Truth Translations of Basic Relevant Logics

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ABSTRACT. This paper is submitted in memory of Meyer’s old friend and longtime collaborator Hugues Leblanc, to whom it is dedicated. It explores the consequences of the classical first-order semantics for the basic positive relevant logic $B+$ of [44] and its classical extension $CB$. We study entailments $A \leq C$, where $A$ and $C$ are built up from variables and $\rightarrow, \land, \lor, \lnot$. The mated classical first-order theory $CBMODEL$ has the usual vocabulary, based on a ternary relation symbol ‘$R$’ (for the relevant accessibility relation) and, for each propositional variable ‘$p$’, a corresponding unary predicate ‘$P$’. There is an infinite run of individual variables ‘$w$’, etc. We use the semantical truth-conditions of [44] to define an extended unary predicate $[B]$ for each formula of $B+$, on the following recursive rubric:

- $[p]w = Pw$, where $P$ is the unary predicate corresponding to the propositional variable $p$;
- $[C \land D]w = [C]w \land [D]w$;
- $[C \lor D]w = [C]w \lor [D]w$;
- $[\neg C]w = \neg([C]w)$;
- $[C \rightarrow D]w = \forall x \forall y(Rwxy \supset ([C]x \supset [D]y))$.

THEOREM. $CB \vdash A \rightarrow C$ iff $CBMODEL \vdash \forall w([A]w \supset [C]w)$.

The theorem, near enough, is from [44]. But note that it translates relevant PROVABILITY into simple first-order VALIDITY. Put otherwise, $B+$ (and its conservative BOOLEAN extension $CB$) reduces RELEVANT LOGIC to CLASSICAL LOGIC simpliciter. (The modal analogue is the minimal normal modal logic $K$.)

1 Introduction

A. R. Anderson posed in [2] the question, “Is there a (formal) semantics for the system $E$ of (relevant) entailment?” [45] and subsequent papers gave an affirmative answer to this question. The general method was a “worlds semantics”, roughly in the style of Kripke’s [26]. The major innovation was that, just as modal logicians introduced a binary relation to explicate the unary necessity and possibility operators, so we required a ternary relation
R to spell out formally the semantics of (irreducibly) binary connectives like the → of relevant implication and entailment.

The relational semantics that Meyer proposed with Routley was (or might as well have been) classical. This suggests that the study of propositional relevant logics is also the study of appropriate classical first-order theories. The same can be said of the theories of types proposed by researchers into Combinatory Logic (henceforth, CL) and Lambda Calculus (henceforth λ). Especially exciting here is the first-order theory CBMODEL with no proper axioms. We shall show that, on a well-motivated translation, this classical theory exactly contains not only relevant logic but also intersection, union and Boolean type theories.

The original aim of our relational semantical effort was to provide a well-motivated analysis for such famous relevant logics as the Anderson-Belnap systems R and E of [3]. But even as Kripke moved on from the boyhood delight of taking on S5 in his [24] to a later plethora of modal logics, just so the supply of relevant logics multiplied. The analogy can be pushed further. There is a minimal (normal) modal logic, which has come (in, e.g., Bull and Segerberg [10]) to be called K (for Kripke). This is the system without special assumptions, save those that arise from the method. Other modal logics arise on adding specific semantic postulates—e.g., that the binary relation be transitive and reflexive, which suffices for S4.

There is a similar minimal positive relevant logic. It is the system B+ of [44]. And B+ and its conservative Boolean extension CB are the systems of relevant entailment that this paper will (mainly) be about. We pause briefly to think of the other positive relevant logics that arise, when additional semantical postulates are added to those that produce B+. What are these postulates? And what may we take them to say?

2 An Example: R+

As an example, pick the system R. More accurately, pick its positive fragment R+ (the care and feeding of negation being no immediate concern). R+ stands to B+ (more or less) as S4 stands to K. As S4 demands binary reflexivity, so R+ imposes, for all “worlds” w,

\[ \text{Ternary Reflexivity Postulate.} \quad R_{www} \quad \text{TRP} \]

A further relevant postulate takes a little more work. First, note that, while relational products of binary relations are themselves binary relations, we can build up 4-ary, 5-ary and in general n-ary relations by taking products of 3-ary ones. Not only that, but there are distinct ways of thus relating
4 or more things.\(^1\) We shall distinguish the following:

\[
\begin{align*}
Rxyzw &= \exists a (Rxya \land Razw) \\
Rx(yz)w &= \exists b (Rxbw \land Ryzb)
\end{align*}
\]

The extra \(R^+\) postulate then turns out to be, for all \(w, x, y, z,\)

Pasch Postulate. \quad \text{\(Rwxyz \Rightarrow Rwyxz\)} \quad \text{PP}

And \(R^+\) is the system that one gets if one imposes TRP and PP on top of the underlying semantical machinery for \(B^+\) in [44].

Glance again at the special \(R^+\) postulates. Does anything occur to you? In the first place, TRP seems to be doing some duplicating. Is there a link, perhaps, to the duplicating combinator \(\lambda x.xx\) (alias \(\Delta\) or \(\text{WI}\) or \(\text{SII}\) for \(\text{CL}\) fans)? If you thought this you’re right. In the second place, PP seems to be doing some permuting. And the permuting that it is doing is, near enough, that induced by Curry’s combinator \(C\) (alias \(\lambda wxy.wyx\) for the \(\lambda\) crowd).

These connections between candidate relevant axioms and matching combinators show up also in the formulation of other relevant logics. The principle on which the matching occurs is that the candidate axioms count as the types, on the Curry-Howard isomorphism, of the corresponding combinators. (This is the formulas-as-types interpretation, \textit{in spades}!) So pleased were we with these connections that [32] toyed with the thought that \(\text{CL}\) was The Key to the Universe. ([36], of which numerous friends are co-authors, has already been given at a couple of conferences. The written version is in preparation; think of the present essay as an introduction thereto.)

There are, however, more connections than were dreamt of in Curry’s philosophy. Consider, one more time, the \(R^+\) postulate TRP. Its combinator twin is \(\text{WI}\). But \(\text{WI}\), on the analysis of Curry & Feys [17], has no type. The reason for this is that Curry-Feys types all correspond to pure \(\rightarrow\)-formulas. In fact, as Coppo, Dezani, Pottinger, Sallé, Venneri, Barendregt, Ronchi \textit{et al.} saw quite independently\(^2\) (a few years later, but more deeply—cf. [11, 15, 37, 16, 5, 42]), by also making \(\land\) explicit we get a matching type for \(\lambda x.xx\). Thinking propositionally, add the axiom scheme

\[^1\text{We used to write 4-ary versions of } R \text{ as } R^2. \text{ As this would become tedious by about } R^9541920, \text{ we elect here (as Dunn also has done) simply to use ‘} R \text{ ‘ again for the various relational products.}\]

\[^2\text{After [32], Meyer sought to enlist his friends in the } \text{CL} \text{ community in working out the subject matter of the present essay. For a few years, no dice! Then in 1986, as he was dwelling yet again on the combinatory glories of } B^+, \text{ he was informed by Hindley, ‘But that’s already been done.’ Thanks to him, accordingly, for calling the Torino work on intersection types to our attention. (And it hadn’t all been done!)}\]
AxWI \((A \rightarrow B) \land A \rightarrow B\)

It is well known that, on the extension of the Curry–Howard isomorphism to intersection types, AxWI marries the pure duplicating combinator WI.

