifying the input formulas. The method makes use of a subroutine to compute the *core*, that is defined as the set of formulas characterizing the minimal sphere. The computation of the core needs in turn the consistency checking procedure. Thus there is a mutual recursive definition between the procedure for checking consistency and the procedure to compute the core.

Groeneboer and Delgrande [14] have developed a tableau method for the conditional logic VN which is based on the translation of this logic into the modal logic S4.3.

[26] have defined a labelled tableaux calculus for the logic CE and some of its extensions. The flat fragment of CE corresponds to the nonmonotonic preferential logic P and admits a semantics in terms of preferential structures (possible worlds together with a family of preference relations). The tableau calculus makes use of pseudo-formulas, that are modalities in a hybrid language indexed on worlds. In that paper it is shown how to obtain a decision procedure for that logic by performing a kind of loop checking.

Finally, complexity results for some conditional logics have been obtained by Friedman and Halpern [17]. Their results are based on a semantic analysis, by an argument about the size of possible countermodels.

In the future, we intend to continue our work in two directions:

1. We want to investigate if it is possible to develop sequent calculi based on the selection function semantics for stronger conditional logics. If this is possible, we would like to extend CondLean to support these stronger systems.

2. We hope to increase the efficiency of our theorem prover CondLean by experimenting standard refinements and heuristics.

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A Appendix: cut elimination in systems with (CEM)

We present an alternative way to show that cut is admissible in systems allowing the (CEM) rule. In particular, we present a direct proof by using the invertible rule:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} (CEM)$$

where $y \neq z$. The rules (CS) and (ID) are also replaced by the following invertible ones:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \qquad (\Gamma, x \xrightarrow{A} y \vdash \Delta)[x/u, y/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta} (CS)$$

$$\frac{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta} (ID)$$

We are not able to conclude the proof for systems allowing both (CEM) and (MP), therefore finding a cut-free sequent calculus for these systems will be part of our future researches. More details on the problem occurring with these systems are presented at the end of this section.

First of all, we need to extend our notation by representing with $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ a sequent containing *any* number of transitions labelled with the formula A; moreover, given $u : A \vdash u : A^{'}$ and $u : A^{'} \vdash u : A$, we denote with $\Gamma^\star \vdash \Delta^\star$ the sequent obtained by replacing *any* number of transitions labelled with A with the same transitions labelled with $A^{'}$ in $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$.

Definition A.1: We denote with

$$\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$$

a sequent $\Gamma \vdash \Delta$ such that n transitions $x_1 \xrightarrow{A} y_1, x_2 \xrightarrow{A} y_2, ..., x_n \xrightarrow{A} y_n \in \Gamma$ and m transitions $u_1 \xrightarrow{A} v_1, u_2 \xrightarrow{A} v_2, ..., u_n \xrightarrow{A} v_m \in \Delta$, where $n, m \geq 0$.

Definition A.2: If $u : A \vdash u : A'$ and $u : A^{'} \vdash u : A$ are derivable, then we denote with

$$\Gamma^\star \vdash \Delta^\star$$

the sequent obtained by replacing in $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ *any* number of transitions $x_i \xrightarrow{A} y_i$ with $x_i \xrightarrow{A^{'}} y_i$ and *any* number of transitions $u_j \xrightarrow{A} u_j$ with $u_j \xrightarrow{A^{'}} u_j$.

As an example, given $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j] = x : A, x \xrightarrow{A} y, y \xrightarrow{A} z \vdash$, we use $\Gamma^\star \vdash \Delta^\star$ to denote one of the following sequents: (1)$x : A, x \xrightarrow{A} y, y \xrightarrow{A} z \vdash$, (2)$x : A, x \xrightarrow{A'} y, y \xrightarrow{A} z \vdash$, (3)$x : A, x \xrightarrow{A} y, y \xrightarrow{A'} z \vdash$, (4)$x : A, x \xrightarrow{A'} y, y \xrightarrow{A'} z \vdash$. In (1) no transitions $x_i \xrightarrow{A} y_i$ have been replaced, in (1) the transition $x \xrightarrow{A} y$ has been replaced by $x \xrightarrow{A'} y$, and so on.

