A Tableaux Calculus for KLM Preferential and Cumulative Logics

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Abstract. We present tableaux calculi for some logics of default reasoning, as defined by Kraus, Lehmann and Magidor. We give a tableaux proof procedure for preferential logic **P** and for loop-cumulative logic **CL**. Our calculi are obtained by introducing suitable modalities to interpret conditional assertions. Moreover, they give a decision procedure for the respective logics and can be used to establish their complexity.

1 Introduction

In the early 90' [10] Kraus, Lehmann and Magidor (from now on KLM) proposed a formalization of non-monotonic reasoning that was early recognized as a landmark. Their work stemmed from by two sources: the theory of nonmonotonic consequence relations initiated by Gabbay [6] and the preferential semantics proposed by Shoham [12] as a generalization of Circumscription. Their works lead to a classification of nonmonotonic consequence relations, determining a hierarchy of stronger and stronger systems.

According to the KLM framework, defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals or assertions of the form $A \mid\sim B$ whose reading is *normally (or typically) the A's are B's*. The operator "$\mid\sim$" is nonmonotonic, in the sense that $A \mid\sim B$ does not imply $A \land C \mid\sim B$. For instance K may contain the following set of conditionals: $adult \mid\sim work, adult \mid\sim taxpayer, student \mid\sim adult, student \mid\sim \neg work, student \mid\sim \neg taxpayer, retired \mid\sim adult, retired \mid\sim \neg work$, whose meaning is that adults typically work, adults typically pay taxes, students are typically adults, but they typically do not work, nor they do pay taxes, and so on. Observe that if $\mid\sim$ were interpreted as classical (or intuitionistic) implication, we simply would get $student \mid\sim \bot$, $retired \mid\sim \bot$, i.e. typically there are not students, nor retired people, thereby obtaining a trivial knowledge base. One can derive new conditional assertions from the knowledge base by means of a set of inference rules.

In KLM framework, the set of adopted inference rules defines some fundamental types of inference systems, namely, from the weakest to the strongest: Cumulative (**C**) , Loop-Cumulative (**CL**), Preferential (**P**) and Rational logic (**R**). All these systems allow one to infer new assertions from K without incurring in the trivialising conclusions of classical logic: regarding our example, in none of them, one can infer *student* $\mid\!\sim$ *work* or *retired* $\mid\!\sim$ *work*. In cumulative logics (both **C** and **CL**) one can infer *adult* $\wedge$ *student* $\mid\!\sim \neg$*work* (giving preference to more specific information), in Preferential logic **P** one can also infer that *adult* $\mid\!\sim \neg$*retired* (i.e. typical adults are not retired). In the rational case **R**, if one further knows that *adult* $\not\mid\!\sim \neg$*married* (i.e. it is not the case the adults are typically unmarried), one can also infer that *adult* $\wedge$ *married* $\mid\!\sim$ *work*.

From a semantic point of view, to the each logic (**C**, **CL**, **P**, **R**) corresponds one kind of models, namely, possible-world structures equipped with a preference relation among worlds or states. More precisely, for **P** we have models with a preference relation (an irreflexive and transitive relation) on worlds. For the stronger **R** the preference relation is further assumed to be *modular*. For the weaker logic **CL**, the preference relation is defined on *states*, where a state can be identified, intuitively, with a set of worlds. In the weakest case of **C**, the preference relation is on states, as for **CL**, but it is no longer assumed to be transitive. In all cases, the meaning of a conditional assertion $A \mid\!\sim B$ is that B holds in the *most preferred* worlds/states where A holds.

In KLM framework the operator "$\mid\!\sim$" is considered as a meta-language operator, rather than as a connective in the object language. However, it has been readily observed that KLM systems **P** and **R** coincide to a large extent with the flat (i.e. unnested) fragments of well-known conditional logics, once we interpret the operator "$\mid\!\sim$" as a binary connective [3], [2], [9]. The connections with conditional and modal logic have been studied further in two directions. A recent result by Halpern and Friedman [4] has shown that preferential and rational logic are quite natural and general systems: surprisingly enough, the axiom system of preferential (likewise of rational logic) is complete with respect to a wide spectrum of semantics: from ranked models, to parametrized probabilistic structures, ϵ-semantics and possibilistic structures. The result of Halpern and Friedman can be explained by the fact that all these structures are examples of *plausibility structures* (or plausibility ordered structures), and the truth in them is captured by the axioms of preferential (or rational) logic. These results, and their extensions to the first order setting [5] are the source of a renewed theoretical interest in KLM framework.

On the other hand, Lamarre and then Boutillier [2] have shown that preferential and rational logic (but not the weaker **C** and **CL**) could be translated into standard modal logics, although the proposed translation is not the most natural.

Even if KLM was born as an inferential approach to nonmonotonic reasoning, curiously enough, there has not been much investigation on deductive mechanisms for these logics. Lehmann has proved that deciding whether a conditional is entailed by a set of positive conditionals is a CoNP problem in both the preferential and the rational case. Decidability of **P** and **R** can be also proved by mapping them into standard modal logics [2]. A tableaux proof procedure for cumulative logic has been given in [1]. To the best of our knowledge, for **CL** no decision procedure was known before the present work.