From our present perspective, R+ is a typical (positive) relevant logic. In its pure implicational part R→, it is just Church’s weak implication [13, 14]. Also unsurprisingly, R→ mirrors Church’s preferred λI version of λ as pure intuitionist implication J→ mirrors λK.3 Also, R+ is just R→, dressed up by Anderson and Belnap in the distributive lattice clothes that extend E→ to E+ via Ackermann [1].4

3 A little formal housekeeping

Logic is the science of inference. Does Logic then, like other sciences, aim to set out and codify a body of truths? We answer “Yes”. And “No”. No, because the job of Logic is to clarify Right Reason. How the biologists, or the astronomers, or the politicians reason correctly from their premisses is our first order of business as logicians. But also Yes, since Logic itself is formulated as a deductive discipline, with its own axioms and rules of inference.

The tension between our “Yes” and our “No” is reflected in alternative choices of formal syntax for the language of Logic. The first essays in the newfangled algebra of logic, by Boole and his successors, were relational. Later in the 19th century came Frege and Russell and that crew, who introduced an assertional way of stipulating Logic. In particular, when we say that A implies B, should we formalize that relationally by writing something like

\((\rho)\) \hspace{1em} A \leq B,

or assertionally with

\((\alpha)\) \hspace{1em} \vdash (A \rightarrow B)?

Much ink, not to mention an entire literature about use-mention confusion, has been spilled by the strife of \((\rho)\) with \((\alpha)\). Suffice it to say here that the likes of \((\alpha)\) go with our “Yes” above; while \((\rho)\) inclines to our “No”. For “Yes” says that Logic is a body of truths—bearing a prefaced turnstile to say so; while getting on in theories is Riding the Rails of a suitable \(\leq\).

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3 Curry once observed, in commenting on [14], that Church never much liked the combinator K. For Church’s λ views, see [12]. For putting meat on the bones of the λI → R→ connection, see Helman [23]. Should we call this the Church–Helman isomorphism, as the λK → J→ connection has attained fame as the Curry–Howard isomorphism?

4 Further DeMorgan lattice extensions take R→ all the way to R, as, in effect, [1] took E→ to E.
To set up our truth translations, we recall and adapt some formal machinery in previous papers by Meyer and by Routley, especially [44], [32] and [33]. We invite you to skip (but not too lightly) through this section on first reading, since more motivation is coming.

We formulate our logics in two versions. The first, in the sense of Curry’s [18], will be assertional. We use ⊢ as a unary predicate. Where A is a formula of our language, ⊢ A will be an (elementary) statement. Axioms and theorems are all statements. Our second formulations will be, also in the sense of Curry [18], relational systems. Where A and C are formulas, A ≤ C will be an (elementary) statement. Theorems in this formulation will be of the form A ≤ C. There is a simple translation from the assertional systems to the relational ones, given by ⊢ A → C iff A ≤ C, for each system. Our main focus here will be on the relational formulations.

In [44], B+ was axiomatized on p. 193. Given a countable stock of atoms p, q, r, etc., we here\(^5\) specify formulas (i.e., Curry’s obs) A of its language L+ by

\[
A ::= p | A \land A | A \lor A | A \rightarrow A
\]

Everyone knows that the main business of logic is entailment—keeping track of what follows from what. In line with motivating remarks above, this gives the relational formulations of our logics some priority. We pronounce ‘A ≤ C’ as ‘A entails C’.

Against the odds, perhaps, in these licentious times, modern logicians resist the temptation to be relational more often than they yield to it. When so resisting, formulas of any (well-formed) shape are theorem candidates. The logical grammar that we have in mind, still following Curry [18], is a little different. A formula, standing alone, functions linguistically as a noun. But the theorems of a system, stated in what Curry calls the U-language, are statements. To make a sentence from a noun, one requires a verb. Our candidate verbs are the prefixed ⊢ for the assertional systems, and the infixed ≤ for the relational ones.

Nonetheless, we introduce ≤ in the assertional systems as an overloaded particle. We may write a statement of the form ⊢ A → B as A ≤ B.\(^6\) This facilitates easier reading of formulas. Other advantages of the ≤ notation are (a) it makes immediate contact with algebraic counterparts of logics and (b) it stresses the essential unity of relevant logics and intersection (and union) theories of types. In pursuit of these themes, we have introduced relational

\(^5\)We ignore, for now, the Ackermann constant t and the (contrasting) Church constant T.

\(^6\)Including binary particles added or defined below, the full increasing scope order is o, ∧, ∨, ←, →, ↔, ⊃, ≡, otherwise associating to the left and using dots as in Curry [18].
systems for which ‘\(\vdash A\)’ is ill-formed; here the elementary statements of our language are themselves of the form \(A \leq B\).

The full language \(L\) adds classical \(\neg\) to \(L^+\). So our Backus-Naur specification for all of \(L\) is

\[
A ::= p|\neg A|A \land A|A \lor A|A \rightarrow A
\]

Other particles may be added to taste. We assume definitions in \(L\) of other connectives—e.g.,

\[
D \supset A \supset B = \text{df } \neg A \lor B \quad D =. A \equiv B = \text{df } (A \supset B) \land (B \supset A)
\]

We now make explicit which are the statements of our systems. When \(A\) is a formula, \(\vdash A\) shall be an elementary statement of the language we may call \(L^\vdash\). When \(A\) and \(B\) are formulas, \(A \leq B\) shall be an elementary statement of \(L^{\leq}\). We reserve the right to insert ‘\(\oplus\)’ into the notation and otherwise decorate it for particular purposes. (We also reserve the right to be sloppy when pedantry would be irksome; readers who catch us will be awarded 3 Mars bars). For now, note merely that \(L^{\leq}\) may be considered a sublanguage of \(L^\vdash\) (on the convention that the entailment relation holds between \(A\) and \(B\) just in case the implication \(A \rightarrow B\) is a logical truth. In symbols, \(A \leq B\) iff, as a matter of logic, \(\vdash A \rightarrow B\).) But \(L^\vdash\) is not a sublanguage of \(L^{\leq}\) (on which point a pleasant subtlety of our truth translations will eventually rest). Until further notice, we assume that the elementary statements are those of \(L^\vdash\).

A positive model structure (+ms) was a triple \(K = \langle 0, K, R \rangle\), where \(K\) is a set, \(0 \in K\) and \(R\) is a ternary relation on \(K\). A binary relation \(\subseteq\) was defined by\(^7\)

\[
d \subseteq. a \subseteq b = \text{df } R0ab,
\]

and the following postulates were laid down,\(^8\) for all \(a, b, c, x \in K\):

\begin{align*}
p1. & \quad \subseteq \text{ is reflexive and transitive.} \\
p2. & \quad a \subseteq x \text{ and } Rxbc \Rightarrow Rabc \\
p3. & \quad b \subseteq x \text{ and } Raxc \Rightarrow Rabc \\
p4. & \quad x \subseteq c \text{ and } Ra0x \Rightarrow Rabc
\end{align*}

\(^7\)We have previously used < where we write \(\subseteq\) here. But as \(\subseteq\) is a semantical notion meaning subtheory, near enough, to stick with < now would have invited confusion with \(\leq\) (a logical notion meaning entails).

\(^8\)These postulates are from [32], which expands on those of [44].
Also important are the notions of interpretation, verification, and validity. Let \( 2 = \{0, 1\} \) be the usual truth-values \{false, true\}, and let \( K \) be a +ms. A possible interpretation is just any function \( I \) from \( L^+ \times K \) to \( 2 \). Fixing \( I \) and \( K \) in context, we may write, for a formula \( A \) and \( w \in K \),

\[
[A]_w
\]

for \( I(A, w) = 1 \). We use other truth-functional particles and quantifiers over \( K \) to continue the story, to specify the conditions on which a possible interpretation \( I \) is an interpretation.