We prove that cut is admissible in systems SeqCEM{+CS}{+ID} by "splitting" the notion of cut in two propositions, as stated by the following:

Theorem A.3 (Admissibility of cut for SeqCEM{+CS}{+ID}): In systems SeqCEM{+CS}{+ID}:

(A) if $\Gamma \vdash \Delta, F$ and $\Gamma, F \vdash \Delta$ are derivable, so $\Gamma \vdash \Delta$ (cut);

(B) if (I) $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$, (II) $u : A \vdash u : A'$ and (III) $u : A' \vdash u : A$ are derivable, so $\Gamma^\star \vdash \Delta^\star$.

Proof. By double mutual induction on the complexity of the cut formula and on the height of the derivation. To prove (A), the induction on the height is intended as usual as the sum of the height of the premises of the cut inference; to prove (B), the induction on the height is intended as the height of the derivation of $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$. We have several cases:

- (Base for (A)): if one of the two premises is an axiom, then either $\Gamma \vdash \Delta$ is an axiom, or the premise which is not an axiom contains two copies of F and $\Gamma \vdash \Delta$ can be obtained by contraction, which is admissible (Theorem 4.8);
- (Base for (B)): if $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ is an axiom, then we distinguish two subcases:
  - ⋆ $x : P \in \Gamma[x_i \xrightarrow{A} y_i], \Delta[u_j \xrightarrow{A} v_j]$: we conclude that $\Gamma^\star \vdash \Delta^\star$ is derivable, since $x : P \in \Gamma^\star, \Delta^\star$ (only transition formulas can be modified by applying the proposition (B));
  - ⋆ $x : \bot \in \Gamma[x_i \xrightarrow{A} y_i]$: as in the other case, we conclude since $x : \bot \in \Gamma^\star$;
- (Inductive step for (A)): we distinguish all cases:
  1. the last step of *one* of the two premises is obtained by a rule in which F is *not* the principal formula. We distinguish two cases.
     (i) The sequent where F is not principal is derived by any rule (R), except the (EQ) rule. This case is standard, we can permute (R) over the cut: i.e. we cut the premise(s) of (R) and then we apply (R) to the result of cut. As an example, consider the case when $F = x \xrightarrow{A} y$ is the cut formula, and it is principal only in the right

derivation, introduced (forward) by an application of (CEM). In the left derivation (CS) is applied to another transition $x \xrightarrow{A'} y$ (consider $\Sigma(u) = \Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{(1)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y, x : A' \quad (2)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u), u \xrightarrow{A} u}{(5)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}(CS) \quad \dfrac{(3)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \quad (4)(\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta)[y/v, z/v]}{(6)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta}(CEM)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(cut)$$

From (6) we obtain a proof of (at most) the same height of $(6')\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A'$, by Theorem 3.7. Applying Lemma 3.6 to (6) we obtain a proof of no greater height of $(6'')\Gamma'(u), u \xrightarrow{A'} u, u \xrightarrow{A} u \vdash \Delta(u)$.
For the inductive hypothesis on the height, we cut (1) with $(6')$, obtaining $(7)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A'$; then we cut (2) with $(6'')$, obtaining a proof of $(8)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u)$. The initial cut is replaced as follows:

$$\dfrac{\dfrac{(1)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y, x : A' \quad (6')\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A'}{(7)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A'}(cut) \quad (8)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(CS)$$

(ii) If one of the sequents, say $\Gamma \vdash \Delta, F$ is obtained by the (EQ) rule, where F is not principal, then also $\Gamma \vdash \Delta$ is derivable by the (EQ) rule and we are done;

2. F is the principal formula in the last step of *both* derivations of the premises of the cut inference. There are six subcases: F is introduced (a) by $(\rightarrow$ L), $(\rightarrow$ R), (b) by $(\Rightarrow L), (\Rightarrow R)$, (c) by (EQ), (d) by (ID) on the left and by (EQ) on the right, (e) by (CS) on the left and by (EQ) on the right, and (f) by (CEM) on the left and by (EQ) on the right. The list is exhaustive, since we do not consider systems allowing both (MP) and (CEM).

