# Bounding normalization time through intersection types

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# Intersection types and complexity

- I.T. with idempotence: used for semantic purposes, i.e. for building filter models for $\lambda$-calculus $\rightsquigarrow$ modulo equivalence $\sigma \wedge \sigma = \sigma$
- I.T. without idempotence: recently used to reason about **complexity** of $\beta$-reductions throught type assignment systems for $\lambda$-calculus
  - Terui (2006): any normalizing $\beta$-reduction sequence for `M` is bounded by the size of the type derivation for `M`
  - Lengrand (2011): the measure of a principal type derivation for `M` corresponds to the maximal length of a normalizing $\beta$-reduction sequence for `M`

# Our aim

- Using I.T. without idempotence nor associativity to express the functional dependence of the length of a normalizing $\beta$-reduction sequence for M on the size of M itself
- We take inspiration from the system STA (Gaboardi-RDR 2007) characterizing PTIME
  - STA gives a bound on the number of normalization steps from M to its normal form, in the form $|\mathtt{M}|^{d+1}$
  - $d$ is a measure depending on the type derivation for it (the *depth*)
  - Since for every normalizing term there is a type derivation with minimal depth, this bound depends on M

## System STI (1)

- Soft Type assigning system with Intersection
- The system assigns types to strongly normalizing $\lambda$-terms
- Intersection with commutativity but without idempotence nor associativity

**Our aim**

Proving a normalization bound on any reduction sequence for a given (strongly normalizing) term

# System $\mathtt{STI}$ (2)

- $\lambda$-terms: $\mathtt{M} ::= \mathtt{x} \mid \mathtt{MM} \mid \lambda\mathtt{x}.\mathtt{M}$
- $\beta$-reduction: $(\lambda\mathtt{x}.\mathtt{M})\mathtt{N} \to \mathtt{M}[\mathtt{N}/\mathtt{x}]$
- $\mathtt{STI}$ types:

$$\mathtt{A} ::= \mathtt{a} \mid \sigma \to \mathtt{A} \qquad \text{(linear types)}$$

$$\sigma ::= \mathtt{A} \mid \underbrace{\sigma \wedge \ldots \wedge \sigma}_{n} \quad (n > 1) \qquad \text{(intersection types)}$$

- Context: finite set of assumptions $\mathtt{x} : \sigma$
  - $\Gamma \wedge \Delta = \{\mathtt{x} : \sigma \wedge \tau \mid \mathtt{x} : \sigma \in \Gamma, \mathtt{x} : \tau \in \Delta\}$
  - $\Gamma, \Delta$ if $\mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta) = \emptyset$
- $\Pi \triangleright \Gamma \vdash \mathtt{M} : \sigma$ denotes a derivation $\Pi$ with conclusion $\Gamma \vdash \mathtt{M} : \sigma$.

## STI rules

$$\frac{}{\mathtt{x : A} \vdash \mathtt{x : A}}\ (Ax) \qquad \frac{\Gamma \vdash \mathtt{M} : \sigma \quad \mathtt{x} \notin \mathsf{dom}(\Gamma)}{\Gamma, \mathtt{x : A} \vdash \mathtt{M} : \sigma}\ (w)$$

$$\frac{\Gamma, \mathtt{x} : \sigma \vdash \mathtt{M : A}}{\Gamma \vdash \lambda \mathtt{x.M} : \sigma \to \mathtt{A}}\ (\to I) \qquad \frac{\Gamma \vdash \mathtt{M} : \sigma \to \mathtt{A} \quad \Delta \vdash \mathtt{N} : \sigma \quad \Gamma \# \Delta}{\Gamma, \Delta \vdash \mathtt{MN : A}}\ (\to E)$$

$$\frac{\Gamma_1 \vdash \mathtt{M} : \sigma_1 \quad \ldots \quad \Gamma_n \vdash \mathtt{M} : \sigma_n \quad n > 1}{\Gamma_1 \wedge \ldots \wedge \Gamma_n \vdash \mathtt{M} : \sigma_1 \wedge \ldots \wedge \sigma_n}\ (\wedge_n)$$

$$\frac{\Gamma, \mathtt{x}_1 : \sigma_1, \ldots, \mathtt{x}_n : \sigma_n \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x} : \sigma_1 \wedge \ldots \wedge \sigma_n \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \ldots, \mathtt{x}/\mathtt{x}_n] : \tau}\ (m)$$