These conditions are of two sorts. First, there are truth-conditions, which require (as usual) the value of a compound formula \( A \) at \( w \) on \( I \) to depend on values of its immediate subformulas. For the moment these are \( T \land, T \lor \) and \( T \rightarrow \), spelled out below. Second, there is a hereditary condition \( H \), familiar from semantic investigations of other non-classical logics.\(^9\)

\[
H. \ a \subseteq b \Rightarrow ([A]_a \Rightarrow [A]_b), \text{ for all formulas } A \text{ and } a, b \in K.
\]

As it turns out, it suffices inductively for the full condition \( H \) merely to impose it on atoms, via

\[
Hp. \ a \subseteq b \Rightarrow ([p]_a \Rightarrow [p]_b), \text{ for all atoms } p \text{ and } a, b \in K.
\]

An interpretation \( I \) of \( L^+ \) in the +ms \( K \) is then any possible interpretation satisfying the truth-conditions and \( Hp \). Since 0 is intended as a real (or logical) world, we say that \( A \) is verified on \( I \) just in case \( A[0] \). \( A \) is valid in the +ms \( K \) iff \( A \) is verified on all interpretations \( I \) in \( K \). And finally \( A \) is positively valid iff \( A \) is valid in all +ms. Theorem 2 of \([44]\) delivered the following happy result (among others):

\[
\text{Soundness and completeness of } B^+. \quad \vdash A \text{ is a theorem of } B^+ \quad \text{iff } A \text{ is positively valid.}
\]

4 Basic relevant logics

When relevant logics first appeared, they were dismissed by many as deplorably weak. More lately, systems like \( R \) have taken flak as unconscionably strong.\(^{10}\) There is a simple remedy for such complaints: if you don’t like all the axioms, change or delete some. This has an obvious semantical counterpart: if you don’t like all the postulates, change or delete some.\(^{11}\)

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\(^9\)In particular, Kripke [27] on intuitionist logic.

\(^{10}\)Cf., e.g., Brady [9].

\(^{11}\)You can even, if you would like, add some postulates (with their corresponding axioms, of course).
But when, you ask, will it ever end? The deletion process terminates, we affirm, when all the special postulates are gone, and only those principles remain that belong to the method. From [45] on, the following have been central:

1. This is a worlds semantics.
2. Logical entailment is interpreted as truth-preservation across worlds.
3. Extensional particles are interpreted extensionally.
4. Intensional particles are interpreted intensionally, using the ternary relation $R$.
5. There is no universally preferred relevant logic.

We now say a bit about each of these 5 points.

Ad 1. We follow Leibniz (and Kripke) in referring to the points of our model structures as worlds. We insist that they need not be possible worlds. And there are many other things to call these points—theories, states, setups, cases, posets, filters—in accordance with the ontological predilections, terminological idiosyncrasies and grant applications of various authors.\textsuperscript{12}

Ad 2. That $A$ entails $B$ says, on an old story, that $B$ is true everywhere that $A$ is true. This is also the relevant semantical story.

Ad 3. Belnap has in conversation laid it down, and we have long agreed, that to interpret the extensional particles and, or, and not extensionally is to fix their truth-values at a world $w$ on the values of their components at $w$. This motivates the truth-conditions on an interpretation $I$ that we may write as follows:

$$T \land. \quad [A \land B]_w = [A]_w \land [B]_w$$
$$T \lor. \quad [A \lor B]_w = [A]_w \lor [B]_w$$
$$T \neg. \quad [\neg A]_w = \neg [A]_w$$

The $\neg$ here in question is Boolean negation.\textsuperscript{13}

\textsuperscript{12}Even before Kit Fine [22], we leaned to theories ourselves. Cf. Routley–Meyer [45].

\textsuperscript{13}Boolean negation was a newfangled particle in relevant logics, conservatively introduced (for $R+$ and $R$) by us in [33] and [34] respectively. Original equipment for these logics was the (not exactly extensional) DeMorgan negation $\sim$. 
Ad 4. The intensional particles are *whatever we want them to be*. Certainly the (relevant) implication \(\rightarrow\) is expected to be among them. In weaker relevant logics, \(\rightarrow\) counts as a *left-to-right* conditional.\(^{14}\) It is also possible to have a *right-to-left* conditional, which we will symbolize by \(\leftarrow\).\(^{15}\) Other candidates are fusion \(\circ\), fission \(+\), traditional necessity \(\Box\) and possibility \(\Diamond\), etc. If fusion is *commutative*, then \(B \leftarrow A\) is *the same* as \(A \rightarrow B\). Here are the two most salient semantical truth-conditions on *relevant* particles:

\[
\begin{align*}
T \rightarrow [A \rightarrow B] &= \forall x \forall y (Rwxy \Rightarrow ([A]x \Rightarrow [B]y)) \\
T o. [A \circ B] &= \exists x \exists y (Rxyw \land [A]x \land [B]y)
\end{align*}
\]

The relevant particles are interpreted *intensionally*. To fix for example the value of \(A \rightarrow B\) at \(w\), it is necessary to look at worlds *beyond* \(w\). Somewhere in between is the DeMorgan negation \(\sim\). Invoking the Routley \(*\) of [47], its truth-condition on the semantics of [45] (etc.) is just

\[
T \sim [\sim A] = \neg ([A]^{*})
\]

Ad 5. It was once an open question which was the *correct* modal logic. C. I. Lewis (a bit unwillingly), Kripke [26] and their cohorts turned this question on its head, by showing that one could season and salt modal logics to taste. Relevant logics have a similar history. Even [3] introduces a number of relevant logics (a bit unwillingly, since it *claims* to prefer \(E\)).

Seasoning and salting relevant logics is *not* our present business. That business is rather getting to the *core* of the semantical analysis hitherto laid down. As this is *logic*, the analysis and critique of *inference*, this core will concentrate on the *valid* entailments \(A \leq B\). At the *elemental* level, we thus view (relevant) implication (as we have since Meyer [28]) as *essentially relational*.

Appealing to the *Semantic Entailment* lemma of previous papers, which also plays a role below, we lay down

\[
T \leq [A] \leq [B] = \forall x ([A]x \Rightarrow [B]x). \quad (\text{SemEnt})
\]

\(^{14}\)We adopt the well-chosen terminology of Restall [41]. This is, as [41] also notes, a *left residual* in the sense of Birkhoff [8]. It is also the *arrow of entailment* in the relevant logic literature.

\(^{15}\)We reverse the arrow with reluctance. The idea is that \(B \leftarrow A\) shall be our notation for ‘\(A\ right-implies B\)’, a *right residual* in the [8] sense. Dunn, in his [20], accurately notes that there is no consistency in the literature about which is the “left” residual and which is the “right”. We follow our [32]. Dunn’s [21] (sob!) *misread* [32], intimating that it had assumed that fusion is commutative. [32] made no such general assumption, and very clearly considered both residuals.
The present concentration on semantic entailment, although always a key element in relevant analysis (especially in *soundness* proofs), only became *explicit* in the (so-called) *simplified semantics* of Priest and Routley [39] and Restall [40]. We carry it a little further here, by dropping the “real world” 0 and the set of “logical worlds” to which 0 belongs.

This takes cares of items 1 and 2 above. We turn now to 3, which enjoins us to *respect* whatever truth-functional particles are present. Thinking of worlds as *theories*,16 the truth-condition $T \land$ is built-in. But $T \lor$ (and $T \neg$, when present) require thought. In C.S. Peirce’s *long run*, $A \lor B$ needs the support of at least one of $A, B$. In the long run, at least as a *regulative ideal*,17 we expect $\neg A$ to be *present* iff $A$ is *absent*. So we prefer prime theories, which resolve the disjunctions they contain. And we prefer negation *consistent* and *complete* theories, which, when $\neg$ is present, contain exactly one of the pair $A, \neg A$, for each formula $A$.

