Interactive Realizers and Monads
Stefano Berardi and Ugo de'Liguoro
May 17, 2010

Abstract
We propose a realizability interpretation of a system for quantifier free arithmetic which is equivalent to the fragment of classical arithmetic without nested quantiers, called here $\text{EM}_1$-arithmetic. We interpret classical proofs as interactive learning strategies, namely as processes going through several stages of knowledge and learning by interacting with the “environment” and with each other. We give a categorical presentation of the interpretation through the construction of two suitable monads.

1 Introduction
Since the discovery, by G"odel, Gentzen, Kreisel and others, of interpretations of the classical into the intuitionistic logic, it has been held that classical proofs should have a constructive significance, which logicians and computer scientists may envisage as their computational content. This soon appeared, and still remains, one of the more promising fields of application of logic to mathematics and computer science, with a positive contribution by logic rather than the finding of negative results only.

The impressive amount of work by several authors carried out so far has produced a rich deal of suggestive ideas and results supporting them; however, in our opinion, a convincing account of the nature of such a content is still missing, either because of its indirect description through translations and interpretations, or because the proposed approaches are based on almost magic properties of formal systems (think of G"odel’s dialectica interpretation and of its usage together minimal logic, or Friedman’s A-translation), or because of the too fine grained analysis, often at the level of logical connectives and quantiers on which they rely. The discomfort with such descriptions of the constructive content of classical proofs becomes more evident when looking at the “algorithms” one obtains by means of the known techniques, which are, worse than inefficient, impossible to follow and likely far away from the ideas on which the proofs were based.

We wish to go a further step into a recently emerged approach, which in our view originates from the game theoretic account of logic and arithmetic, and especially Coquand’s semantics of evidence in [12]. We look for an explicit description of the constructive meaning of proofs, possibly relating the obtained algorithms to the proof ideas they should come from. The basic intuition is that of understanding proofs as strategies, by which a finite agent learns how to ensure the truth of the proof conclusion. We emphasize here the point of learning in the limit as the distinctive feature of the approach we are developing. Learning is an interactive process in which (general) hypothesis are raised and tested against (particular) facts, possibly realizing that some wrong guesses have been made and reacting in some clever way. Also learning is an intrinsically unbounded process, in the sense that its goal is never definitely attained: or at least, the learning agent could not be able to decide to have been eventually successful.

A rough account of this follows the lines of Tarski semantics of formulas and of Kleene realizability interpretation of quantiers, but for the crucial case of the existential quantification. As a basic assumption let us take for granted that quantier free sentences (hence without any occurrence of variables, neither bound nor free) can be checked against truth by direct computation. A strategy for $\forall x A(x)$ is
a uniform method telling how to learn about \( A(c) \) for any possible choice of the (individual denoted by) the constant \( c \). The learning of the truth of \( \exists x A(x) \), instead, involves the guessing of a \( c \) such that \( A(c) \) can be learned to be true, without being \( c \) computable in general. By this reason a guess cannot be regarded as a definite choice, rather as a tentative one: if and when an evidence should occur against its correctness, the learner has to be instructed on how to backtrack her guesses and try some new guess.

Consider for concreteness the excluded middle law, \( \text{EM} \), that we write in the suggestive form: \( \exists x A(x) \lor \forall y \neg A(y) \). Rephrasing in the learning scenario Coquand’s dialogic interpretation of this law, the learning of \( \text{EM} \) begins by assuming \( \forall y \neg A(y) \), which is by the above the strategy of learning \( \neg A(c) \) for any arbitrary choice of \( c \). If an evidence occurs that \( A(c') \) should be true for some specific \( c' \), then the learner changes her mind and backtracks to the point where the previous assumption was made; she takes now \( A(c') \) as her new guess suspending at the same time her assent to all the consequences that have been drawn from the incompatible assumption that \( \neg A(c') \).

This process involves a memory, recording of all the previous guesses and of the logical dependencies among them, and it could be very complex depending on the logical complexity of \( A \). If it is a quantifier free closed formula, however, \( A \) is decidable by our basic assumption, so that the discovery that \( A(c') \) actually holds for some \( c' \) is definite, and all the consequences drown from its negation can be simply forgotten: this is what is called 1-backtracking in [7]. In the case of a more complex formula \( A \) the learner might realize to have been wrong in discarding \( \forall y \neg A(y) \), and even to have been wrong in her belief to have been wrong: in other words she might be in place to resume her own rejected guesses, but also to resume her negative attitude toward some of them, and so on.

A learning process of this guise is not guaranteed to terminate in principle, and it would be contradictory with the undecidability of the Halting problem to ask that, even if the learning process comes out from a sound proof, it should reach a perfect knowledge within a finite number of steps and effectively. It is at this point that we resort to Gold’s idea of learning in the limit (see [17, 18]). A sound learning strategy (a winning strategy in the game theoretic jargon) should guarantee that the learner can be wrong in her guessing only finitely many times, so that she will eventually hold her final guesses without any further change of her mind because of the discovery of some counterexample. The trick is that, while the process of generating and discarding the guesses is effective, one accepts that, except in particular cases, it is undecidable whether the definite guess has been reached or not.

Learning in the limit in the sense of Gold is not sufficient for interpreting proofs of the whole classical arithmetic. It corresponds to the 1-backtracking fragment which we call here \( \text{PRA + EM}_1 \), and which is essentially Hayashi’s Limit Computable Mathematics [20]. We have studied in [8] the concept of limit in the general case of unbounded backtracking, thought always of well-founded depth; but a precise description of the interpretation of formal proofs into learning strategies is a quite challenging task, especially because we do not want to alter the form of the conclusion, nor the proof itself by translating and forcing them into some normal form, because classically equivalent formulas and proofs might embody different constructive ideas.