(a) $F = x : A \rightarrow B$ is introduced by $(\rightarrow L)$ on the left and by $(\rightarrow R)$ on the right as follows:

$$\dfrac{\dfrac{(1)\Gamma, x : A \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : A \rightarrow B}\ (\rightarrow R) \qquad \dfrac{(2)\Gamma \vdash \Delta, x : A \qquad (3)\Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta}\ (\rightarrow L)}{\Gamma \vdash \Delta}\ (cut)$$

This cut is replaced by two cuts on the subformulas A and B as follows (the sequent $(2^{'})$ is obtained by weakening from (2)):

$$\dfrac{\dfrac{(2^{'})\Gamma \vdash \Delta, x : A, x : B \qquad (1)\Gamma, x : A \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : B}\ (cut) \qquad (3)\Gamma, x : B \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

(b) $F = x : A \Rightarrow B$ is introduced by $(\Rightarrow R)$ and $(\Rightarrow L)$. Then we have

$$\dfrac{\dfrac{(1)\Gamma, x \xrightarrow{A} z \vdash \Delta, z : B}{(2)\Gamma \vdash \Delta, x : A \Rightarrow B}\ (\Rightarrow R) \qquad \dfrac{(3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \quad (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{(5)\Gamma, x : A \Rightarrow B \vdash \Delta}\ (\Rightarrow L)}{\Gamma \vdash \Delta}\ (cut)$$

where z does not occur in Γ, Δ and $z \neq x$; By Lemma 3.6, we obtain that $(1^{'})\Gamma, x \xrightarrow{A} y \vdash y : B, \Delta$ is derivable by a derivation of no greater height than (1); moreover, we can apply the height-preserving admissibility of weakening (Theorem 3.7) to (2) in order to obtain a proof of no greater height of $(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y$ and of $(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B$.
First, we can make the following cut, which uses the inductive hypothesis on the height:

$$\dfrac{(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y \qquad (3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y}{(6)\Gamma \vdash \Delta, x \xrightarrow{A} y}\ (cut)$$

Applying the height-preserving admissibility of weakening to (6), we have a proof of no greater height of $(6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B$. Thus we can replace the initial cut as follows:

$$\dfrac{\dfrac{\begin{array}{c}(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B \\ (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta\end{array}}{\Gamma, y : B \vdash \Delta}\ (cut) \qquad \dfrac{\begin{array}{c}(1^{'})\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B \\ (6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B\end{array}}{\Gamma \vdash \Delta, y : B}\ (cut)}{\Gamma \vdash \Delta}\ (cut)$$

The upper cut on the left uses the induction hypothesis on the height, the others the induction hypothesis on the complexity of the cut formula.

(c) $F = x \xrightarrow{A} y$ is introduced by (EQ) in both premises, we have

$$\dfrac{\dfrac{(1)\, u : A' \vdash u : A \quad (2)\, u : A \vdash u : A'}{\Gamma', x \xrightarrow{A'} y \vdash x \xrightarrow{A} y, \Delta}(EQ) \quad \dfrac{(3)\, u : A \vdash u : A'' \quad (4)\, u : A'' \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{A''} y, \Delta'}(EQ)}{\Gamma', x \xrightarrow{A'} y \vdash x \xrightarrow{A''} y, \Delta'}(cut)$$

where $\Gamma = \Gamma', x \xrightarrow{A'} y$, $\Delta = x \xrightarrow{A''} y, \Delta'$. (1)-(4) have been derived by a shorter derivation; thus we can replace the cut by cutting (1) and (3) on the one hand, and (4) and (2) on the other, which give respectively

$$(5)\ u : A' \vdash u : A'' \text{ and } (6)\ u : A'' \vdash u : A'.$$

Using (EQ) we obtain $\Gamma', x \xrightarrow{A'} y \vdash \Delta', x \xrightarrow{A''} y$.

(d) $F = x \xrightarrow{A} y$ is introduced on the left by (ID) rule, and it is introduced on the right by (EQ). Thus we have

$$\dfrac{\dfrac{u : A' \vdash u : A \quad u : A \vdash u : A'}{(2)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \quad \dfrac{(1)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y, y : A \vdash \Delta}{x \xrightarrow{A} y, \Gamma', x \xrightarrow{A'} y \vdash \Delta}(\mathrm{ID})}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(\mathrm{cut})$$