In this work we begin our investigation of tableaux procedures for KLM logics, by considering the cases of **P** and **CL**. The investigation of tableaux calculi for the weakest **C** and the strongest **R** is left for future work. Our approach is based on a sort of run-rime translation of **P** conditional assertions into modal logic G. The idea is simply to interpret the preference relation as an accessibility relation: a conditional $A \mid\!\sim B$ holds in a model if B is true in all A-worlds w that are minimal. An A-world is minimal if all smaller worlds are not A-worlds. The relation with G is motivated by the fact that we assume, following KLM, the so-called *smoothness condition*, which is related to the *limit assumption*. This condition ensures indeed that A-minimal worlds exist, by preventing an infinitely descending chain of worlds. This condition is therefore ensured by the finite-chain condition on the accessibility relation (as in modal logic G). Therefore, our interpretation of conditionals is different from the one proposed by Boutilier, who rejects the smoothness condition, and as a consequence, gives a more complicate interpretation of **P** into modal logic S4. Moreover, we do not give a formal translation of **P** in G, we appeal to the correspondence as far as it is needed to derive the tableaux rules for **P**.

We are able to extend our approach to the case of **CL** by using a second modality which takes care of states. More precisely, we show that we can map **CL**-models into **P**-models with an additional modality; this fact seems to have remained unnoticed and might be of independent interest. In both cases, **P** and **CL**, we can define a decision procedure and obtain also a complexity bound for these logics, namely that they are both CoNP. In case of **CL** this bound is new, at the best of our knowledge.

The plan of the paper is as follows: in section 2, we recall preferential logic **P**. In section 3, we give a simple tableaux calculus for **P**, and we prove its soundness and completeness. In section 4 we improve the calculus to obtain a decision procedure for **P** and we prove that it is CoNP. In section 5, we give a similar calculus for logic **CL**, obtaining a decision procedure and a CoNP complexity bound for it. In section 6, we discuss some related works and we suggest some future developments.

2 Preferential Logic P

The language of KLM logics consists just of assertions $A \mid\!\sim B$. We consider a richer language allowing boolean combinations of assertions and propositional formulas.

The language $\mathcal{L}$ of **P** is defined from a set of propositional variables ATM, the boolean connectives and the conditional operator $\mid\!\sim$. The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\sim B \in \mathcal{L}$; if A is a boolean combination of formulas of $\mathcal{L}$, $A \in \mathcal{L}$.

The axiomatization of **P** consists of all axioms and rules of propositional calculus together with the following axioms and rules:

- REF. $A \mid\sim A$ (reflexivity)
- LLE. If $\models A \leftrightarrow B$, then from $A \mid\sim C$ infer $B \mid\sim C$ (left logical equivalence)
- RW. If $\models A \rightarrow B$, then from $C \mid\sim A$ infer $C \mid\sim B$ (right weakening)
- CM. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \wedge B \mid\sim C)$ (cautious monotonicity)
- AND. $((A \mid\sim B) \wedge (A \mid\sim C)) \rightarrow (A \mid\sim B \wedge C)$
- OR. $((A \mid\sim C) \wedge (B \mid\sim C)) \rightarrow (A \vee B \mid\sim C)$

The semantics of **P** is defined by considering possible world structures with a preference relation (a strict partial order) $w < w'$ whose meaning is that w is preferred to w'. We have that $A \mid\sim B$ holds in a model $\mathcal{M}$ if B holds in all *minimal worlds* with respect to the relation $<$ where A holds. This definition makes sense provided minimal worlds for A exist whenever there are A-worlds. This is ensured by the *smoothness condition* in the next definition.

Definition 1 (Semantics of P). *A model is a triple* $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ *where:* $\mathcal{W}$ *is a non-empty set of items called worlds;* $<$ *is an irreflexive and transitive relation on* $\mathcal{W}$*;* V *is a function* $V : \mathcal{W} \longmapsto pow(ATM)$*, which assigns to every world* w *the set of atoms holding in that world. We define the truth conditions* $(\mathcal{M}, w \models A)$ *as follows:*

- $\mathcal{M}, w \models A$ *for the boolean cases is defined in the obvious way;*
- *Let* A *be a propositional formula; we define* $Min_<(A) = \{w \mid \mathcal{M}, w \models A$ *and* $\forall w' < w \quad \mathcal{M}, w' \not\models A\}$*;*
- $\mathcal{M}, w \models A \mid\sim B$ *if for all* $w' \in Min_<(A)$*, then* $\mathcal{M}, w' \models B$*.*

We assume that the relation $<$ *satisfies the following condition, called* smoothness*: if* $\mathcal{M}, x \models A$ *then* $x \in Min_<(A)$ *or* $\exists y \in Min_<(A)$ *such that* $y < x$*.*
We say that a formula A *is* valid in a model $\mathcal{M}$*, denoted with* $\mathcal{M} \models A$*, if it respects the truth conditions in all worlds of the model, i.e.* $\mathcal{M}, w \models A$ *for every* $w \in \mathcal{W}$*. A formula is* valid *if it is valid in every model* $\mathcal{M}$*.*

Given a model $\mathcal{M}$, we denote by $[A]$ the set of worlds w such that $\mathcal{M}, w \models A$.

We also introduce in $\mathcal{L}$ a modality $\Box$, whose intuitive meaning is as follows: $\Box A$ holds in a world w if A holds in all the worlds w' such that $w' < w$:

Definition 2 (Truth condition of modality $\Box$). *We define the truth condition of a boxed formula as follows:*

$$\mathcal{M}, w \models \Box A \textit{ iff for every } w' \in \mathcal{W} \textit{ if } w' < w \textit{ then } \mathcal{M}, w' \models A$$

Observe that for any formula A we have that $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \wedge \Box \neg A$. It is easy to see that $\Box$ has the properties of the modal system G: the accessibility relation (i.e. Rxy iff $y < x$) is transitive and does not have infinite ascending chains.

3 The Tableaux Calculus for Preferential Logic P

In this section we present a tableaux calculus for **P** called $\mathcal{T}\mathbf{P}$. To save space, we only give propositional rules for $\neg$ and $\wedge$.