Being aware of the difficulty, in this paper we address the issue in the limited case of the quantifier free fragment of Heyting arithmetic, \( \text{HA} \), known as primitive recursive arithmetic, \( \text{PRA} \). As it is explained in the next section, we add to such a theory the existential quantification only to the extent of expressing \( \text{EM}_1 \), which is \( \text{EM} \) when \( \exists x A(x) \) is \( \Sigma^0_1 \), that is when \( A \) is decidable. We do this by adding Skolem functions to the language \( \mathcal{L}_0 \) of \( \text{PRA} \), in a way which is reminiscent of Hilbert’s \( \varepsilon \)-terms, so that one does not need to consider nested quantifiers in the technical development. We then rephrase \( \text{EM}_1 \) by means of new axioms implying that, for each primitive recursive predicate \( P \) of arity \( k+1 \) there is a \( k \)-ary function symbol \( \varphi_P \) such that

\[
P(x, y) \rightarrow P(\bar{x}, \varphi_P(\bar{x})).
\]

The effect of \( \varphi_P \) is to choose a \( y \) making \( P(\bar{x}, y) \) true if such a \( y \) exists depending on the parameters \( \bar{x} \); \( \varphi_P(\bar{x}) = 0 \) (or any other default value) otherwise. But the function denoted by \( \varphi_P \) does not need to be computable; we view its values as guesses for \( y \) instead, and accept the idea that the individual denoted
by a term including the symbols \( \varphi P \) might change while the learner’s finite knowledge of the standard model grows.

We represent a state of knowledge by a finite set of atomic closed formulas \( P(\vec{m}, n) \) which have been found to be true by the learner in the standard recursion theoretic model of \( \text{PRA} \), and such that they uniquely define a finite part of the relative Skolem functions \( \varphi P \). Therefore by simply taking finite supersets of some given state of knowledge as its extensions, we see that there are in general many incompatible ways of enlarging the learner knowledge, which correspond to the possible choices for defining the guessing functions \( \varphi P \). To account for the dependency of the meaning of terms and formulas on the state of knowledge we interpret the individuals and the statements of the theory into functions from states to natural numbers and booleans respectively. Because of this a formula which was deemed false in some state \( s \) might become true in some state \( s' \supseteq s \); but note that also the opposite is possible.

The way out from such a seemingly chaotic situation is the redefinition of the concept of individual as a dynamic object. An individual \( \alpha \), also called a strongly convergent function, is a mapping from the set of states \( S \) to some set of values \( X \), such that given any countable sequence of states \( s_0 \subseteq s_1 \subseteq s_2 \subseteq \cdots \) which is weakly increasing w.r.t. the extension relation, \( \alpha(s_i) \) is eventually constant. An individual is a dynamic or perhaps an epistemic concept, since it clearly evolves along the history of a learning process depending on the actual experiments made by the learner. In the case of a formula \( A \) a more concrete description of the learning strategy is a searching procedure that, given some initial state \( s_0 \), produces consistent extensions \( s_1, s_2, \ldots \) of it, such that the meaning of \( A \), which is a function in \( S \rightarrow B \) (where \( B \) is the set of truth values) eventually becomes true.

We prove that under a natural lifting of the standard interpretation of terms and formulas from the language \( L_0 \), into functions from \( S \) to \( \mathbb{N} \) and from \( S \) to \( B \), terms and formulas always denote dynamic individuals, provided that the variables occurring in them are also interpreted by individuals in the new sense. We eventually obtain the desired result of having the whole language \( L_1 \) of the theory \( \text{PRA} + \text{EM}_1 \) uniformly interpreted into the same kind of mathematical objects.

Describing the model in terms of type theory, individuals inhabit the types of the shape \( SX = S \rightarrow X \); the meaning of \( S \) is that of a strong monad in the sense of [27], which we call the state monad, that can be seen as a type constructor. If we take for simplicity the category of sets as our base interpretation category, then \( S \) is an endofunctor of \( \text{Set} \) for which there exist an (injective) inclusion \( \eta^S_X : X \rightarrow SX \) and an extension map \( \varepsilon^S : (X \rightarrow SY) \rightarrow (SX \rightarrow SY) \), satisfying a suitable universality condition. \( SX \) does contain more than individuals, but we characterize the functions of the shape \( f^{s^\alpha} \) as those treating the state as a more concrete approximation of the definition of truth, by using only the value \( \alpha(s) \) in the evaluation of \( f^{s^\alpha}(\alpha, s) \). Since \( \alpha \) is the denotation of a term \( t \) or of a formula \( A \), this amounts to evaluate all the parts of \( t \) and \( A \) into the same state \( s \), which is a sort of global state: by this we say that a function \( f^{s^\alpha} \) has a global state, and call it global for short. Global functions have the pleasant property of sending individuals to individuals, and to be determined by their behaviour over the image of \( \eta^S_X(X) \) into \( SX \). The desired result of having terms and formulas denoting individuals is obtained by interpreting function, predicate and even connective symbols into global functions.

The relation with 1-backtracking is apparent from the fact that an individual might change its value only finitely many times along some given sequence of states of knowledge; this is further clarified by the fact that for each tuple of numbers \( \vec{m} \) the value of \( \varphi P(\vec{m}, n) \) is definitely set to \( n \) by adding the information \( P(\vec{m}, n) \) to the current state; but we have to allow for a finite number of changes, and not just one, because the arguments of \( \varphi P \) might well be variables (hence dynamic individuals) and because of the possible nesting of the \( \varphi P \) symbols, by which the tuple of arguments \( \vec{m} = \vec{\alpha}(s) \) (for a vector \( \vec{\alpha} \) of individuals and some newly reached state \( s \)) in \( \varphi P(\vec{m}) \) might also change.