These preferences have given shape to the ternary *relational semantics* for relevant logics. A simpler *operational semantics*, along the lines of Urquhart [49] or Fine [22], works fine in sorting out the relevant behavior of $\to$ and $\land$. But [49] breaks down for $R^+$ in the presence of $T \lor$. And [22] introduces the prime theory apparatus too, after which it is *not pretty*.18

We turn now to point 4—the ternary relation $R$ itself, and its role in accounting for the relevant $\to$ and its suite. Many efforts have been made to give *intuitive content* to this relation—to *picture* it and otherwise to get up close and personal. We blush to confess that we have been associated with some of these efforts; as to others, which permute the order of the terms and which attempt binary19 reductions and the like, our conscience is clear. The intuition that remains is that $T \to$ is the *formal instrument* which looks after the informal operation of *modus ponens*. For what $Rxyz$ *says* (as the completeness proofs make clear) is that $x$ *likes* $B \to C$ if and only if, whenever $y$ *likes* $B$ then $z$ *likes* $C$. That is a step, from left to right, of modus ponens for $\to$; a contrapositive move then takes us us from right

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16I.e., *filters*, from the lattice-theoretic viewpoint.

17We are *far from suggesting* that such regulative ideals should dominate *logic in prac-tice*. To the contrary, much nonsense has emanated from the insistence that theories be *negation-consistent*, on penalty of concluding *absolutely everything*. But if $\neg$ is *Boolean*, one had better be *consistent*!

18But it is *fine*. While [45] and its suite somewhat anticipated Fine [22], it was (we think) the same calculus of theories that provided the mathematical background for his work and ours.

19Some binary reductions in fact work well—cf. [4] for Dunn’s work on $RM$, for example. But nothing works (at least as we see it) to dispatch the *general* relevant ternary $R$ in favor of what can be cooked up from 2-place relations. Satisfyingly, as both Dunn and R. Wille have called to our attention, C.S. Peirce also took *3-place* relations to be the *most fundamental* ones.
to left.

While ternary R is all about modus ponens for → (the one rule that you can count on while all others dissolve around you), it gives rise to other (conservatively addable) useful connectives. We have set out To, which provides a fusion operator to bunch premises in an intensional way as $T \land$ bunches them by extensional conjunction. Clearly there are several more fusion-like connectives $o'$, depending which of $w, x, y$ take the place of 1, 2, 3 in the rubric

$$To. \ [Ao'B]w = \exists x \exists y (R123 \land [A]x \land [B]y).$$

Just so, there are other arrow-like connectives $\to'$, fooling similarly with the rubric

$$T\to'. \ [A \to' B]w = \forall x \forall y (R123 \Rightarrow ([A]x \Rightarrow [B]y))$$

EXERCISE. Locate (what we have called) ← among the →'.

We have allowed that there may be many intensional particles. Some are directly definable in terms of → and the truth-functions. (Boring exercise: set out $T \leftrightarrow$, where $A \leftrightarrow B = df(A \to B) \land (B \to A).$) Others result from a little intensional fiddling—cf. $T \sim$ above. Things get more lively if we mix relevant and irrelevant particles. Nothing prevents (and Anderson–Belnap motivating remarks suggested) that the entailment of $E$ be parsed as the strict relevant implication $\Box (A \to B)$, borrowing an → from R and a $\Box$ from S4.20 The obvious thing to do semantically is to have two accessibility relations (as in [43], [31]), a ternary $R$ to look after → and a binary $S$ to explicate $\Box$. (After that, the sky is the limit.)

Our final point above was that there should be no preferred relevant logic. Yet, in a way, we have already come up with one. It is the logic that one gets out of the first-order theory of any ternary relation $R$, merely imposing those that we choose out of the above truth-conditions. (Or so it is our purpose to convince you below.) Without negation, that logic is $B^+$. To extend $B^+$ to any other positive relevant logic, simply impose the additional postulates on $R$ that go with that logic. (Those additional conditions may involve a particular “real world” 0, or a set $O$ of such worlds.) Similarly, there is a natural minimal classical relevant logic, imposing $T\sim$ as well. We call that system $CB$ in this paper.21

20This thought, though tempting, was only partially correct. Kripke [25] lucked out, since the $A \to B$ of $E\to$ is exactly translatable in $R^\uparrow$, as $\Box (A \to B)$. The same luck holds when DeMorgan $\sim$ is added. But this luck runs out, as Maksimova demonstrated, when a non-theorem of $E^+$ is proved on $R^\uparrow$ translation.

21We will apply the technique of our [33] below, which will show $CB$ a conservative extension of $B^+$. 
5 A basic Boolean excursion

In this section, we attend to the claim that the addition of classical negation is conservative over $B+$ as defined above. This claim has been made before—by ourselves and others. Here, very carefully, we shall prove it. And then we shall draw the appropriate corollaries.

Recall from section III the modeling of $B+$ in $+ms$, on which the system is sound and complete. This modeling depends by $p1$ on a reflexive, transitive $\subseteq \triangleright$ (defined by $R0ab$), equipped with replacement properties by $p2$–$p4$. And every interpretation must satisfy also a (monotone increasing) hereditary condition as well.

We can get all that if we simply take the $\subseteq$ in question to be the equality relation $=$. We have $p1$, since equality is reflexive and transitive. We have $p2$–$p4$, since equality licenses replacements. And we have $H$, since $a = b \Rightarrow I(A,a) = I(A,b)$ trivially. So, for now, let us try out the postulate

$p0. a \subseteq b$ iff $a = b$.

Let us call any appropriate structure $K = \langle 0, K, R \rangle$ that satisfies $p0$ a $b+ms$. We have already noted that any $b+ms$ is certainly a $+ms$. So, by soundness, each theorem $\vdash A$ of $B+$ is valid in every $b+ms$. But what of completeness, relative to the $b+ms$? This requires, for each non-theorem $A$ of $B+$, that there should be some $b+ms$ $K$ in which $A$ is invalid. And so there is, massaging our model theory along the lines of [33]. We recall again, for all interpretations $I$ in all $+ms$,

Semantic Entailment Lemma (SemEnt). $[A \rightarrow B]0$ iff $\forall x ([A]x \Rightarrow [B]x)$.

THEOREM 1 (Classical Theorem for B+). The following conditions are equivalent, for each formula $A$ of $L+$:

1. $\vdash A$ is a theorem of $B+$
2. $A$ is valid in all $+ms$
3. $A$ is valid in all $b+ms$

Proof. The hard part is to show that (3) implies (2). We proceed by contraposition. Goal: $A$ is invalid in some $b+ms$. Hypothesis: $A$ is invalid in $+ms$ $K = \langle 0, K, R \rangle$. There is an interpretation $I$ in $K$ such that $A$ is false at 0 on $I$. We define a $b+ms$ $K' = \langle 0', K', R' \rangle$ as follows:

(a) $0'$ is a new element disjoint from $K$
(b) \( K' = K \cup \{0'\} \)

(c) For all \( a, b, c \in K \), we have \( R'abc \iff Rabc \)

(d) For all \( a' \in K' \), we have \( R'0'a'd' \)

(e) Otherwise for \( a', b', c' \in K' \), \( \neg R'a'b'c' \)

We next define an interpretation \( I' \) in \( K' \), determined by its values on atoms and truth-conditions:

(f) \( I'(p, a) = I(p, a) \) for all \( a \in K \) and atoms \( p \)

(g) \( I'(p, 0') = I(p, 0) \) for all atoms \( p \)

We now show, by structural induction on \( D \), the generalized

\( I'(D, a) = I(D, a) \) for all \( a \in K \)

\( I'(D, 0') = I(D, 0) \)

Assume on Inductive Hypothesis (IH) that (f) and (g) hold at all \( a \in K' \) for all subformulas of an arbitrary formula \( D \). This is clear by (f) and (g) when \( D \) is some atom \( p \). We invoke the (IH) to show the same for compound \( D \). The interesting case is \( D = B \rightarrow C \).