# Observations

- Terms are built in a linear form
- The $(m)$ rule controls the number of contractions, so the growth of the reduction time (SLL- Lafont)
- Contraction on the right side of a derivation is handled by rule $(\wedge_n)$, which is parametric in $n$

Also, we can distinguish between

- constructive rules: $(Ax)$, $(\to I)$, $(\to E)$
- non-constructive rules: $(w)$, $(\wedge_n)$, $(m)$

# Strong normalization

STI is sound and complete w.r.t. strong normalization:

- if a term is typable in STI then it is strongly normalizing (soundness)

*Proof.* Termination of reduction does not depend on the strategy

- all strongly normalizing terms can be typed in STI (completeness)

*Proof.* Following the guidelines of Neergaard (2005):

1. Longest reduction strategy
2. Subject expansion under perpetual strategy
3. Completeness follows as a corollary

# Measures (1)

### Size $|\Pi|$

- if the last rule of $\Pi$ is $(Ax)$, then $|\Pi| = 1$
- if the last rule of $\Pi$ is a rule with $n$ premises $\Pi_i$, then $|\Pi| = \left(\sum_{i=1}^{n} |\Pi_i|\right) + 1$

### Size $|\mathtt{M}|$

- $|\mathtt{x}| = 1$
- $|\lambda\mathtt{x}.\mathtt{M}| = |\mathtt{M}| + 1$
- $|\mathtt{MN}| = |\mathtt{M}| + |\mathtt{N}| + 1$

# Measures (2)

$$\frac{\Gamma, \mathtt{x}_1 : \sigma_1, \ldots, \mathtt{x}_n : \sigma_n \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x} : \sigma_1 \wedge \ldots \wedge \sigma_n \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \ldots, \mathtt{x}/\mathtt{x}_n] : \tau} \ (m)$$

**Rank $\mathtt{rk}(\Pi) = \max$ between $1$ and rank of any $(m)$ rule in $\Pi$**

- Rank of $(m) = k \leq n$ variables $\mathtt{x}_i$ s.t. $\mathtt{x}_i \in \mathrm{FV}(\mathtt{M})$

**Degree $\mathtt{d}(\Pi) = \max$ nesting of $(\wedge_n)$ rules in $\Pi$**

**Weight $\mathtt{W}(\Pi, r)$ of $\Pi$ w.r.t. $r$**

- $(\wedge_n)$ is the last applied rule $\Rightarrow \mathtt{W}(\Pi, r) = r \cdot \max_{i=1}^{n} \mathtt{W}(\Sigma_i, r)$
- $(w)$ or $(m)$ is the last applied rule $\Rightarrow \mathtt{W}(\Pi, r) = \mathtt{W}(\Sigma, r)$

# Intersection tree

- $\mathtt{M}$ is an instance of $\mathtt{N}$ if it is obtained from $\mathtt{N}$ by renaming a subset of its free variables
- Let $(\delta)$ be a sequence of applications of rules $(w)$ and $(m)$
- intersection tree: maximal (sub)proof of the shape defined inductively as
  - If the last rule of $\Sigma$ is a constructive rule, then

$$\cfrac{\Sigma}{\Gamma \vdash M : \sigma}\ (\delta)$$

    is an empty intersection tree with one leaf $\Sigma$
  - If $\Sigma_i$ is a (possibly empty) intersection tree $(1 \leq i \leq n)$, then

$$\cfrac{\cfrac{\Sigma_i \triangleright \Gamma_i \vdash \mathtt{M} : \sigma_i \quad (1 \leq i \leq n)}{\Gamma_1 \wedge \ldots \wedge \Gamma_n \vdash \mathtt{M} : \sigma_1 \wedge \ldots \wedge \sigma_n}\ (\wedge_n)}{\Gamma' \vdash \mathtt{M}' : \sigma}\ (\delta)$$

    is an intersection tree and its leaves are the leaves of all the $\Sigma_i$

# Weighted subject reduction

### Weighted substitution

Let $\Pi \triangleright \Gamma, \mathtt{x} : \sigma \vdash \mathtt{M} : \tau$ and $\Sigma \triangleright \Delta \vdash \mathtt{N} : \sigma$, with $\Gamma \# \Delta$ and $\mathtt{x} \notin \mathsf{dom}(\Delta)$, then $S(\Sigma, \Pi) \triangleright \Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{x}] : \tau$ and $\mathtt{W}(S(\Sigma, \Pi), r) \leq \mathtt{W}(\Pi, r) + \mathtt{W}(\Sigma, r)$, for every $r \geq \max\{\mathtt{rk}(\Pi), \mathtt{rk}(\Sigma)\}$