The interpretation of the language \( L_1 \) into individuals is just a first ingredient of the model; indeed the natural question arises of what can we say about the meaning of a formula \( A \) when \( \text{PRA} + \text{EM}_1 \vdash A \). A simple minded answer is that it is an individual always converging to true. But this is not the case: indeed it is not difficult to find an \( A \) and an \( s \in S \) such that \( \text{PRA} + \text{EM}_1 \vdash A \) and \( A \) is false in \( s \); then
the sequence $\sigma(i) = s$ for all $i \in \mathbb{N}$ is weakly increasing, and $A$ is definitely false along $\sigma$. The problem cannot be overcome by restricting to strictly increasing sequences, since the state $s$ can be extended by adding information which is irrelevant to $A$ (say $P(s, n)$ for a predicate $P$ not occurring in $A$). A subtler property holds instead: if $\text{PRA} + \text{EM}_1 \vdash A$ then for any $s$ we can effectively find an extension $s' \supseteq s$ such that $A$ is true in $s'$, namely the subset of states in which $A$ is true is cofinal in $S$ w.r.t. the extension ordering.

The main result of the present paper is a semi-constructive proof of this claim\(^1\). Suppose that $\text{PRA} + \text{EM}_1 \vdash A$. Given an arbitrary initial state $s_0$ we construct a weakly increasing sequence $\sigma = s_0, s_1, \ldots$ such that $A$ is eventually true along $\sigma$. The construction is a learning process which is however not blind search: it is the proof of $A$ that embodies the searching strategy, which is at the same time continuous to depend only on a finite amount of information, coherent to produce sensible answers, and strongly convergent to ensure that the desired goal of making $A$ true will be eventually reached.

An interactive realizer is a function $r \in S(S) = S \rightarrow S$ which satisfies the above requirements to interpret a proof. By profiting of the improved exposition of the model in [4] w.r.t. its very first presentation in [6] and [9], we see a realizer $r$ as computing a state $r(s)$ which is compatible with $s$ and includes only what is needed to proceed toward the validation of a formula $A$. Suppose that the free variables of $A(x)$ are just $x$ (an inessential restriction), and let $\alpha \in \mathbb{S}\mathbb{N}$ be the individual interpreting $x$ given by some environment mapping $\xi : \text{Var} \rightarrow \mathbb{S}\mathbb{N}$; then we say that $r$ forces $\alpha$ into $A$, and write $r \vdash \alpha : A$, if for any $s \in \mathbb{S}$, if $r(s)$ is the empty set (the trivial state $\bot$) then $A(\alpha(s))$ is true in $s$. We call the subset of such states the prefix points of $r$, written $\text{Prefix}(r)$, since the definition we have adopted immediately implies that $r(s) = \bot$ if and only if $r(s) \subseteq s$. The existence of a search by means of $r$ out of an arbitrary starting state $s_0$ implies that $\text{Prefix}(r)$ is cofinal in $\mathbb{S}$, which is the way we understand the definite catch of the values of $\alpha$ into the set $\{n \in \mathbb{N} \mid A(n) \text{ is true}\}$.

There is a subtle difficulty with this concept of forcing, which otherwise could be confused with the homonym Kripke’s relation between possible worlds and formulas. We have observed above that even if the knowledge grows, the best that we can hope is that a formula which is a theorem of $\text{PRA} + \text{EM}_1$ will become eventually true along certain (weakly) increasing sequence of states, that are comparable to the branches of a tree-form Kripke model. This is clearly weaker than Kripkes’s forcing, which is monotonic along the paths of a model formed by the accessibility relation; since a statement is valid in a Kripke model if it is forced at the root of the tree of possible worlds, it is forced by every possible world in the model. Also terms take a fixed denotation in Kripke models as soon as they come to “existence” in a possible world, so that they have been also called “rigid designators” in [22]. We observe that this exactly mirrors the intuitionistic view of the validity of existential assertions, whose construction is, according to the Brouwer-Heyting-Kolmogorow interpretation, a pair of an individual (in the standard sense) and of a constructive proof that it satisfies a given property, where the individual part, as well the proof, must be known at once.

In the case of our model, on the contrary, the denotations of terms and formulas change with the state; but the state is the only way for a realizer to force $\alpha$ into the goal $A$: it is like pointing at some target which is moved by a side effect of the shots themselves, that can be eventually hit only through an interactive process of trials and errors. The interaction, therefore, depends on factors that are independent of the proof, like the given $\alpha$ and the starting state of knowledge: this is our account of why the interpretation of a non constructive proof of $A$ is an unpredictable (morally non-deterministic) process, whose actual behavior depends on the interaction with the “environment”, namely any other proof using $A$ as a lemma and interacting with the same state.

Coming back to the interpretation of proofs we now sketch how a realizer can be constructed out of them. In our model the forcing relation relative to some statement $A$ is a binary relation between realizers and dynamic individuals, so that it is included into $S(S) \times S(\mathbb{N})$: we consider $\mathcal{R}X = S(\mathbb{N}) \times S(S)$.

\(^1\)A constructive proof can be given, however, and it can be used to produce effective bounds to the time complexity of the extracted algorithms. See [3].
for historical reasons and because of the isomorphism with the object part of the side effect monad \( \mathcal{E}X = S \to (\mathbb{N} \times S) \). Indeed it turns out that \( \mathcal{R} \) is itself a monad, whose extension operation \( \mathcal{R}^\alpha \) is defined by means of a binary operation \( \otimes^S \) over \( S(S) \), allowing to combine two realizers \( r \) and \( r' \) into a new one \( r \otimes^S r' \). We call the so obtained realizer the merge of \( r \) and \( r' \), which is nothing else than the resulting interaction between \( r \) and \( r' \), where each of them is engaged in satisfying its own goal, namely the premises of an inference rule in the proof. This operation can be constructed in several ways, but it is axiomatically definable as the lifting to \( S(S) \) of a binary operation \( \otimes \) over \( S \) which is a monoid with unit \( \bot \) and satisfying a few additional requirements. As a consequence, \( \otimes^S \) is a monoidal operation too, that is essential to combine the realizers of subproofs to obtain a compositional interpretation of a proof in terms of interactive realizers.