Ad (f'). Let \( R'' \) be whichever of \( R, R' \) is appropriate. Unpack \( T \rightarrow \) by \( \forall x \forall y (R''axy \Rightarrow (\llbracket B \rrbracket x \Rightarrow \llbracket C \rrbracket y)) \). How, given (IH), can this value differ for \( a \in K \) on \( I' \) as opposed to \( I \)? (The quantifiers on the \( I' \) side range over \( K' \); on the \( I \) side, over \( K \); and \( R'' \) is \( R' \) on the \( I' \) side and \( R \) on the \( I \) side.) Assume \( B \rightarrow C \) is false at \( a \) on \( I \); it must also be false there on \( I' \) (since we have the same counterexample on (IH), given that \( R' \) is an extension of \( R \)). Assume then for \textit{reductio} that \( B \rightarrow C \) is true at \( a \) on \( I \) but false at \( a \) on \( I' \). Then there are \( x', y' \in K' \) such that \( R'ax'y' \) and \( \llbracket B \rrbracket x' \) but not \( \llbracket C \rrbracket y' \) on \( I' \). Since this never happens back in \( K \) on \( I \), at least one of \( x', y' \) must be \( 0' \). But the “Otherwise” clause (e) above in the definition of \( R' \) then enforces \( \neg R'ax'y' \) (since \( a \) is old). Contradiction!

Ad (g'). Again assume (IH). Suppose that \( B \rightarrow C \) is true at \( 0' \) on \( I' \). Then, by SemEnt, for all \( x' \in K' \) we have \( \llbracket B \rrbracket x' \Rightarrow \llbracket C \rrbracket x' \). Since \( K \subseteq K' \), we have on IH also \( \forall x(\llbracket B \rrbracket x \Rightarrow \llbracket C \rrbracket x) \) on \( I \) in \( K \), whence \( \llbracket B \rrbracket 0 \) on \( I \). For the converse, suppose \( B \rightarrow C \) false at \( 0' \) on \( I' \). Then, by SemEnt, there is an \( x' \in K' \) such that \( \llbracket B \rrbracket x' \) but not \( \llbracket C \rrbracket x' \) on \( I' \). If this \( x' \in K \), then by (IH) and SemEnt we have \( B \rightarrow C \) also false at 0 on \( I \). The remaining possibility is \( \llbracket B \rrbracket 0' \) and not \( \llbracket C \rrbracket 0' \) on \( I' \). But under \( g' \) of the inductive hypothesis this also sets \( I(B, 0) \) true and \( I(C, 0) \) false, whence again \( B \rightarrow C \) is false at 0 on \( I \).
We return to our original formula $A$, which was false at 0 on $I$ in the $+ms K$. By $(g')A$ is also false at $0'$ on $I'$ in the $b+ms K' = (0', K', R')$. In a nutshell, the denial of (ii) implies the denial of (iii). Contraposing as promised, (3) implies (2). This concludes what it is necessary to say, and the theorem is proved. ■

Not only is the theorem proved, but also the chief obstacle has been removed to the relevant accommodation of classical semantical negation $\neg$. That obstacle lay in the hereditary condition $H$, which says that if $a \subseteq b$ in a $+ms$ then $[A]a \Rightarrow [A]b$ for all formulas $A$ on any interpretation $I$. Our postulates and truth-conditions for $B+$ (and other logics, even those with DeMorgan $\sim$) sufficed in [45] and elsewhere to establish all of $H$ from its assumption $Hp$ above on atoms. But $T\neg$ resists the structural induction. Happily it can no longer resist in the presence of $p0$, as [33], [34] make clear.

**DEFINITION 2 (Classical CB Definition).** A statement $\vdash B$ of $L+$ is a theorem of $CB+$ iff $B$ is valid in all $b+ms$.

Adding classical $\neg$, governed by $T\neg$, to the minimal system $B+$ has done no harm.22

**THEOREM 3 (Classical conservation theorem).** $CB\vdash$ is a conservative extension of $B+\vdash$.

**Proof.** By the classical theorem for $B+$. Enough said. ■

### 6 Basic consecution logics

Confession is good for the soul. For the good of our souls, we make one. The “real world” 0, although it is not doing very much, is nonetheless doing a little in the basic semantics of this paper. So to keep our promise to drop 0 from the semantics, it is necessary also to take a thoroughgoing relational view of what relevant logics are about.

That the ghost of 0 does not return to haunt us with anomalies, we henceforth prefer the relational language $L\leq$ to the assertional language $L\vdash$ that we have assumed until further notice. (This is the further notice.) We also trade in our basic logics like $B+\vdash$ and $CB\vdash$ for entailment logics $B+\leq$ and $CB\leq$.

Another reason for the shift is to find a more perfect union between the logical ideas (on which we are concentrating here) and the type-theoretic ideas developed by the $\lambda$ and $CL$ communities. For these ideas have been worked out (e.g., in [5]) in terms of a binary relation $\leq$ on types.

---

22As the late W.V. Quine might have put it. The Relevantist of Belnap and Dunn [7] might interject that it has done no good, either. For an anti-relevantist view, see Meyer [29].
On the interpretation of types as formulas, this relation (as far as it goes) is exactly \( B + \leq \) entails.

Formulas \( B \) of the Boolean sublanguage \( BL \) of \( L \) have the following Backus-Naur specification:\(^{23}\)

\[
B ::= p | \neg B | B \land B | B \lor B
\]

A Boolean tautology is any formula \( B \) of \( BL \) which is a truth-table tautology. By extension, we also call a formula \( A \) of \( L \) a Boolean tautology if \( A = s(B) \) for some substitution \( s \) of a Boolean tautology \( B \) of \( BL \).\(^{24}\)

We now lay down axioms and rules for \( CB \leq \) and \( B + \leq \).

B1. \( A \leq C \), when \( A \supset C \) is a Boolean tautology

B2. \( (A \supset B) \land (A \supset C) \leq A \supset B \land C \)

B3. \( (A \supset C) \land (B \supset C) \leq A \lor B \supset C \)

R1. \( B \leq C \Rightarrow (A \leq B \Rightarrow A \leq C) \)

R2. \( A' \leq A \) and \( B \leq B' \Rightarrow A \supset B \leq A' \supset B' \)

R3. \( A \leq B \) and \( A' \leq B' \Rightarrow A \land A' \leq B \land B' \)

R4. \( A \leq B \) and \( A' \leq B' \Rightarrow A \lor A' \leq B \lor B' \)

R5. \( A \leq B \Rightarrow \neg B \leq \neg A \)

Note the redundancy of R4 in the presence of R3 and R5. Clearly these axioms and rules confer a distributive lattice structure on \( B + \leq \), and a Boolean lattice structure on \( CB \leq \).

7 Soundness and Completeness of the Relational Systems

We keep first our promise to drop 0 from the semantics. (In view of SemEnt, and after the classical moves just above, it is evident that 0 isn’t really doing very much at this basic level.) Combined with our first promise recall that we now view our basic systems as relational. A basic model structure (henceforth, just \( bms \)) is then just a pair \( K = \langle K, R \rangle \), where \( K \) is a non-empty set and \( R \) is a ternary relation on \( K \). A possible interpretation is again a function \( I \) assigning one of true, false to each formula \( A \) at each \( w \in K \). Such a function \( I \) is an interpretation if it satisfies the applicable truth-conditions from among \( T \land, T \lor, T \neg \) and \( T \supset \) above. An interpretation \( I \) is completely determined by its values on all atoms \( p \) at all worlds \( w \in K \).

\(^{23}\)In brief, a formula \( B \) of \( L \) is in \( BL \) if there are no arrows in \( B \).