### Weighted subject reduction

$\Pi \triangleright \Gamma \vdash \mathtt{M} : \sigma$ and $\mathtt{M} \underset{\beta}{\longrightarrow} \mathtt{M}'$ imply there is $\Pi' \triangleright \Gamma \vdash \mathtt{M}' : \sigma$ s.t. $\mathtt{W}(\Pi', r) < \mathtt{W}(\Pi, r)$ for every $r \geq \mathtt{rk}(\Pi)$

Here we must take into account that a $\beta$-redex (in the term) may appear in finitely many copies in the type derivation $\rightsquigarrow$ **parallel reductions**

# Example (1)

We want to reduce $(\lambda x.xx)((\lambda y.y)z)$ to normal form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x_1 : A \vdash x_1 : A}\,(Ax) \quad \dfrac{}{x_2 : a \vdash x_2 : a}\,(Ax)}{x_1 : A, x_2 : a \vdash x_1 x_2 : a}\,(\to E)}{x : A \wedge a \vdash xx : a}\,(m)}{\vdash \lambda x.xx : (A \wedge a) \to a}\,(\to I) \qquad \dfrac{\Sigma_1 \rhd z : A \vdash (\lambda y.y)z : A \quad \Sigma_2 \rhd z : a \vdash (\lambda y.y)z : a}{\Sigma \rhd z : A \wedge a \vdash (\lambda y.y)z : A \wedge a}\,(\wedge_2)}{z : A \wedge a \vdash (\lambda x.xx)((\lambda y.y)z) : a}\,(\to E)$$

$\Sigma$ ends by a non empty intersection tree $\rightsquigarrow$ there are two “virtual“ copies of the same redex

$$\dfrac{\dfrac{\dfrac{}{y : A \vdash y : A}\,(Ax)}{\vdash \lambda y.y : A \to A}\,(\to I) \quad \dfrac{}{z : A \vdash z : A}\,(Ax)}{\Sigma_1 \rhd z : A \vdash (\lambda y.y)z : A}\,(\to E) \quad \text{and} \quad \dfrac{\dfrac{\dfrac{}{y : a \vdash y : a}\,(Ax)}{\vdash \lambda y.y : a \to a}\,(\to I) \quad \dfrac{}{z : a \vdash z : a}\,(Ax)}{\Sigma_2 \rhd z : a \vdash (\lambda y.y)z : a}\,(\to E)$$

# Example (2)

If we reduce $(\lambda y.y)z$ we get the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{x_1 : A \vdash x_1 : A}}\,(Ax) \quad \dfrac{}{\mathtt{x_2 : a \vdash x_2 : a}}\,(Ax)}{\mathtt{x_1 : A, x_2 : a \vdash x_1 x_2 : a}}\,(\to E)}{\mathtt{x : A \wedge a \vdash xx : a}}\,(m)}{\vdash \lambda\mathtt{x.xx : (A \wedge a) \to a}}\,(\to I) \qquad \dfrac{\dfrac{}{\mathtt{z : A \vdash z : A}}\,(Ax) \quad \dfrac{}{\mathtt{z : a \vdash z : a}}\,(Ax)}{\mathtt{z : A \wedge a \vdash z : A \wedge a}}\,(\wedge_2)}{\Pi \triangleright \mathtt{z : A \wedge a \vdash (\lambda x.xx)z : a}}\,(\to E)$$

where both the redexes of $\Sigma_1$ and $\Sigma_2$ have been reduced

# Measure of reduction

If $\Pi \triangleright \Gamma \vdash \mathtt{M} : \sigma$ and $\mathtt{M} \xrightarrow[\beta]{n} \mathtt{M}'$, then

i) $n < |\mathtt{M}|^{\mathtt{d}(\Pi)+1}$

ii) $|\mathtt{M}'| < |\mathtt{M}|^{\mathtt{d}(\Pi)+1}$

Observe that

- The exponent of the function dependends on the term
- The bound on the normalization procedure can easily be exponential $\rightsquigarrow$ no bound on the complexity!
- The proof given above is independend on a given reduction strategy $\rightsquigarrow$ strong normalization soundness