Definition 3 (The calculus $\mathcal{T}\mathbf{P}$). *The rules of the calculus manipulates sets of formulas G. In the rules we write G, A to denote $G \wedge A$. Moreover, given G we define the following:*

- $G^{\Box} = \{\Box A \mid \Box A \in G\}$
- $G^{\Box^{\downarrow}} = \{A \mid \Box A \in G\}$
- $G^{\mid\!\sim^{+}} = \{A \mid\!\sim B \mid A \mid\!\sim B \in G\}$
- $G^{\mid\!\sim^{-}} = \{\neg(A \mid\!\sim B) \mid \neg(A \mid\!\sim B) \in G\}$
- $G^{\mid\!\sim^{\pm}} = G^{\mid\!\sim^{+}} \cup G^{\mid\!\sim^{-}}$

The tableaux rules are given in Figure 1.

$$(\mathbf{AX})\ G, A, \neg A \qquad\qquad (\neg)\ \frac{G, \neg\neg A}{G, A}$$

$$(\mid\!\sim^{+})\ \frac{G, A \mid\!\sim B}{G, \neg A, A \mid\!\sim B \qquad G, \neg\Box\neg A, A \mid\!\sim B \qquad G, B, A \mid\!\sim B}$$

$$(\mid\!\sim^{-})\ \frac{G, \neg(A \mid\!\sim B)}{A, \Box\neg A, \neg B, G^{\mid\!\sim^{\pm}}} \qquad\qquad (\Box^{-})\ \frac{G, \neg\Box A}{G^{\Box}, G^{\Box^{\downarrow}}, G^{\mid\!\sim^{\pm}}, \neg A, \Box A}$$

$$(\wedge^{+})\ \frac{G, A \wedge B}{G, A, B} \qquad\qquad (\wedge^{-})\ \frac{G, \neg(A \wedge B)}{G, \neg A \qquad G, \neg B}$$

Fig. 1. Tableaux system $\mathcal{T}\mathbf{P}$.

$$\cfrac{\cfrac{\cfrac{(A \mid\!\sim C) \wedge \neg(A \wedge C \mid\!\sim C)}{A \mid\!\sim C, \neg(A \wedge C \mid\!\sim C)}\ (\wedge^{+})}{A \mid\!\sim C, A \wedge C, \Box\neg(A \wedge C), \neg C}\ (\mid\!\sim^{-})}{A \mid\!\sim C, A, C, \neg C, \Box\neg(A \wedge C)}\ (\wedge^{+})$$

$$\times$$

Fig. 2. A derivation of $(A \mid\!\sim C) \wedge \neg(A \wedge C \mid\!\sim C)$ in $\mathcal{T}\mathbf{P}$.

The system is sound and complete with respect to the semantics.

Theorem 1 (Soundness of $\mathcal{T}\mathbf{P}$). *The system $\mathcal{T}\mathbf{P}$ is sound with respect to the semantics, i.e. if there is a closed tableaux for a set G, then G is unsatisfiable.*

To prove the completeness of $\mathcal{T}\mathbf{P}$ we have to show that if A is unsatisfiable, then there is a closed tableaux starting with A. We prove the contrapositive, that is: if there is no closed tableaux for A, then there is a model satisfying A. To build this model, the idea is to select sets with certain properties from *possibly different* open tableaux for A. This proof is inspired by [8].
First of all, we distinguish *static* and *dynamic* rules. $(\mid\!\sim^-)$ and $(\Box^-)$ are called *dynamic*, since their conclusion represents another world with respect to the premise; the other rules are called *static*, since the world represented by premise and conclusion(s) is the same. Moreover, we have to introduce the *saturation* of a set of formulas G. Given a set of formulas G, we say that it is saturated if all the static rules have been applied.

Definition 4 (Saturated sets). *A set of formulas G is saturated with respect to the static rules if the following conditions hold:*

- *if $A \wedge B \in G$ then $A, B \in G$;*
- *if $\neg(A \wedge B) \in G$ then $\neg A \in G$ or $\neg B \in G$;*
- *if $\neg\neg A \in G$ then $A \in G$;*
- *if $A \mid\!\sim B \in G$ then $\neg A \in G$ or $\neg\Box\neg A \in G$ or $B \in G$.*

Lemma 1. *Given a finite set of formulas G, there is a finite and saturated set $G' \supseteq G$. Moreover, if G is consistent, then G' is consistent.*

By Lemma 1, we can think of having a function which, given a consistent set G, returns one fixed consistent saturated set, denoted by $\mathtt{SAT}(G)$. Moreover, we denote by $\mathtt{APPLY}(G, F)$ the result of applying to G the rule for the principal connective in F. In case the rule for F has more conclusions (the case of a branching), we suppose that the function APPLY chooses one consistent conclusion in an arbitrary but fixed manner.

Theorem 2 (Completeness of $\mathcal{T}$P). $\mathcal{T}\mathbf{P}$ *is complete with respect to the semantics.*

Proof. As mentioned above, we assume that no tableaux for G_0 is closed, then we construct a model for G_0. We build X, the set of worlds of the model, as follows:

1. initialize $X = \{\mathtt{SAT}(G_0)\}$;
2. choose $G \in X$;
3. for each formula $\neg(A \mid\!\sim B) \in G$, be $G_{\neg(A\mid\sim B)} = \mathtt{SAT}(\mathtt{APPLY}(G, \neg(A \mid\!\sim B)))$. If $G_{\neg(A\mid\sim B)} \notin X$, then $X = X \cup G_{\neg(A\mid\sim B)}$;
4. for each formula $\neg\Box A \in G$, be $G_{\neg\Box A} = \mathtt{SAT}(\mathtt{APPLY}(G, \neg\Box A))$;
   - add the relation $G_{\neg\Box A} < G$;
   - if $G_{\neg\Box A} \notin X$, then $X = X \cup G_{\neg\Box A}$.
5. repeat from 2 until all elements in X have been considered.