\(^{24}\)For \( L + \), the Boolean tautologies that count are instances of \( A \supset B \), where \( A \) and \( B \) are themselves built up from atoms under just \( \land \) and \( \lor \). They are not particularly Boolean, since this fragment is the same for the intuitionist logic \( J \) and indeed the first-degree entailments \( Dfde \) of the relevant logics \( R, E \) and \( T \) of [3].
More adjustment is required in the vocabulary of verification. Since all statements of our relational language \( L \leq \) are now of the form \( A \leq C \), all thoughts of a real world 0 at which exactly the logical truths shall always be true are now emphatically beside the point. Even so, we may retain an imaginary 0 as a façon de parler. And we may still write \([B]0\) for \( B \) is verified on \( I \). Our manner of speaking applies, in the first instance, only to \( \rightarrow \) formulas, letting \( A \rightarrow C \) stand in for the honest statement \( A \leq C \). So, given an interpretation \( I \) in a bms \( K \) we say, in the abbreviated notation introduced above,

\[
V \rightarrow \ [A \rightarrow C]0 \iff \forall x ([A]x \Rightarrow [C]x)
\]

When \( A \rightarrow C \) is verified on \( I \), we may also say that \( A \) entails \( C \) on \( I \). This is in complete conformity to the SemEnt principle laid down in section 4 above. \( A \) entails \( C \) in a bms \( K \) iff \( A \) entails \( C \) on all interpretations \( I \) in \( K \); and \( A \) basically entails \( C \) iff \( A \) entails \( C \) in all bms.

We note immediately

**THEOREM 4 (Basic soundness theorem).** For all formulas \( A, C \) of \( L \leq \), if \( A \leq C \) is a theorem of \( CB \leq \) then \( A \) basically entails \( C \).

**Proof.** Consider an arbitrary bms \( K \) and an interpretation \( I \) therein. Show by deductive induction that if \( A \leq C \) in \( CB \leq \) then \( A \) entails \( C \) on \( I \). Enough said.\(^{25}\)

Semantical completeness of the basic relational systems requires (as ever) a little more work. We shall treat \( B^+ \leq \) and \( CB \leq \) together in setting out machinery. As ever, a theory shall be any set \( S \) of formulas closed under entailment and conjunction. I.e., \( S \) is a theory iff, for all formulas \( A \) and \( B \) in the language, we have both

(\( \leq \) E) \( A \leq B \Rightarrow (A \in S \Rightarrow B \in S) \)

and

(\( \land \) I) \( A \in S \) and \( B \in S \Rightarrow A \land B \in S \)

The calculus of theories shall be the structure \( CT = \langle CT, o, \subseteq \rangle \), where \( CT \) is the set of all theories, \( \subseteq \) is set inclusion, and \( o \) is the fusion operation defined by

\(^{25}\)Evidently the soundness theorem holds a fortiori of \( B^+ \leq \).
Some theories are more *truth-like* than others. Among the conditions that we might like a theory \( S \) to satisfy, for all formulas \( A \) and \( B \), are the following:

\[
\begin{align*}
\forall E) & \quad A \lor B \in S \Rightarrow A \in S \lor B \in S \\
\neg E) & \quad \neg A \in S \Rightarrow A \notin S \\
\neg I) & \quad A \notin S \Rightarrow \neg A \in S
\end{align*}
\]

Note that the converse \( \lor I \) principle to \( \lor E \) holds automatically by Axiom B1 above. A theory \( S \) that admits \( \lor E \) is called *prime*; that admits \( \neg E \), *consistent*; that admits \( \neg I \), *complete*. And a prime (and consistent and complete, if \( \neg \) is present) \( S \) is *truth-like*.²⁷

As an exercise, show that the fusion of any two theories is itself a theory. Alas, the fusion of truth-like theories is unlikely to be itself truth-like. We accordingly move to the *canonical relational structure* \( \text{CPT} \) of prime theories. \( \text{CPT} = (\text{CPT}, \text{CTR}) \), where \( \text{CPT} \) is the set of all non-trivial prime theories, and \( \text{CTR} \) is the *canonical ternary relation* defined on the set \( CT \) of all theories by

\[
\text{CTR}_{xyz} = \text{df} \quad xoy \subseteq z,
\]

for all theories \( x, y, \) and \( z \).

Readers, we hope, will see now where we are heading. The canonical structure \( \text{CPT} \) will be the \( K = (K, R) \) for our semantical completeness proofs. At least it is clear that \( \text{CPT} \) is a bms. For \( \text{CPT} \) is a non-empty set, and \( \text{CTR} \) is a ternary relation when restricted to that set. We next concoct a *canonical interpretation* \( \text{CI} \), by setting, for all formulas \( A \) and prime theories \( x \),

\[
\text{DCI} \quad \text{CI}(A, x) = \text{true} \iff A \in x
\]

Oops—that definition merely makes \( \text{CI} \) a possible interpretation. To make it moreover an interpretation, we need the

²⁶The term *fusion* is due to Fine. But the notion was introduced earlier by Powers in his \[38\], as *modus ponens product*. And indeed \( \text{SoT} \) consists of all results of applying \( \rightarrow E \), taking major premises from \( S \) and minor ones from \( T \).

²⁷The term *truth-like* comes from \[4\], to characterize any theory that respects appropriate truth-conditions (e.g., \( T \lor \) ) on such truth-functional particles as belong to its vocabulary. But a preference for truth-like theories goes back much further, at least to our \[45\].

²⁸A theory \( S \) is non-trivial iff \( S \) is neither the empty set nor the set of all formulas.
LEMMA 5 (Canonical interpretation lemma). *CI respects the truth-conditions* $T \land, T \lor, T \to, T \neg$.

**Proof.** The members of CPT are non-trivial prime theories. $T \land$ and $T \lor$ (and $T \neg$ in the Boolean case) are built into the definition of a truth-like theory. So only $T \to$ requires serious verification. We must show on CI, for all formulas $B$ and $C$ and truth-like non-trivial theories $w$,

(a) \[ [B \to C]w = \forall x \forall y (CTRwxy \Rightarrow ([B]x \Rightarrow [C]y)), \]

where the quantifiers range over CPT. Definitions reduce this to

(b) \[ B \to C \in w \text{ iff } \forall x \forall y (wox \subseteq y \Rightarrow (B \in x \Rightarrow C \in y)). \]

Left to right is obvious. But the other direction is interesting. Assume $B \to C \notin w$. It will suffice to find $x, y$ in CPT such that (i) $wox \subseteq y$, (ii) $B \in x$ and (iii) $C \notin y$. There are at any rate theories $x_0$ and $y_0$ in CT that clearly satisfy these conditions: just set $x_0 = \{D : B \leq D\}$, which is the principal theory $[B]$ determined by $B$; and set $y_0 = wox_0$. The final trick is to blow up $x_0$ and $y_0$ to truth-like theories $x$ and $y$ so that (i)–(iii) continue to hold. The method is to start with $y_0$, extending it to a prime theory $y$ while keeping $C$ out. One then applies the Squeezing lemma (e.g., of our [44]) to extend $x_0$ to a prime $x$ so that (i) continues to hold.$^{29}$ And the lemma is proved.■

THEOREM 6 (Basic completeness theorem). The following conditions are equivalent for $B+ \leq$ and $CB\leq$:

1. $A \leq C$

2. $A$ entails $C$ on the canonical interpretation $CI$ in the bms $CPT$.

3. $A$ basically entails $C$.