The interactive realizability theorem establishes that for any theorem \( A \) of \( \text{PRA} + \text{EM}_1 \) and any interpretation \( \alpha \) of its free variables into individuals, there is a realizer \( r(\alpha) \) such that \( r(\alpha) \vdash \alpha : A \). As a byproduct we establish that, given a formula \( A(x,y) \) of \( \text{PRA} \) such that \( \text{PRA} + \text{EM}_1 \vdash A(x, \varphi_p(x)) \), where \( \varphi \) is the primitive recursive predicate defined by \( A \), we can extract an effective searching method out of the very proof of \( A \), which is capable of forcing the interpretation of the free variables in \( A \) to values that satisfy \( A \), possibly in some extension of the given state of knowledge, and actually in the interpretation of \( A \) in the standard classical model of arithmetic. The value computed by \( \varphi_p \) is such that \( A \) holds, but it is not necessarily the best chosen one w.r.t. all possible usages of \( A \). By merging the realizer of \( A \) with that of some proof that uses \( A \) as a lemma, we obtain a new realizer that may resume the search processes of its components, possibly leading to a new valuation of the \( \varphi_p \). The so obtained searching method is then an interactive algorithm reflecting the structure of the proof, that often embodies a cleaver and efficient idea of how to search for a partial, but locally sufficient knowledge of the otherwise infinite classical model of arithmetic.

2 Primitive recursive arithmetic plus EM₁ axiom

The theory of primitive recursive arithmetic, called \( \text{PRA} \) in [30] (see vol. 1, chapter 3, section 2), is essentially the quantifier free fragment of Heyting arithmetic with equality. The language \( \mathcal{L}_0 \) of \( \text{PRA} \) contains free variables for natural numbers, the constants 0 and \( \text{succ} \) for zero and successor respectively; further it includes a function symbol \( f \), e.g., for each primitive recursive function, the symbol \( = \) for equality and the connectives \( \neg, \wedge, \vee \) and \( \to \). To this list we add symbols for primitive recursive predicates \( P, Q, \ldots \) each with a fixed arity.

For presenting \( \text{PRA} \) we consider the following deductive system: the logical axioms are those of \( \text{IPC} \), the intuitionistic predicate calculus, plus the axioms for equality; the non logical axioms include the defining equations of all primitive recursive functions and \( \neg \text{succ}(0) = 0 \). As explained in [30], the formula \( \text{succ}(x) = \text{succ}(y) \to x = y \) is derivable, and needs not to be assumed as an axiom. Inference rules are:

\[
\begin{align*}
A \to B & \quad A & \quad A(x) & \quad A(x) \to A(succ(x)) \\
B & \quad MP & \quad A(t) & \quad \text{SUB} & \quad A(y) & \quad \text{IND}
\end{align*}
\]

where in rule SUB the premise \( A(x) \) has been derived from hypothesis not containing \( x \).

By \( A(x) \) we mean that \( x \) possibly occurs in \( A \), and \( A(t) \) denotes the same as the more explicit writing \( A[t/x] \), namely the substitution of \( t \) for \( x \) in \( A \). Although there are no bound variables in the formulas of \( \mathcal{L}_0 \), we speak of the sets \( \text{FV}(t) \) and \( \text{FV}(A) \) of the free variables occurring in \( t \) and \( A \) respectively.

In [30] a more general quantifier free version of the induction rule is considered, namely:

\[
\begin{align*}
A(0) & \quad A(x) \to A(succ(x)) \\
A(t) & \quad \text{IND}
\end{align*}
\]

This rule is however admissible by the rules IND and SUB above, by choosing a fresh \( y \) in the conclusion of IND.
The standard model of PRA interprets terms into \( \mathbb{N} \), the set of natural numbers, and function and predicate symbols into their recursion theoretic counterparts. In the next sections we make use of the simply typed lambda-calculus with numerals and recursors, known as Gödel system \( \mathbf{T} \) (see e.g. \cite{16}), for describing our interpretation and constructions: it is then natural to see the standard model of PRA inside the set theoretic model of the typed lambda-calculus, and to consider predicates as denoting functions with values in the set \( \mathcal{B} = \{ \text{true}, \text{false} \} \) instead of number theoretic functions with values in \( \{ 0, 1 \} \) as usual. Because of the absence of quantifiers, it is routine to show that any formula \( A \in \mathcal{L}_0 \) with \( \text{FV}(A) \subseteq \{ x_1, \ldots, x_k \} \) defines a \( k \)-ary primitive recursive predicate.

PRA is a fragment of the constructive arithmetic HA; however all the instances of the excluded middle which are expressible in the language \( \mathcal{L}_0 \) are derivable in this theory.

**Proposition 2.1** For all \( A \in \mathcal{L}_0 \) it is the case that \( \text{PRA} \vdash A \lor \neg A \).

**Proof.** See \cite{30}, Prop. 2.9. \( \blacksquare \)

As a consequence we could freely assume the axioms of CPC, the classical propositional calculus, in place of IPC. The essential point here is that the absence of quantifiers makes excluded middle into an intuitionistically acceptable principle w.r.t. to an oracle evaluating the function symbols occurring in \( A \). Indeed at the hearth of the proof of Proposition 2.1 is the fact that we can prove e.g. \( f(\vec{x}) = g(\vec{x}) \lor f(\vec{x}) \neq g(\vec{x}) \) by simultaneous induction (which is admissible in PRA), which is possible only because this formula does not express that \( f \) and \( g \) are either equal or different functions, as this last statement requires the existential quantification.