This procedure terminates, since the number of possible sets of formulas that can be obtained by applying $\mathcal{TP}$'s rules to an initial finite set G is finite. We construct the model $\mathcal{M} = \langle X, <_X, V \rangle$ for G as follows:

- $<_X$ is the transitive closure of the relation $<$;
- $V(G) = \{P \mid P \in G\}$

In order to show that $\mathcal{M}$ is a preferential model for G, we prove the following facts:

Fact 1 *The relation $<_X$ is acyclic.*

Proof. If there were a loop we would have the following situation: there are G_1 and G_3 in X, $G_3 <_X G_1$, and G_1 is obtained again from G_3 by applying step 4 (i.e. $G_1 <_X G_3$). This situation, presented in Figure 3, will never happen. Indeed, since $G_3 <_X G_1$, G_3 has been generated by a sequence of applications of $(\Box^-)$, starting from an initial application of $(\Box^-)$ to some formula $\neg\Box\alpha$ in G_1. By the $(\Box^-)$ rule, $\Box\alpha \in G_3$. If G_1 were to be generated again from G_3 by an application of $(\Box^-)$, then $\Box\alpha \in G_1$, which contradicts the fact that G_1 is consistent. $\Box$

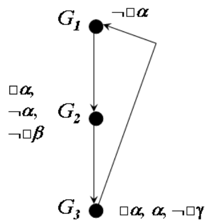

Fig. 3.

Fact 2 *For all formulas F and for all sets $G \in X$ we have:*

- *if $F \in G$ then $\mathcal{M}, G \models F$;*
- *if $\neg F \in G$ then $\mathcal{M}, G \not\models F$.*

Proof. By induction on the structure of F. If F is an atom P, then $\mathcal{M}, G \models P$ by definition of V. If F is $\neg P$, then $\neg P \in G$ implies that $P \notin G$ as G is not closed; thus, $\mathcal{M} \not\models P$ (by definition of V). For the inductive step we only consider the case of $(\neg) \mathrel{|\!\sim}$ and $(\neg)\Box$:

- $\Box A \in G$. Then, for all $G_i <_X G$ we have $A \in G_i$ since G_i has been generated by a sequence of applications of $(\Box^-)$, and by definition of $(\Box^-)$. By inductive hypothesis $\mathcal{M}, G_i \models A$, whence $\mathcal{M}, G \models \Box A$.
- $\neg\Box A \in G$. By construction there is a G' s.t. $G' <_X G$ and $\{\Box A, \neg A\} \subseteq G'$. By inductive hypothesis $\mathcal{M}, G' \models \neg A$. Thus, $\mathcal{M}, G \not\models \Box A$.

- $A \mathrel{|\!\sim} B \in G$: we have to show that $\mathcal{M}, G \models A \mathrel{|\!\sim} B$. Let $H \in Min_{<_X}(A)$; one can observe that (1)$\neg A \in H$ or (2)$\neg\Box\neg A \in H$ or (3)$B \in H$, since H is saturated. (1) cannot be, since by inductive hypothesis $\mathcal{M}, H \models A$. If (2), by the construction of $\mathcal{M}$ there exists a set $H' <_X H$ such that $A \in H'$. By inductive hypothesis $\mathcal{M}, H' \models A$, which contradicts $H \in Min_{<_X(A)}$. Thus it must be (3)$B \in H$, so that by the inductive hypothesis $\mathcal{M}, H \models B$.
- $\neg(A \mathrel{|\!\sim} B) \in G$: by construction of X, there exists $G' \in X$ such that $A, \Box\neg A, \neg B \in G'$. By inductive hypothesis we have that $\mathcal{M}, G' \models A$ and $\mathcal{M}, G' \models \Box\neg A$. It follows that $G' \in Min_{<_X}(A)$. Furthermore, always by induction, $\mathcal{M}, G' \not\models B$.

□

By the above Lemma the proof of the completeness of $\mathcal{TP}$ is over, since $\mathcal{M}$ is a model for the initial set G_0. □

A relevant property of the calculus that will be useful to estimate the complexity of logic **P** is the so-called *disjunction property* of conditional formulas:

Proposition 1 (Disjunction property). *If there is a closed tableaux for $G, \neg(A \mathrel{|\!\sim} B), \neg(C \mathrel{|\!\sim} D)$, then there is a closed tableaux either for $G, \neg(A \mathrel{|\!\sim} B)$ or for $G, \neg(C \mathrel{|\!\sim} D)$.*

The reason why this property holds is that the $(\mathrel{|\!\sim}^-)$ rule discards all the other formulas that could have been introduced by a previous application of this rule.

4 Decision Procedure for P

In this section we analyze the calculus $\mathcal{TP}$ in order to obtain a decision procedure for logic **P**. We also give explicit complexity bound for it.

First, we observe that the above calculus $\mathcal{TP}$ does not ensure a terminating proof search; in fact, the $(\mathrel{|\!\sim}^+)$ rule can be applied without any control, and this is a potential cause of the construction of infinite branches. The intuition is as follows: one needs to apply the $(\mathrel{|\!\sim}^+)$ rule on $A \mathrel{|\!\sim} B$ *only if any formulas introduced by the rule* (i.e. $\neg A$, $\neg\Box\neg A$ and B) *is not already in the premise.* We also need to reformulate the rules for the boolean connectives, in such a way that the formula to which the rule is applied is maintained in the conclusion. This reformulation is needed to avoid to reconsider a positive conditional $A \mathrel{|\!\sim} B$ once that its propositional subformulas ($\neg A$ and B) have been added to the current set of formulas. Since these subformulas might be later decomposed, if we do not keep them we would not be able to block the re-application of $(\mathrel{|\!\sim}^+)$ on $A \mathrel{|\!\sim} B$.