From (2) we can obtain a proof of no greater height of $(2')\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta, x \xrightarrow{A} y$ by weakening (Theorem 3.7); moreover, by label substitution (Lemma 3.6) and weakening we can find a derivation of no greater height than $u : A' \vdash u : A$'s of $(3)\Gamma', x \xrightarrow{A'} y, y : A' \vdash y : A, \Delta$. First, we replace the following cut by inductive hypothesis on the height:

$$\dfrac{(2')\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta, x \xrightarrow{A} y \qquad (1)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y, y : A \vdash \Delta}{(4)\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta}(cut)$$

From (4), we can find a proof of $(4')\Gamma', x \xrightarrow{A'} y, y : A, y : A' \vdash \Delta$ by weakening, thus we replace the initial cut as follows:

$$\dfrac{\dfrac{(3)\Gamma', x \xrightarrow{A'} y, y : A' \vdash y : A, \Delta \qquad (4')\Gamma', x \xrightarrow{A'} y, y : A, y : A' \vdash \Delta}{\Gamma', x \xrightarrow{A'} y, y : A' \vdash \Delta}(cut)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(ID)$$

The above cut can be replaced by inductive hypothesis on the complexity of the cut formula;

(e) $F = x \xrightarrow{A} y$ is derived on the left by (CS) and on the right by (EQ). Thus we have (we denote with $\Sigma(u)$ the substitution $\Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{(1) u : A \vdash u : A' \qquad (2) u : A' \vdash u : A}{(5)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \qquad \dfrac{\begin{array}{c}(3)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A \\ (4)\Gamma'(u), u \xrightarrow{A'} u, u \xrightarrow{A} u \vdash \Delta(u)\end{array}}{\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta}(CS)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(cut)$$

First, we can replace the cut below:

$$\dfrac{(5')\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y, x : A \qquad (3)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A}{(6)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A}(cut)$$

by inductive hypothesis on the height. $(5')$ is obtained from (5) since weakening is height-preserving admissible. We replace the initial cut as follows:

$$\dfrac{\dfrac{\begin{array}{c}(6')\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A, x : A' \\ (1')\Gamma', x \xrightarrow{A'} y, x : A \vdash \Delta, x : A'\end{array}}{\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A'}(cut) \qquad \dfrac{\begin{array}{c}(5'')\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u), u \xrightarrow{A} u \\ (4)\Gamma'(u), u \xrightarrow{A'} u, u \xrightarrow{A} u \vdash \Delta(u)\end{array}}{\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u)}(cut)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(CS)$$

where $(6')$ is obtained from (6) by weakening, $(5'')$ from (5) by label substitution and $(1')$ from (1) by weakening and label substitution. The cut on the left can be replaced by inductive hypothesis on the complexity of the cut formula; the cut on the right can be replaced by inductive hypothesis on the height.

(f) $F = x \xrightarrow{A} y$ is derived on the left by (CEM) and on the right by (EQ). Thus we have:

$$\dfrac{\dfrac{(1) u : A \vdash u : A' \qquad (2) u : A' \vdash u : A}{(5)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \qquad \dfrac{(3)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (4)(\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta)[y, z/u]}{\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta}(CEM)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(cut)$$

where $y \neq z$. Since we have (1) and (2), by mutual induction we can apply proposition (B) on (3), obtaining a derivation of the sequent

$(3')\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A'} z$ (we replace $x \xrightarrow{A} y$ with $x \xrightarrow{A'} y$ and $x \xrightarrow{A} z$ with $x \xrightarrow{A'} z$). $(5')(\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y)[y, z/u]$ is obtained by Lemma 3.6 and has no greater height than (5). We can replace the above cut as follows:

$$\dfrac{(3')\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A'} z \qquad \dfrac{\begin{array}{c}(5')(\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y)[y, z/u] \\ (4)(\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta)[y, z/u]\end{array}}{(6)(\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A'} y \vdash \Delta)[y, z/u]}\,(cut)}{\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A'} y \vdash \Delta}\,(CEM)$$

from which we conclude by contraction, which is admissible (Theorem 3.9). Notice that one instance of $x \xrightarrow{A'} y$ is introduced in (6) by weakening (Theorem 3.7).