Let us call \( \text{EM}_1 \) the following schema, with \( A \in \mathcal{L}_0 \) such that \( \text{FV}(A) \subseteq \vec{x}, y \):

\[
(\text{EM}_1) \quad \forall \vec{x}. \exists y \ A(\vec{x}, y) \lor \forall y \neg A(\vec{x}, y).
\]

\( \text{EM}_1 \) is just an instance of the law of excluded middle where \( \exists y A(\vec{x}, y) \) is a \( \Sigma^0_1 \) formula with parameters, and it is called the \( \Sigma^0_1 \)-LEM principle in the hierarchy studied by Akama et alii in \cite{1}. \( \text{EM}_1 \) uses nested quantifiers, hence it is not expressible by a formula in \( \mathcal{L}_0 \). To find a quantifier free equivalent of \( \text{EM}_1 \) let us consider its classically equivalent prenex and skolemized normal form:

\[
\forall \vec{x}, y. \ A(\vec{x}, \varphi(\vec{x})) \lor \neg A(\vec{x}, y),
\]

which on turn is (classically) equivalent to

\[
\forall \vec{x}, y. \ A(\vec{x}, y) \rightarrow A(\vec{x}, \varphi(\vec{x})).
\]

Then we split it into two quantifier free axiom schemata, for reasons that will be apparent from the technical development in the subsequent sections of this work:

\[
(\chi) \quad P(\vec{x}, y) \rightarrow \chi P(\vec{x})
\]

\[
(\varphi) \quad \chi P(\vec{x}) \rightarrow P(\vec{x}, \varphi P(\vec{x}))
\]

where \( P \) is a primitive recursive predicate of arity \( k + 1 \) and \( \varphi P \) and \( \chi P \) are a function and a predicate symbol of arity \( k \) respectively. By axioms (\( \chi \)) and (\( \varphi \)) the actual meaning of \( \chi P(\vec{x}) \) is the predicate \( \exists y. P(\vec{x}, y) \). Concerning the meaning of \( \varphi P \) we note that the derivable implication \( P(\vec{x}, y) \rightarrow P(\vec{x}, \varphi P(\vec{x})) \) is an instance of the critical axiom of Hilbert’s \( \varepsilon \)-calculus \cite{21}, writing \( \varphi P(\vec{x}) \) in place of \( \varepsilon y P(\vec{x}, y) \), with the restriction that \( P \) has to be primitive recursive. Asking that \( P \) is primitive recursive is equivalent to
the restriction that the $A$ in $\text{EM}_1$ has to be a formula in $\mathcal{L}_0$, since by the above remark, these formulas exactly define primitive recursive predicates.

Let $\mathcal{L}_1$ be the language of the quantifier free predicate calculus defined as $\mathcal{L}_0$ by adding the new symbols $\varphi_P$ and $\chi_P$ to the list of function and predicate symbols respectively for each predicate symbol $P$ of $\mathcal{L}_0$: then $\mathcal{L}_0 \subseteq \mathcal{L}_1$, and the definitions of free variables and substitution apply to $\mathcal{L}_1$ terms and formulas unchanged. Finally, with a slight abuse of notation, we call $\text{PRA}$ of $L_\phi$ exactly define primitive recursive predicates.

Let $\text{PRA}$ of $L$, namely the restriction that the $A$ functions are only recursive in the halting problem. On passing we note that Proposition 2.1 still holds, $\chi$ are indeed intuitionistically unacceptable principles. As a matter of fact the $\text{PRA}$ can be argued by taking in the axioms ($\text{PRA}$ axioms of $\text{PRA}$ of $L_\phi$ is an exhaustive enumeration of primitive recursive predicates, in the sense that it is the case that $\varphi_P$ predicates and the $\varphi_P$ functions are only recursive in the halting problem. On passing we note that Proposition 2.1 still holds, namely $\text{PRA} \vdash A \lor \neg A$ for any $A \in \mathcal{L}_1$, and by the same proof.

3 The state monad and a constructive interpretation of $\text{PRA} + \text{EM}_1$ formulas

Let $R_0, R_1, \ldots$ be a denumerable list of predicate symbols in the language of $\text{PRA}$. We assume that it is an exhaustive enumeration of primitive recursive predicates, in the sense that $i$ is the Gödel number of a definition in $\text{PRA}$ of some primitive recursive predicate, associated to $R_i$. The $R_i$ are called simply predicates, for which we shall freely use letters $P, Q, \ldots$ possibly with indexes. We write $P \equiv Q$ if and only if both $P$ and $Q$ refer to the same $R_i$, i.e. to the same syntactical definition of the relative predicate: hence $\equiv$ is decidable. As said in Section 2, by the standard model we mean the standard classical interpretation of $\text{PRA}$, thought seen in the set theoretic model of system $T$. Except when treating of the semantic interpretation mapping, we write ambiguously $P(\vec{m}, n)$ for $P(\vec{m}, n)$, where $P$ is the primitive recursive predicate interpreting $P$.

Definition 3.1 (States of Knowledge) A state of knowledge (shortly a state) is a finite set:

$$s = \{\langle P_1, \vec{m}_{i_1}, n_{i_1} \rangle, \ldots, \langle P_l, \vec{m}_{i_l}, n_{i_l} \rangle\},$$

such that $P_1, \ldots, P_l$ are predicate symbols, and each $P_i$ is a predicate of arity $k_i + 1$, where $k_i$ is also the length of $\vec{m}_{i_i}$, and:

1. (model condition): $P_i(\vec{m}_{i_i}, n_{i_i})$ is true in the standard model for all $i = 1, \ldots, l$;

2. (consistency condition): if $P_i \equiv P_j$ and $\vec{m}_{i} = \vec{m}_{j}$ then $n_{i} = n_{j}$.