The terminating calculus $\mathcal{TP}^{\mathbf{T}}$ is presented in Figure 4.

The calculus $\mathcal{TP}^{\mathbf{T}}$ is sound and complete with respect to the semantics; we briefly show the completeness, as the soundness is immediate.

$$(\mathrel{|\!\sim}^+)\ \frac{G, A \mathrel{|\!\sim} B}{G, \neg A, A \mathrel{|\!\sim} B \qquad G, \neg\Box\neg A, A \mathrel{|\!\sim} B \qquad G, B, A \mathrel{|\!\sim} B}\ \text{if } G \cap \{\neg A, \neg\Box\neg A, B\} = \emptyset$$

$$(\wedge^+)\ \frac{G, A \wedge B}{G, A \wedge B, A}\ \text{if } A \notin G \qquad (\wedge^+)\ \frac{G, A \wedge B}{G, A \wedge B, B}\ \text{if } B \notin G$$

$$(\wedge^-)\ \frac{G, \neg(A \wedge B)}{G, \neg(A \wedge B), \neg A \qquad G, \neg(A \wedge B), \neg B}\ \text{if } \neg A, \neg B \notin G \qquad (\neg)\ \frac{G, \neg\neg A}{G, \neg\neg A, A}\ \text{if } A \notin G$$

Fig. 4. The calculus $\mathcal{TP}^{\mathbf{T}}$. Omitted rules $(\Box^-)$ and $(\mathrel{|\!\sim}^-)$ are the same as in Figure 1.

Theorem 3 (Completeness of $\mathcal{TP}^{\mathbf{T}}$). *The calculus $\mathcal{TP}^{\mathbf{T}}$ is complete with respect to the semantics.*

Proof. We show that if G is derivable in $\mathcal{TP}$ (i.e. there exists a closed tableaux with root G), then G is derivable in $\mathcal{TP}^{\mathbf{T}}$. The prove is by induction on the height of the derivation of G in $\mathcal{TP}$. If G is an axiom, we are done; otherwise, consider the rule applied to G: if it is $(\Box^-)$ or $(\mathrel{|\!\sim}^-)$, then we can conclude since the two rules are identical in the two calculi. The same for an application of $(\mathrel{|\!\sim}^+)$; indeed, if $(\mathrel{|\!\sim}^+)$ is applied to $G', A \mathrel{|\!\sim} B$, then we have to distinguish two cases: (i) $\neg A, \neg\Box\neg A, B \notin G'$: in this case we can easily conclude by an application of $(\mathrel{|\!\sim}^+)$ in $\mathcal{TP}^{\mathbf{T}}$; (ii) $\neg A \in G'$ or $\neg\Box\neg A \in G'$ or $B \in G'$: in this case, the $(\mathrel{|\!\sim}^+)$ rule is not applicable, by the restriction given in $\mathcal{TP}^{\mathbf{T}}$. However, we are analyzing the case when, in $\mathcal{TP}$, the $(\mathrel{|\!\sim}^+)$ rule has been applied on $G', A \mathrel{|\!\sim} B$ and one of the three conclusions is the same as the set $G', A \mathrel{|\!\sim} B$, therefore this rule application is useless and can be removed. By this fact, we can conclude the proof.

For the rules related to boolean connectives the proof is easy and left to the reader. As an example, consider $(\wedge^+)$ applied to $G', A \wedge B$ and just observe that if $A, B \notin G'$, then we obtain a proof in $\mathcal{TP}^{\mathbf{T}}$ by two applications of $(\wedge^+)$, otherwise at least one of the conclusions of the rule is identical to the premise, therefore it is useless to add both subformulas. □

In order to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search, we define a complexity measure on a set of formulas G, denoted by $m(G)$, which consists of four measures c_1, c_2, c_3 and c_4 in a lexicographic order. Before giving the formal definition, we need to define the following:

Definition 5 (Saturation multiset $\mathcal{S}$ of G). *Given a set of formulas G, we define the saturation multiset $\mathcal{S}$ as follows:*

$$\mathcal{S} = \{F \in G \mid (F = A \wedge B \text{ and } (A \notin G \text{ or } B \notin G)) \text{ or }$$
$$(F = \neg(A \wedge B) \text{ and } (\neg A \notin G \text{ and } \neg B \notin G)) \text{ or } ((F = \neg\neg A) \text{ and } (A \notin G))\}$$

Notice that $\mathcal{S}$ is a multiset; in particular, if $F = A \wedge B$ and both $A \notin G$ and $B \notin G$, then two instances of $F = A \wedge B$ belong to $\mathcal{S}$, whereas if $A \notin G$ and $B \in G$, then only one instance of $F = A \wedge B$ belongs to $\mathcal{S}$.

We write $A \mathrel{|\!\sim} B \in_+ G$ (resp. $A \mathrel{|\!\sim} B \in_- G$) if $A \mathrel{|\!\sim} B$ occurs positively (resp. negatively) in G, where positive and negative occurrences are defined in the standard way. We also denote with $cp(F)$ the complexity of a formula F.

Definition 6 (Lexicographic order). *We define $m(G) = \langle c_1, c_2, c_3, c_4 \rangle$ where:*

- $c_1 = \mid \{A \mathrel{|\!\sim} B \in_- G\} \mid$;
- $c_2 = \mid \{A \mathrel{|\!\sim} B \in_+ G \mid \Box\neg A \notin G\} \mid$;
- $c_3 = \mid \{A \mathrel{|\!\sim} B \in_+ G \mid \{\neg A, \neg\Box\neg A, B\} \cap G = \emptyset\} \mid$;
- $c_4 = \sum_{F \in \mathcal{S}} cp(F)$.