- (Inductive step for (B)): we consider all cases:

1. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of ($\rightarrow$ R), i.e.:

$$\dfrac{\Gamma[x_i \xrightarrow{A} y_i], x : A \vdash \Delta[u_j \xrightarrow{A} v_j], x : B}{\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j], x : A \rightarrow B}\,(\rightarrow R)$$

By inductive hypothesis, we have that $\Gamma^\star, x : A \vdash \Delta^\star, x : B$ is derivable, from which we conclude by an application of ($\rightarrow$ R);

2. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of ($\rightarrow$ L), i.e.:

$$\dfrac{\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j], x : A \qquad \Gamma[x_i \xrightarrow{A} y_i], x : B \vdash \Delta[u_j \xrightarrow{A} v_j]}{\Gamma[x_i \xrightarrow{A} y_i], x : A \rightarrow B \vdash \Delta[u_j \xrightarrow{A} v_j]}\,(\rightarrow L)$$

We can apply the inductive hypothesis on both premises, obtaining derivations for $\Gamma^\star \vdash \Delta^\star, x : A$ and $\Gamma^\star, x : B \vdash \Delta^\star$, from which we obtain a derivation of $\Gamma^\star, x : A \rightarrow B \vdash \Delta^\star$ by an application of ($\rightarrow$ L);

3. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of ($\Rightarrow$ R); the proof is ended as follows:

$$\dfrac{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j], y : B}{\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j], x : A \Rightarrow B}\,(\Rightarrow R)$$

By inductive hypothesis on the premise, we can replace any transition $x_i \xrightarrow{A} y_i$ with $x \xrightarrow{A'} y$ and *we do not replace the transition* $x \xrightarrow{A} y$: we can conclude as follows:

$$\frac{\Gamma^\star, x \xrightarrow{A} y \vdash \Delta^\star, y : B}{\Gamma^\star \vdash \Delta^\star, x : A \Rightarrow B} \, (\Rightarrow R)$$

4. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of ($\Rightarrow$ L); the proof is ended as follows:

$$\frac{\Gamma[x_i \xrightarrow{A} y_i], x : A \Rightarrow B \vdash \Delta[u_j \xrightarrow{A} v_j], x \xrightarrow{A} y \qquad \Gamma[x_i \xrightarrow{A} y_i], x : A \Rightarrow B, y : B \vdash \Delta[u_j \xrightarrow{A} v_j]}{\Gamma[x_i \xrightarrow{A} y_i], x : A \Rightarrow B \vdash \Delta[u_j \xrightarrow{A} v_j]} \, (\Rightarrow L)$$

As we made in the previous case, we apply the inductive hypothesis on the two premises, obtaining proofs of the sequents $\Gamma^\star, x : A \Rightarrow B \vdash \Delta^\star, x \xrightarrow{A} y$ and $\Gamma, x : A \Rightarrow B, y : B \vdash \Delta^\star$, from which we conclude by an application of ($\Rightarrow$ L);

5. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of (EQ), applied on transitions $x \xrightarrow{F} y \in \Gamma$ and $x \xrightarrow{F'} y \in \Delta$. If F, F' are both different from A, then we have that $(1) x \xrightarrow{F} y \vdash x \xrightarrow{F'} y$ is derivable, thus we conclude that $\Gamma^\star \vdash \Delta^\star$, since we can add any $x_i \xrightarrow{A'} y_i$ and $u_j \xrightarrow{A'} v_j$ to (1) by weakening, which is admissible. If $F = A$ and $F' = A'$, then we can obviously conclude, since both $\Gamma^*, x \xrightarrow{A'} y \vdash \Delta^*, x \xrightarrow{A} y$ and $\Gamma^*, x \xrightarrow{A'} y \vdash \Delta^*, x \xrightarrow{A'} y$ are derivable. Otherwise, we have that $F = A$ and $F' \neq A'$ (the case when $F' = A$ and $F \neq A'$ is symmetric), thus:

$$\frac{(2) u : A \vdash u : A'' \qquad (3) u : A'' \vdash u : A}{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j], x \xrightarrow{A''} y} \, (EQ)$$

We have also that $(*) u : A \vdash u : A'$ and $(**) u : A' \vdash u : A$. We can apply the proposition (A) as follows (inductive hypothesis on the complexity of the cut formula):

$$\frac{(**) u : A' \vdash u : A \qquad (2) u : A \vdash u : A''}{(4) u : A' \vdash u : A''} \, (cut)$$

$$\frac{(3) u : A'' \vdash u : A \qquad (*) u : A \vdash u : A'}{(5) u : A'' \vdash u : A'} \, (cut)$$