We call $S$ the set of states of knowledge.

A state of knowledge is a finite piece of information about the standard model of $\text{PRA}$: it says for which tuples of natural numbers the predicates $P_i$ are known to be true (by the model condition). The consistency condition implies that in each state of knowledge $s$ there exists at most one witness $n$ of the existential statement $\exists y. P(\vec{m}, y)$ for each predicate $P$ and tuple of natural numbers $\vec{m}$. This $n$ will be the value of $\varphi_P(\vec{m})$ in the state $s$.

States of knowledge can be presented as a structure $(S, \subseteq, \bot, \cup)$, where $(S, \subseteq)$ is the partial order defined by $s \subseteq s'$ if and only if $s \subseteq s'$; it has a bottom element $\bot = \emptyset$ and join of compatible states, $s \cup s' = s \cup s'$, where $s, s'$ are compatible, written $s \uparrow s'$, if $s, s' \subseteq s''$ for some $s'' \in S$, or equivalently if whenever $P(\vec{m}, n) \in s$ and $P(\vec{m}, n') \in s'$ it is the case that $n = n'$. $S$ is also closed under (arbitrary) intersections, and it is downward closed w.r.t. $\subseteq$, namely subset inclusion.
The set \( S \) is decidable, for which it is essential that the equality \( P_i \equiv P_j \) is an identity of definitions, being the equivalence of primitive recursive predicates undecidable. By the finiteness of the states \( s \in S \), the order and the compatibility relations are computable, as well as the join of two compatible states.

The language \( \mathcal{L}_1 \) of \( \text{PRA} + \text{EM}_1 \) adds to \( \mathcal{L}_0 \) the symbols \( \chi_P \) and \( \varphi_P \) for each predicate symbol \( P \) of \( \mathcal{L}_0 \). To interpret the theory \( \text{PRA} + \text{EM}_1 \) we begin by giving meaning to these symbols.

**Definition 3.2** For each predicate symbol \( P \) of arity \( k + 1 \), let \( [\chi_P] : \mathbb{N}^k \times S \rightarrow \mathbb{B} \) be defined by:

\[
[\chi_P](\vec{m}, s) = \begin{cases} 
  \text{true} & \text{if } (P, \vec{m}, n) \in s \text{ for some } n, \\
  \text{false} & \text{otherwise}.
\end{cases}
\]

Similarly define \( [\varphi_P] : \mathbb{N}^k \times S \rightarrow \mathbb{N} \) by:

\[
[\varphi_P](\vec{m}, s) = \begin{cases} 
  n & \text{if } (P, \vec{m}, n) \in s \text{ for some } n, \\
  0 & \text{otherwise}.
\end{cases}
\]

Because of the consistency condition in Definition 3.1, the value of \( [\varphi_P](\vec{m}, s) \) is unique. However there exist states \( s \) such that \( [\varphi_P](\vec{m}, s) \neq [\varphi_Q](\vec{m}, s) \) even if \( P \) and \( Q \) are equivalent as predicates, though they have different indexes. In this case \( P \) (and its equivalent \( Q \)) denotes a non functional predicate; \( \varphi_P \) and \( \varphi_Q \) are also different symbols, that in some models denote distinct Skolem functions.

Clearly both \( [\chi_P](\vec{m}, s) \) and \( [\varphi_P](\vec{m}, s) \) are computable. Note that the decidability of \( [\chi_P](\vec{m}, s) \) makes the default value 0 of \( [\varphi_P](\vec{m}, s) \) effectively distinguishable from its possible proper value 0, according to the fact that \( (P, \vec{m}, 0) \in s \) or not. In any case the meaning of \( \varphi_P \) is a total computable function.

**Lemma 3.3 (Monotonicity of \( [\chi_P] \))** Let \( s \subseteq s' \), for \( s, s' \in S \):

1. if \( [\chi_P](\vec{m}, s) = \text{true} \) then \( [\chi_P](\vec{m}, s') = \text{true} \);
2. the inverse implication does not hold in general.

**Proof.** Immediate: for the second claim take any \( P, \vec{m}, n \) such that \( P(\vec{m}, n) \) is true, \( s = \bot \), \( s' = \{ (P, \vec{m}, n) \} \).

In order to extend the standard interpretation of \( \text{PRA} \) to a constructive interpretation of \( \text{PRA} + \text{EM}_1 \), though in a richer model, we let the meaning of any term and formula depend on an extra parameter in \( S \), even if this is essential only when symbols \( \varphi_P \) or \( \chi_P \) occur in the term or formula.

To this aim we use a (strong) monad; following [27] we present monads as Kleisli triples. We quickly summarize the needed concepts and definitions: see [27] for a treatment of strong monads and of their use for giving the semantics to “computational” types. Let \( |C| \) be the class of objects of the category \( C \):

**Definition 3.4 (Kleisli Triple)** A Kleisli triple \((T, \eta, \gamma)\) over a category \( C \) is given by a mapping \( T : |C| \rightarrow |C| \) over the objects of \( C \), a family of morphisms \( \eta_X : X \rightarrow TX \in C \) for each \( X \in |C| \), and a mapping \( \gamma \) such that \( f^* : TX \rightarrow TY \in C \) whenever \( f : X \rightarrow TY \in C \), and the following equations hold:

1. \( f^* \circ \eta_X = f \),
2. \( \eta^*_X = \text{Id}_{TX} \),
3. \( g^* \circ f^* = (g^* \circ f)^* \), where \( g : Y \rightarrow TZ \in C \).
The mapping \( \eta_X \) is the inclusion of \( X \) into \( TX \); \( f^* \) is called the extension of the morphism \( f : X \to TY \) to the morphism \( f^* : TX \to TY \). A triple \( (T, \eta, \mu) \) defines a new category, the Kleisli category \( \mathcal{C}_T \), whose class of objects is \( |\mathcal{C}| \), and morphisms are \( \mathcal{C}_T[X, Y] = \mathcal{C}[X, TY] \). The identity \( \text{Id}_{\mathcal{C}_X} \) over \( X \in |\mathcal{C}| \) is \( \eta_X \in \mathcal{C}_T[X, X] = \mathcal{C}[X, TX] \), and composition is given by: \( f \circ g = f^* \circ g \). Under this reading, the clauses of Definition 3.4 imply that \( \mathcal{C}_T \) is a well defined category.