We consider the lexicographic order given by $m(G)$.

Intuitively, c_2 represents the number of positive conditionals in G *which can still create a new world*. The application of $(\Box^-)$ reduces c_2: indeed, if $(\mathrel{|\!\sim}^+)$ is applied to $A \mathrel{|\!\sim} B$, this application introduces a branch containing $\neg\Box\neg A$; when a new world is generated by an application of $(\Box^-)$ on $\neg\Box\neg A$, it contains A and $\Box\neg A$. If $(\mathrel{|\!\sim}^+)$ is applied to $A \mathrel{|\!\sim} B$ once again, then the conclusion where $\neg\Box\neg A$ is introduced is closed, by the presence of $\Box\neg A$ in that branch. c_4 represents the *saturation degree*, i.e. the sum of the complexities of formulas to which one can still apply propositional rules.

In order to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search we need the following:

Lemma 2. *Let G' be obtained by an application of a rule of $\mathcal{TP}^{\mathbf{T}}$ to a premise G. Then, $m(G') < m(G)$.*

Now we have all the elements to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search:

Theorem 4 (Termination of $\mathcal{TP}^{\mathbf{T}}$). *$\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search.*

We conclude this section with a complexity analysis of $\mathcal{TP}^{\mathbf{T}}$, in order to prove that validity in **P** belongs to **CoNP**. First of all, notice that we could take advantage of the disjunction property (Proposition 1). By this property we can reformulate the $(\mathrel{|\!\sim}^-)$ rule as follows:

$$\frac{G, \neg(A \mathrel{|\!\sim} B)}{A, \Box A, \neg B, G^{\mathrel{|\!\sim}^+}} \; (\mathrel{|\!\sim}^-)$$

This rule reduces the length of a branch at the price of making the proof search more non-deterministic. Moreover, observe that with this reformulation the parameter c_1 would no longer be needed.

We give a non-deterministic algorithm for testing validity in **P** that: (i) takes a set of formulas G as input; (ii) returns `false` iff G is satisfiable; (iii) generates a branch starting with G of polynomial length and requiring only a polynomial number of *guesses* in the size of G. The algorithm is defined below; in brackets we give the complexity of each operation, considering that $n = \mid G \mid$. Moreover, we assume that *after each step* from 1. to 4., a procedure `CHECK`, obviously of polynomial complexity, is invoked to verify if the current set of formulas is an axiom and, in that case, the execution terminates returning `true`:

input G: set of formulas;
output boolean;
begin
  1. $G' \longleftarrow$ guess one branch applying only propositional rules; $(O(n))$
  2. $G' \longleftarrow$ guess one $\neg(A_j \mid\!\sim B_j) \in G'$ and apply $(\mid\!\sim^-)$ on it; $(O(n))$
  3. **for** each $A_i \mid\!\sim B_i \in G'$ **do**
    3.1. $G' \longleftarrow G' \cup$ one of $\neg A_i, \neg\Box\neg A_i$ and B_i; $(O(1))$
    3.2. $G' \longleftarrow$ guess one branch applying only propositional rules; $(O(n))$
  4. **if** $\neg\Box\neg A_1, \ldots \neg\Box\neg A_k \in G'$
    4.1. $G' \longleftarrow$ guess one $\neg\Box\neg A_i$ and apply $(\Box^-)$ on it; $(O(1))$
    4.2. **repeat** from **3.**;
  5. **if** G' is an axiom **return** true;
    else return false;
end

Observe that when the algorithm exit the cycle no rule is applicable to the current set of formulas. Moreover, it can be shown that one can repeat the cycle between instructions 3. and 4. at most n times. Therefore, the complexity of the cycle is $O(n^2)$.

Theorem 5 (Complexity of P). *Validity of preferential logic* **P** *is* **CoNP**.

Proof. A is not valid in **P** iff the algorithm returns `false`. Since the algorithm is NP, non validity is NP; therefore, validity is CoNP. □

5 Loop Cumulative Logic CL

In this section we develop a tableaux calculus for **CL**. First we define a mapping of KLM cumulative models to preferential models which are essentially obtained by extending **P**-models with an additional accessibility relation R. Then, we define tableaux calculus $\mathcal{T}$**CL** for **CL** and we show that the calculus $\mathcal{T}$**CL** can be turned into a terminating calculus thus providing a decision procedure for **CL** and a CoNP complexity bound for validity in **CL**.

Let us first recall the axiomatization of the system **CL** and the notion of *cumulative models* as given by KLM in [10]. The axiomatization of the system **CL** can be obtained by the axiomatixation of the system **P** by removing the axiom OR and by adding the following infinite set of axioms LOOP:

$$(LOOP)\ (A_0 \mid\sim A_1) \land (A_1 \mid\sim A_2)...(A_{n-1} \mid\sim A_n) \land (A_n \mid\sim A_0) \rightarrow (A_0 \mid\sim A_n)$$

Notice that these axioms are derivable in **P**.