**Proof.** (1) $\Rightarrow$ (3) is the content of the basic soundness theorem above. (3) $\Rightarrow$ (2) by the canonical interpretation lemma. We conclude by showing (2) $\Rightarrow$ (1). Assume (2). For reductio, assume it a non-theorem that $A \leq C$. Consider the principal theory $[A]$ determined by $A$. On the reductio hypothesis, $C \notin [A]$. But, as in the proof of the last lemma, $[A]$ can be extended to a truth-like theory $x$ such that (i) $A \in x$ and (ii) $C \notin x$. This contradicts (2) and ends the proof.■

$^{29}$These tasks are most elegantly accomplished using Belnap’s pair-extension lemma; cf. [4, p. 124].
8 Truth translations

We now introduce our truth translations of formulas $A$ of $L$ into classical first-order Predicate Logic $PL$. Our classical vocabulary will be as usual, based on a single ternary relation symbol $R$ and countably many monadic predicate symbols $P_i$, $0 < i < \omega$. We call a formula of $PL$ with exactly one free variable (but as many bound variables as you like) a unary predicate expression. We may write a unary predicate expression $A$ whose free variable is $x$ as $A(x)$.

The underlying idea of our translation is just this. It is customary, in Kripke-style relational semantics for systems $S$ of non-classical logics, to parse the propositions of such logics as sets of worlds. This is, in effect, to interpret the formulas of $S$ as themselves propositional functions—taking worlds $w \in K$ as arguments and one of the truth-values $\{\text{false}, \text{true}\}$ as values. When curried, our characterization of a possible interpretation $I$ as a function from $L \times K$ to $2$ conforms; such an $I$ is equivalently a function from $L$ to $K \to 2$, where $2$ is the 2-element Boolean algebra.

We note (but shall not further explore in the present highly classical environment) that only laziness or undue attachment to prevailing convention makes $2$ a preferred target algebra for our propositional functions. To the contrary, propositions can have any structure that they ought to have—Heyting algebras, DeMorgan monoids, or what you will. But laziness and attachment to convention will prevail in this paper.

Having confessed our staid inclinations, and having set out the relational semantics for basic relevant logics, we trust that the rest of our plan of translation will quickly become clear to the reader. Consider an arbitrary formula $B$ of $L$. $B$ has a unique construction tree, recording how it has been built up from atoms under the connectives $\land, \lor, \neg, \rightarrow$ and perhaps others. Working our way from the leaves (or tips) of the tree, which are the atoms, we wish to assign a unary predicate expression of $PL$ to each subformula $C$ of $B$. Choosing an appropriate recursive plan, the process will terminate in the assignment of such an expression to $B$ itself.

We mentioned possible interpretations in passing a couple of paragraphs ago. What distinguishes genuine, honest interpretations from merely possible ones is the imposition of truth-conditions like $T \land, T \rightarrow$, etc. (There was also a hereditary condition $H$, but the further moves above assure that it is now satisfied vacuously.) Fidelity to the truth-conditions will accordingly be the key ingredient in the promised recursive plan.

Let us return to the bracket notation introduced in section 3, where we used $'[A]w'$ as shorthand for ‘the formula $A$ is true at the world $w$ on the interpretation $I$. This looks a lot like predicating the formula $A$ of the world $w$. The plot thickens with the truth-conditions of section 4. These say things
like \([\neg A]w = \neg[A]w\) and 
\([A \rightarrow B]w = \forall x \forall y (Rwxy \supset ([A]x \supset [B]y))\). Do we not have in this notation the central clue as to how our truth translations will work?

Indeed, this is the central clue. To each atom \(p\) of \(L\) we assign a unique monadic predicate symbol \(P = \tau(p)\). Making informal use of \(\lambda\) notation, and presupposing \(\eta\)-reduction and rewriting of bound variables, we may write this translation as

\[
\tau(p) = \lambda x. P x, \quad \text{for each atom } p.
\]

Still using \(\lambda\) notation, we recursively extend this translation \(\tau\) to take each formula \(A\) of \(L\) into an (in general, defined) unary predicate expression of \(PL\), on the rubric of the following table:

<table>
<thead>
<tr>
<th>(L) Formula</th>
<th>(PL) Unary Predicate (\tau(A))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B \land C)</td>
<td>(\lambda x. \tau(B) x \land \tau(C) x)</td>
</tr>
<tr>
<td>(B \lor C)</td>
<td>(\lambda x. \tau(B) x \lor \tau(C) x)</td>
</tr>
<tr>
<td>(\neg B)</td>
<td>(\lambda x. \neg \tau(B) x)</td>
</tr>
<tr>
<td>(B \rightarrow C)</td>
<td>(\lambda x. \forall y \forall z (Rxyz \supset ((\tau(B) y \supset \tau(C) z)))</td>
</tr>
</tbody>
</table>

Note how the truth-conditions (\(T \rightarrow\), etc.) have showed up in the \(\tau(A)\).

We now define an entailment translation \(\nu\), by setting

\[
\nu(A \leq C) = \forall x (\tau(A) x \supset \tau(C) x)
\]

To show how simple our truth translation is, we institute again our old bracketed notation, slowly churning out the mated formulas of \(PL\).

**EXAMPLES.**

**Ax \& E.** \(p \land q \leq p\)  \((AxB1, \text{since } p \land q \supset p \text{ is a Boolean tautology.})\)

**Translation.** \(\forall x ([p \land q]x \supset [p]x) = \forall x ([p]x \supset [q]x \supset [p]x) = \forall x (P x \land Q x \supset P x)\)

Henceforth we leave the prefaced \(\forall x\) tacit, to facilitate easier reading of formulas. Next,

**Ax \rightarrow \lor E.** \((p \rightarrow s) \land (q \rightarrow s) \leq p \lor q \rightarrow s\)  \((AxB3)\)

**Translation.**

\[
\begin{align*}
[p \rightarrow s]x \land [q \rightarrow s]x & \supset [p \lor q \rightarrow s]x = \\
[p \rightarrow s]x \land [q \rightarrow s]x \supset & \forall y \forall z (Rxyz \supset (Rxyz \supset (Rxyz \supset (P y \supset S z)) \land \forall y \forall z (Rxyz \supset (Q y \supset S z))))
\end{align*}
\]
Showing these translations theorems of PL is an easy exercise. How about a non-theorem?

**AxWI.** \((p \rightarrow q) \land p \leq q\)

**Translation.** \([[(p \rightarrow q) \land p]x \leq [q]x = [(p \rightarrow q)]x \land [p]x \leq [q]x = \forall y \forall z (Rxyz \supset (Py \supset Qz)) \land Px \supset Qx\)

Clearly the translation is not a PL theorem. That’s good, because AxWI—though a theorem and even an axiom of very many relevant logics—is insufficiently minimal for B+.

**Verification theorem.** Let \(A \leq C\) be any statement in the domain of definition of the verification translation \(\nu\). Then the following conditions are equivalent for \(B+\leq\) and \(CB\leq\):

(i) \(A \leq C\) is a theorem

(ii) \(\nu(A \leq C)\) is a theorem of PL

**Proof.** As the first-order translations merely recapitulate the semantics, the theorem is obvious.

9 **CBMODEL (and its kin)**

We now take leave to rename the first-order logic PL, as formulated above with a single ternary \(R\) and countably many monadic \(P_i\). Henceforth we call it CBMODEL. We take it, as we implicitly did above, as an assertional system, whose elementary statements are of the form \(\vdash A\). The theorems of CBMODEL are just those \(\vdash A\) such that \(A\) is first-order valid (with axioms and rules, if desired, supplied to the reader’s taste). And our main theorem then becomes

**THEOREM 7 (Truth translation theorem).** \(A \leq C\) in \(CB\) iff \(CBMODEL\)-\(\forall x([A]x \supset [C]x)\), subject to the definitional schemes introduced above.

But there is less truth than poetry in our claim in the last section to have dropped 0 from the semantics of B+ and CB. For the ghost of 0 did remain, via verification conditions that simply mirrored the truth-conditions at 0, back when it had as much ontological standing as any other “world” in \(K\). And that ghost would haunt us still further if we proceeded to stronger relevant logics like \(R+\), with its semantical truths like \(Rx0\), for all \(x \in K\).