**Remark 3.5 (Monad)** Monads arose in category theory for the study of adjunctions (see e.g. [25]), but they have shown to be a fruitful concept also to treat algebraic structures. A monad \( T \) is a structure \( (T, \eta, \mu) \) where \( T : \mathcal{C} \to \mathcal{C} \) is a functor, \( \eta : \text{Id}_\mathcal{C} \to T \) and \( \mu : T^2 \to T \) (where \( T^2 = T \circ T \)), called the unit and the multiplication of the monad respectively, are natural transformations such that, for all \( X \in |\mathcal{C}| \):

\[
\mu_X \circ \eta_{TX} = \mu_X \circ T\eta_X = \text{Id}_{TX}, \quad \mu_X \circ \mu_{TX} = \mu_X \circ T\mu_X.
\]

A triple induces a monad and vice versa (see [27] and the references there). In fact, to extend the mapping \( \lambda \) to a functor it suffices to set \( \mu_T = \mu \circ \lambda_T \). Vice versa given the monad \( (T, \eta, \mu) \), one recovers the Kleisli extension by \( f^* = \mu_Y \circ Tf \) for \( f : X \to TY \). Because of this correspondence, we speak of triples and monads interchangeably.

We work in the category of sets \( \text{Set} \), though using only the part of it which models Gödel system \( T \). Actually we conjecture that our constructions could be generalized to any ccc with a natural number object (see [24]).

We use the simply typed \( \lambda \)-calculus as metanotation: sets are denoted as types and morphisms by \( \lambda \)-terms. By \( X \to Y \) we denote the object \( Y^X \), but sometimes also the homset \( \text{Set}[X, Y] ; X \to Y ) \to Z \) abbreviates \( X \to (Y \to Z) \), that is the arrow associates to the right. Because of the well known isomorphism \( X \times Y \to Z \simeq X \to Y \to Z \), the same function will be written both in the uncurried form: \( f(x, y) \) and in the curried one: \( f x y \), according to convenience; also the more familiar notation \( f(x) \) is preferred to \( f x \).

The following is a monad, which we call the state monad:

**Definition 3.6 (The State Monad)** We call the tuple \((S, \eta, \mu^S)\) the state monad, where:

\[
\begin{align*}
S^X &= \text{Set} \to X, & \text{where } X \in |\text{Set}|, \\
\eta^S_X(x) &= \lambda s \in S. x, & \text{for } x \in X, \\
f^*\eta^S_x(\alpha) &= \lambda s \in S. f(\alpha(s), s) & \text{for } f : X \to SY \in \text{Set} \text{ and } \alpha \in SX.
\end{align*}
\]

Until this will not cause confusion, we shall abbreviate \( \eta^S \) by \( \eta \) and \( f^S \) by \( f^* \).

**Proposition 3.7** The tuple \((S, \eta, \mu^S)\) is a Kleisli triple, hence a monad.

**Proof.** By checking the equations of Definition 3.4. For equation (1) let \( x \in X \) and \( f : X \to SY \), then:

\[
(f^* \circ \eta_X)(x) = f^*(\lambda s. x) = \lambda s'. f((\lambda s. x)(s'), s') = \lambda s'. f(x, s') = f(x).
\]

To see (2), for any \( \alpha \in SY \) and \( s \in S \) we have:

\[
(\eta_Y)^* (\alpha, s) = \eta_Y (\alpha(s), s) = (\lambda \_. \alpha)(s) = \alpha(s).
\]

Eventually to check (3) let \( f \) be as above, \( g : Y \to SZ \), \( \alpha \in SX \) and \( s \in S \). Then:

\[
(g^* \circ f^*)(\alpha, s) = g^*(f^*(\alpha), s) = g(f^*(\alpha, s), s) = g(f(\alpha(s), s), s).
\]

On the other hand:

\[
(g^* \circ f)^* (\alpha, s) = (g^* \circ f)(\alpha(s), s) = g^*(f(\alpha(s)), s) = g(f(\alpha(s), s), s).
\]
Remark 3.8 According to Remark 3.5, the action of $S$ over morphisms is:

$$
Sf = (\eta_Y \circ f)^*
= \lambda \alpha \in SX \ s \in \mathbb{S}.((\lambda y \in Y \ s' \in \mathbb{S}. \ y) \circ f)(\alpha(s), s)
= \lambda \alpha \in SX \ s \in \mathbb{S}.f(\alpha(s))
= \lambda \alpha \in SX \ .f \circ \alpha,
$$

for $f : X \to Y$. We call $Sf$ the pointwise extension of $f$. Note that $Sf = \lambda \alpha . f \circ \alpha$, hence it is just the hom-functor $\text{Set}(\mathbb{S}, -)$.