Definition 7 (Cumulative model (KLM)). *A cumulative model is a tuple* $\mathcal{M} = \langle S, l, <, V \rangle$*, where* S *is a set, whose elements are called states;* $l : S \mapsto 2^{\mathcal{U}}$ *is a function that labels every state with a nonempty set of worlds;* $<$ *is an irreflexive and transitive relation on* $\mathcal{W}$*.* V *is a valuation function* $V : \mathcal{U} \longmapsto pow(ATM)$*, which assigns to every world* w *the atoms holding in that world. For* $s \in S$ *and* A *propositional, we let* $s \models\equiv A$ *iff* $\forall w \in l(s) w \models A$*. Let* $Min_<(A)$ *be the set of minimal states* s *such that* $s \models\equiv A$*. We define* $\mathcal{M}, s \models\equiv A \mid\sim B$ *iff* $\forall s' \in Min_<(A)$*, then* $s' \models\equiv B$*. We assume that* $<$ *satisfies the smoothness condition.*

Let $\mathcal{L}_L$ be a modal conditional language obtained by $\mathcal{L}$ by introducing a new modality L as follows: (i) if A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$, $\neg\Box\neg LA \in \mathcal{L}_L$; ($ii$) if A, B are propositional, then $A \mid\sim B \in \mathcal{L}_L$; ($iii$) if A is a boolean combination of formulas of $\mathcal{L}_L$, then $A \in \mathcal{L}_L$.

We can map cumulative models into preferential structures with an additional accessibility relation as defined below:

Definition 8 (CL-preferential structures). *A model has the form* $\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$ *where:* $\mathcal{W}$ *is a non-empty set of items called worlds;* R *is a serial accessibility relation;* $<$ *is an irreflexive and transitive relation on* $\mathcal{W}$ *satisfying the smoothness condition;* V *is a function* $V : \mathcal{W} \longmapsto pow(ATM)$*, which assigns to every world* w *the atomic formulas holding in that world. We add to the truth conditions for preferential models in Definition 1 the following clause:*

$$M, w \models LA \textit{ iff for all } w' \mid Rww',\ w' \models A$$

Moreover, we need to change the truth condition for conditional formulas as follows $\mathcal{M}, w \models A \mid\sim B$ *if* $Min_<(LA) \subseteq [LB]$.

We establish a correspondence between cumulative models (KLM) and CL-preferential models by showing that we are able to map cumulative models to CL-preferential models (and vice-versa) by preserving the satisfiability of conditional formulas.

Proposition 2. *A set of conditional formulas* $\{(\neg)A_1 \mid\!\sim B_1, \ldots, (\neg)A_n \mid\!\sim B_n\}$ *is satisfiable in a cumulative model* $\langle S, l, <, V\rangle$ *iff it is satisfiable in a C-preferential model* $\langle W, R, <, V\rangle$.

We will now present a tableaux calculus for **CL**. The calculus can be obtained from the calculus $\mathcal{T}$**P** for preferential logics, by adding a suitable rule for dealing with the modality L. We define $G^{L^{\downarrow}} = \{A \mid LA \in G\}$.

Our tableaux system $\mathcal{T}$**CL** for **CL** is shown in Figure 5 and is obtained by introducing the new modality L in the rules of $\mathcal{T}$**P** and by adding the new rule (L^-). Observe that rules $(\mid\!\sim^+)$ and $(\mid\!\sim^-)$ have been changed as they introduce the modality L in front of the propositional formulas A and B in their conclusions. The new rule (L^-) is a dynamic rule.

$$(\mid\!\sim^+)\ \frac{G, A \mid\!\sim B}{G, \neg LA, A \mid\!\sim B \qquad G, \neg\Box\neg LA, A \mid\!\sim B \qquad G, LB, A \mid\!\sim B} \qquad (\mid\!\sim^-)\ \frac{G, \neg(A \mid\!\sim B)}{LA, \Box\neg LA, \neg LB, G^{\mid\sim^{\pm}}}$$

$$(\mathbf{L}^-)\ \frac{G, \neg LA}{G^{L^{\downarrow}}, \neg A}\ \text{ where } \{\neg LA\} \text{ may be empty} \qquad (\Box^-)\ \frac{G, \neg\Box A}{G^{\Box}, G^{\Box^{\downarrow}}, G^{\mid\sim^{\pm}}, \neg A, \Box A}$$

Fig. 5. Tableaux system $\mathcal{T}$**CL**. If $\neg LA$ is not in the premise of (L^-) (i.e. $\{\neg LA\} = \emptyset$) the rule allows to step from G to $G^{L^{\downarrow}}$. The boolean rules are omitted.

The proof of the completeness of the calculus can be done as for the preferential case, provided we suitably modify the procedure for constructing a model for a finite consistent set of formulas G of $\mathcal{L}_L$. First of all, we need to modify the definition of saturated sets.

Definition 9 (Saturated sets). *A set of formulas G is saturated with respect to the static rules if, in addition to the conditions for boolean connectives in Definition 4, the following conditions also hold:*

- *if* $A \mid\!\sim B \in G$ *then* $\neg LA \in G$ *or* $\neg\Box\neg LA \in G$ *or* $LB \in G$;
- *if* $LA \in G$ *then* $\neg L\neg A \in G$

For this notion of saturated set of formulas we can still prove Lemma 1 for language $\mathcal{L}_L$.

Theorem 6 (Completeness of $\mathcal{T}$CL). $\mathcal{T}$**CL** *is complete with respect to the semantics.*

Let us now analyze the calculus $\mathcal{T}$**CL** in order to obtain a decision procedure for **CL** logic. First of all, we reformulate the calculus as we made for **P**, obtaining a system called $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$. Rules for boolean connectives are the same as in $\mathcal{T}\mathbf{P}^{\mathbf{T}}$ (see Figure 4). The $(\mid\sim^{+})$ rule can be applied only if none of the subformulas $\neg LA$, $\neg\Box\neg LA$ and LB are already in the current node. The (L^{-}) rule is obviously applicable only if $\{\neg LA\} \cup G^{L^{\downarrow}} \neq \emptyset$. $(\mid\sim^{-})$ and $(\Box^{-})$ are not reformulated.