Nonetheless, 0 is gone from our semantical postulates for B+ and CB. Nor does it haunt us even a little where the central assertions of logic are at issue—i.e., the true entailments. As these correspond exactly to the type theorists’ relational calculi, they too need not mourn for 0.

---

30Hey, AxWI is just conjunctive *modus ponens*. What could be wrong with that? Plenty, as it turns out. Cf. our [35] for its *destruction* of a relevantly formulated naïve set theory.
Still, it would be good to accommodate a wider variety of relevant logics on our scheme of truth translation. It might even be good, for those suckled in Intuitionist or Relevantist or other creeds outworn, to provide a wider range of first-order theories into which relevant logics might be translated. And it would definitely be good to provide such translations for the original formulations of relevant logics as *assertional systems*.

All of these tasks will involve the *resurrection* of 0, or some surrogate thereof, as an *honest* world. In the full Boolean case, we are pushed not merely to PL but to PL=, making equality explicit. And in logics stronger than the minimal basic ones, the *targets* of the translations will be *applied* first-order theories, with *proper axioms* doing duty for the semantical postulates.

We illustrate first with a system CB=MODEL, which stands to CB-MODEL as CB+ stands to CB≤. This too will be a first-order theory, adding a constant 0 and the predicate = to the syntactical equipment of CBMODEL. This theory will have a single proper axiom, namely

\[
\text{Ax0. } \forall x \forall y (x = y \equiv R0xy),
\]

which recapitulates our Booleanizing maneuvers. This leads immediately to a

**Truth translation corollary** \(\vdash A\) in CB+ iff CB=MODEL \(\vdash [A]0\)

**Corollary to corollary** CB\(\vdash A \rightarrow C\) iff CBMODEL \(\vdash \forall x ([A]x \supset [C]x)\)

**Proof.** Again, the first corollary merely recapitulates the semantics. And it and the theorem imply the second corollary, given the Semantic Entailment lemma SemEnt above.

That was so much fun that we will now propose a first-order theory \(\text{CR}=\text{MODEL}\), which will do for the classical conservative extension \(\text{CR}\) of \(R\) what our corollary does for \(CB\). The language of this theory adds the unary (Routley) function symbol \(*\) to the primitives of \(\text{CR}=\text{MODEL}\), with extra postulates \(\text{Ax0, } \forall x Rxx (\text{TRP})\), \(\forall x \forall y \forall z \forall w (\exists \nu (Rx\nu Rz\nu w) \equiv \exists \nu (Rz\nu Rxyw)) \text{ (PP)}\), \(\forall x (x * * = x)\), \(\forall x \forall y \forall z (Rxyz \supset Rxz * y*)\). These last 2 postulates look after DeMorgan \(\sim\), eliminated via the truth-condition \(T\sim\) of section 4.

\(\text{CR}\) is just the system introduced as \(\text{CR}^*\) in our [34], as a *conservative extension* of \(R\).

**Truth translation for \(CR\)** \(\vdash A\) in \(CR\) iff \(\text{CR}=\text{MODEL} \vdash [A]0\)
Similar results hold generally for relevant and other substructural logics, *mutatis mutandis*. And, as noted, we may *forswear* Boolean simplifications (but why?) by making other choices for the first-order logics of the target theories.

10 Summing Up

We present this paper as very much a *work in progress*. Its main idea—that relevant logics can be recast in terms of (as it is put) a *classical first-order metalogic*—has long been evident to workers in the field. Nor is that idea quite so attractive as once it was. Other concerns—such as minimizing the *computational complexity* of proof searches—have come to the fore. (*Not many* of our best friends will shower us with praise for having “reduced” the almost trivial logic $\text{B}+$ to the seriously undecidable $\text{CBMODEL}$.)

Nonetheless, we take the present project to be *worth pursuing*. It was, after all, *unknown*, in the early days of relevant semantical investigation, that these logics could be furnished so smoothly (and indeed conservatively) with a full Boolean classical negation $\neg$. Even the choice of $\text{B}+$ as the minimal positive relevant logic was, in [44] and [32] days, not so clear.

It is clearer now, on several grounds. One is simply the lovely connections between $\text{B}+$ (or at least its $\to, \land$ fragment) that have been worked out independently in the $\lambda$ community. More relevant here is that, as we have recast the basic semantics, the *partial order* relation $\subseteq$ and its special properties are *no longer required*. At root there is just a primitive ternary relation $R$ on the semantic side, with *no* special properties.

Yet much remains to be done. We have exempted (at least so far) the Ackermann and Church sentential constants, fusion $\circ$ and the right-to-left conditional $\leftarrow$ from our classical theories. On the verification side, $\leftarrow$ is

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31 There is a painfully *simple* argument, set out by Meyer in [30], that the semantics of [45] is sound and complete for $R$ when expressed in a theory $\text{RMODEL}$ based on the first-order logic $R^{\forall \exists}$ of [4].

32 The triviality of $\text{B}+$ is *most evident* in its pure $\to$ fragment $\text{B}^\to$. For a formula in this fragment is a theorem iff it is of the form $A \to A$. On the $\text{PL}$ side, the theory of even a single *binary* relation is paradigmatically undecidable. We have a *ternary* one, and all those monadic predicates to boot.

33 Intersection type theorists also add a constant $\omega$, which is relevantly translatable as the (optionally added) Church constant $\text{T}$. (There has been some confusion between Church $\text{T}$ and the hitherto more relevantly important Ackermann $\text{t}$.) Church $\text{T}$ is readily got by Boolean definition as $p \lor \neg p$. So defined, $\text{T}$ satisfies on semantic grounds both the postulates $A \leq \text{T}$ and the (more mysterious) $\text{T} \leq \text{T} \to \text{T}$ of [5].

34 In fact, there is *less* to accommodating such extra particles than one might fear. In view of the semantical completeness results, extra particles like $\leftarrow$ are easy to add conservatively. An earlier draft of this paper showed that fusion $\circ$ also can find a sleeping bag within the tent of classical conservative extension of $\text{B}+$-- The result depended on this: for $\text{B}+\circ$, we can adapt a [33] “metavaluation” argument to show $\text{BoC}$ a theorem.
interested in relations of the shape $Ra0b$, as opposed to those of the form $R0ab$ (which $p0$ trivializes to $a = b$) that look after its left-to-right mate $\rightarrow$. And we have done hardly anything with DeMorgan $\sim$, the original relevant negation. To be sure, [32] proposed one system there called $B$. Routley preferred another $B$ in [46]. Finally, the elimination of 0 is still incomplete. It becomes complete (and is more consonant both with the work on type theories and with logicians’ insistence that their subject is first and foremost about the relation of entailment) on the above plan which presents the systems with a primitive binary $\leq$ rather than a unary $\vdash$.

Probably the most likely applications are to theories of types, in the $\lambda$ logicians’ sense. Already Dezani et al. have noted in [19], as a consequence of [5], that there is a nice model of $\lambda$ in the class of non-empty theories based on the $B+$ fragment (cum extension) $B\land T$ (in $\rightarrow, \land, T$). And it is of course crushingly obvious that classical logic (as here formulated, with ternary $R$) models $\lambda$. For the relation $Rabc$ admits in particular the interpretation $r(a, b) = c$, where $r$ is any binary operation. Well, the application operation of Combinatory Logic is a binary operation; and $\lambda$, which is definable within a system of combinators, has more models these days than one can shake a stick at. Any of these will do.

Still, one would like something a little more sui generis—which illuminates and which does not merely copy known results. To this too we pledge our best endeavours.

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Part II

Probability and Induction