From (4) and (5) we obtain a proof of $x \xrightarrow{A'} y \vdash x \xrightarrow{A''} y$ and, by the admissibility of weakening, we obtain a proof of $\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star, x \xrightarrow{A''} y$, and we are done. If we want to derive $\Gamma^*, x \xrightarrow{A} y \vdash \Delta^*, x \xrightarrow{A''} y$ we have the following proof: $x \xrightarrow{A} y \vdash x \xrightarrow{A''} y$ derives (forward) from (2) and (3) by (EQ), and $\Gamma^*, x \xrightarrow{A} y \vdash \Delta^*, x \xrightarrow{A''} y$ is obtained by weakening;

6. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of (ID): if (ID) has $x \xrightarrow{F} y$, with F different from A, as a principal formula, then we can obviously conclude by applying the inductive hypothesis on the premise. The only interesting case is when (ID) is applied to $x \xrightarrow{A} y$ as follows:

$$\frac{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y, y : A \vdash \Delta[u_j \xrightarrow{A} v_j]}{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j]}\ (ID)$$

We can apply the inductive hypothesis on the premise, obtaining a proof of $(1)\Gamma^\star, x \xrightarrow{A'} y, y : A \vdash \Delta^\star$, to which we can apply the proposition (A) (inductive hypothesis again on the complexity of the cut formula, we omit necessary weakenings and label substitutions) and conclude by an application of (ID) as follows:

$$\cfrac{\cfrac{(**)y : A' \vdash y : A \qquad (1)\Gamma^\star, x \xrightarrow{A'} y, y : A \vdash \Delta^\star}{\Gamma^\star, x \xrightarrow{A'} y, y : A' \vdash \Delta^\star}\ (cut)}{\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star}\ (ID)$$

7. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of (CS): as for (ID), the only interesting case is when (CS) has a transition $x \xrightarrow{A} y$ as a principal formula. The proof is ended as follows:

$$\frac{(1)\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j], x : A \qquad (2)(\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j])[x, y/u]}{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j]}\ (CS)$$

We can apply the inductive hypothesis on both premises, obtaining a proof of $(1')\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star, x : A$ and $(2')(\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star)[x, y/u]$, respectively. We conclude by applying the proposition (A) (inductive hypothesis on the complexity of the cut formula) and by an application of (CS) as follows:

$$\dfrac{\dfrac{(1')\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star, x : A \qquad (*) x : A \vdash x : A'}{\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star, x : A'}\ (cut) \qquad (2')(\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star)[x, y/u]}{\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star}\ (CS)$$

8. $\Gamma[x_i \xrightarrow{A} y_i] \vdash \Delta[u_j \xrightarrow{A} v_j]$ derives from an application of (CEM) as follows:

$$\dfrac{(1)\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j], x \xrightarrow{A} z \qquad (2)(\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j])[y, z/u]}{\Gamma[x_i \xrightarrow{A} y_i], x \xrightarrow{A} y \vdash \Delta[u_j \xrightarrow{A} v_j]}\ (CEM)$$

We can apply the inductive hypothesis on the height to both premises and we can easily conclude as follows:

$$\dfrac{(1')\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star, x \xrightarrow{A'} z \qquad (2')(\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star)[y, z/u]}{\Gamma^\star, x \xrightarrow{A'} y \vdash \Delta^\star}\ (CEM)$$

□

As mentioned above, we are not able to prove that in systems allowing both (CEM) and (MP), i.e. SeqCEM{+CS}{+ID}+MP, cut is admissible. Indeed, extending the proof of cut elimination presented above, we have to consider the case when the cut formula is introduced by (CEM) on the left and by (MP) on the right as follows:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, x : A}{\Gamma \vdash \Delta, x \xrightarrow{A} x}\ (MP) \qquad \dfrac{\Gamma, x \xrightarrow{A} x \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} x \vdash \Delta)[x/u, z/u]}{\Gamma, x \xrightarrow{A} x \vdash \Delta}\ (CEM)}{\Gamma \vdash \Delta}\ (cut)$$

At the moment, we are not able to conclude the proof in this situation, nor to give a counterexample showing that an explicit cut rule is needed to make the calculus complete. We strongly conjecture that cut is admissible even in these systems, and we intend to prove it in our future research.