In order to prove that $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ ensures a terminating proof search we define a measure $m(G) =< c_1, c_2, c_3, c_{3-4}, c_4 >$, defined as in the case[1] of $\mathcal{T}\mathbf{P}^{\mathbf{T}}$, with the addiction of the index c_{3-4} defined as $c_{3-4} =\mid \{(\neg)LA \mid (\neg)LA \in G\} \mid$. Exactly as we made for **P**, we consider a lexicographic order given by $m(G)$, and easily prove that each application of the rules of $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ reduces this measure, as stated by the following:

Lemma 3. *Consider an application of any rule of* $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ *to a premise* G *and be* G' *any conclusion obtained; we have that* $m(G') < m(G)$.

Proof. Identical to the proof of Lemma 2. Just observe that if (L^{-}) is applied, then c_1, c_2 and c_3 are the same in both the premise and the conclusion, but c_{3-4} decreases, since (at least) one formula LA or $\neg LA$ is removed from the conclusion. □

By Lemma 3 we can conclude that $\mathcal{T}\mathbf{CL}^{\mathbf{T}}$ ensures a terminating proof search. Moreover, we easily observe that the disjunction property holds in **CL**; therefore, the algorithm described to give an explicit complexity bound for **P** can be used to prove that validity of **CL** is a CoNP problem. The algorithm is a non deterministic algorithm which generates a branch of polynomial length and evaluates in polynomial time with respect to the length of the initial set of formulas if the branch is closed (returning `true`) or not (returning `false`). The following instructions are situated between instructions 4. and 5.:

4'. if $LA_1, LA_2, ..., LA_u, \neg LA_1, \neg LA_2, ..., \neg LA_w \in G'$
  4'.1 $G' \longleftarrow$ (if $w \neq 0$) guess one $\neg LA_i$ and apply (L^{-}) on it,
    otherwise, if $w = 0$ and $u \neq 0$, then apply (L^{-}); $(O(1))$
  4'.2 $G' \longleftarrow$ guess one branch applying only propositional rules; $(O(n))$

Similarly to the case of **P** we conclude with the following:

Theorem 7 (Complexity of CL). *Validity for* **CL** *is* **CoNP**.

6 Conclusions

In this paper, we have presented some tableaux calculi for some of the KLM logical systems for default reasoning. We have given a tableaux calculus for

[1] These indexes are adapted considering the L modality; for instance, c_2 is defined as follows: $c_2 =\mid \{A \mid\sim B \in_{+} G \mid \Box\neg LA \notin G\} \mid$.

preferential logic **P** and for loop-cumulative logic **CL**. The calculi presented give a decision procedure for the respective logics, whose complexity is CoNP for both **P** and **CL**.

Artosi, Governatori, and Rotolo [1] develop a labelled tableaux calculus for a flat fragment of the conditional logic CU, corresponding to the system **C**. In this paper, we do not treat this system but a stronger version of it, namely **CL**. Furthermore, as a major difference from our approach, the calculus proposed in [1] makes use of labels. The labels are introduced in order to represent possible worlds or sets of possible worlds (for instance the label (W^A, w) stands for any world in $f(A, w)$). In addition to the rules for dealing with "declarative" formulas, they introduce a unification algorithm in order to deal with labels. As a difference from our calculus, the one proposed in [1] contains a cut-rule, called PB, thus is not analytic, unless one can restrict the application of cut in an analytic way. No matter whether or not cut is advantageous, it is clear that we could incorporate it as a heuristic rule in our calculus.

In [7] it is defined a labelled tableaux calculus for the logic **CE** and some of its extensions. The flat fragment of **CE** corresponds to the system **P**. The similarity between the two calculi lies in the fact that both approaches use a modal interpretation of conditionals. The major difference is that the calculus presented here does not use labels, whereas the one proposed in [7] does. This is of course made possible by the fact that the language here is simpler that the one in [7], where there are nested conditionals. A further difference is in the logics considered: whereas [7] considers **CE** and some *stronger* logics, we consider **P** (corresponding to **CE**) and the *weaker* **CL**.

Lehmann and Magidor [11] propose a non-deterministic algorithm that, given a finite set K of conditional assertions $\gamma_i \mid\!\sim \delta_i$ and a conditional assertion $\alpha \mid\!\sim \beta$, checks if $\alpha \mid\!\sim \beta$ is not preferentially entailed by K. This algorithm tries to find a *witness* for a conditional assertion, i.e. a finite sequence of pairs (I_i, f_i), where I_i are sets of indexes and f_i are worlds in a preferential model. They prove that a conditional assertion $\alpha \mid\!\sim \beta$ has a witness iff it is not preferentially entailed by K. They conclude that preferential entailement is CoNP, thus obtaining a complexity result similar to ours. However, it is not easy to compare their algorithm with our calculus, since the two approaches are radically different. As far as the complexity result is concerned, notice that our result is more general than theirs, since our language is richer: we consider boolean combinations of conditional assertions (and also combinations with propositional formulas), whereas they do not. As remarked by Boutilier [2], this more general result is not an obvious consequence of the more restricted one. Moreover, we prove the CoNP result also for the system **CL**. At the best of our knowledge, this result was unknown up to now.

In this paper we only consider two of the logical systems for nonmonotonic reasoning defined by KLM. We plan to extend our calculi to the other KLM systems, namely to the weaker **C** and to the stronger **R**. For **C** we conjecture that a complete calculus is given by a variant of $\mathcal{T}$**CL** in which the ($\Box^-$) rule is weakened so that it does not enforce the transitivity of the preferential relation $<$. Another development of our work will be the extension to first order case. The starting point will be the analysis of first order preferential and rational logics by Friedman, Halpern and Koller in [5].

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