From Remark 3.5 we know that $\mu^S_X = I_{\mathbb{S}X^*}$, so that by definition unraveling we have that $\mu^S$ is the diagonalization of its first argument:

$$
\mu^S_X(\delta, s) = \delta(s, s), \quad \text{for any } \delta \in S^2X \text{ and } s \in \mathbb{S}.
$$

Next we extend the definitions of $[\chi_P]$ and $[\varphi_P]$ in 3.2 and we interpret truth values and numbers expressed by terms and formulas in $L_1$ by elements of $S\mathbb{B}$ and $S\mathbb{N}$ instead of $\mathbb{B}$ and $\mathbb{N}$ respectively. For the sake of discussion let $t$ and $A$ be any term and formula of $L_1$ respectively such that there is at most one free variable occurring in them. In case $t \in L_0$ its standard interpretation is a map in $\mathbb{N} \to \mathbb{N}$; if instead $t \in L_1 \setminus L_0$ then its meaning should have a similar type than $\varphi_P(x)$ (for some binary $P$) that is an arrow $[t] : \mathbb{N} \to S\mathbb{N}$; similarly the semantics of $A \in L_1 \setminus L_0$ should be in $\mathbb{N} \to S\mathbb{B}$. To define a uniform interpretation of $L_1$ we have two possibilities. The first one is to let terms and formulas to have their denotations of type $X \to SY$, namely as arrows of the Kleisli category $\text{Set}_S$: this is the choice preferred in [27]. However in our construction we use types of the form $SX \to SY$ for morphisms, because we want to stress that our “individuals” are certain convergent objects in $SX$ for $X = \mathbb{N}$ or $\mathbb{B}$ (see Section 4). This is in analogy with the common idea that real numbers are the actual individuals of analysis, even if they have been constructed, say, as Cauchy sequences of rational numbers.

In general given the monad $T$ over the category $\mathcal{C}$ we may consider the category $\mathcal{C}^T$ such that $|\mathcal{C}^T| = |\mathcal{C}|$ and $\mathcal{C}^T(X, Y) = \{f^* \mid f \in \mathcal{C}T(X,Y)\}$, where we recall that $\mathcal{C}T(X,Y) = \mathcal{C}(X, TY)$. Then it is straightforward to see that $\mathcal{C}^T$ and $\mathcal{C}_T$ are equivalent categories, so that we can work out the interpretation of $L_1$ in $\text{Set}_S$ without essentially departing from Moggi’s theory of computational types.

More in detail we say that an environment is a map $\xi : \text{Var} \to S\mathbb{N}$ and consistently that the interpretation of a term $t$ of the language $L_1$ should be an element $[t]_{\xi}^S \in S\mathbb{N}$. For the basic cases we have the unproblematic clauses:

$$
[x]_{\xi}^S = \xi(x), \quad [0]_{\xi}^S = \eta_{\mathbb{N}}(0) = \lambda \_ \cdot 0,
$$

where we write $\lambda \_ \cdot \ldots$ for $\lambda s \in \mathbb{S} \ldots$ when $\ldots$ does not depend on $s$.

Suppose that $f$ is a unary functional symbol, whose meaning in the standard model is the (primitive recursive) function $f : \mathbb{N} \to \mathbb{N}$, and that the interpretation $[t]_{\xi}^S$ of the term $t$ has been defined; by taking into account Remark 3.5 about the action of $S$ over $f$ we can define:

$$
[f(t)]_{\xi}^S = (Sf)([t]_{\xi}^S) = \lambda s \in \mathbb{S}.f(\lambda s_{\xi}^S(s)),
$$

that is $[f]^S := Sr$. In the case of predicates we have similarly that, if $P : \mathbb{N} \to \mathbb{B}$ is the interpretation of the unary predicate $P$, then we define $[P]^S := SP = \lambda \alpha \in S\mathbb{N} \ s \in \mathbb{S}.P(\alpha(s))$.

Remark 3.9 The interpretation of a numeral $n \equiv \text{succ}^a0$ is a constant function in $S\mathbb{N}$: if $\text{succ} : \mathbb{N} \to \mathbb{N}$ is the successor function, then for example (by omitting the environment $\xi$):

$$
[[\text{succ}(0)]^S = S(\text{succ})[0]^S = \lambda s \in \mathbb{S}.\text{succ}((\lambda \_ \cdot 0) s) = \lambda \_ \cdot \text{succ}(0) = \lambda \_ \cdot 1.
$$

By the interpretation of 0 and a straightforward induction, we have that $[n]^S = \lambda \_ \cdot n$, for all $n$. 

10
We step to $k$-ary functions and predicates using the construction proposed in [27]. A uniform embedding of $TX_1 \times \cdots \times TX_k$ into $T(X_1 \times \cdots \times X_k)$ has to be provided, which is constructed by means of the concept of tensorial strength. If the reader is not interested to the treatment of product in the theory of monads and computational types, (s)he may skip the Definition 3.10, the Proposition 3.11, and take the equations (1)-(4) below 3.11 as definitions of tuple and of the interpretation of $k$-ary functions.

Recall that a category has enough points if it has a terminal object 1, and for all pair of arrows $f, g : X \to Y$, if $f \circ a = g \circ a$ for all $a : 1 \to X$ (called point of $X$), then $f = g$. The category $\text{Set}$ has obviously enough points (actually it is the typical such category), as the terminal object is any singleton set $\{ * \}$, a point $x : \{ * \} \to X$ is just a constant function $* \mapsto x$ for some $x \in X$ and morphisms are set theoretic maps; consequently we can use a more compact definition of tensorial strength than in the general case of cartesian categories, on the ground of Proposition 3.4 of [27]. We also recall that any category theoretic $\lambda$-model has enough points, so that the following is not really restrictive.

**Definition 3.10 (Tensorial Strength and Strong Monad [27])** Let $(T, \eta, \mu)$ be a monad over a category $\mathcal{C}$ with finite products and enough points, and $!_Y : Y \to 1$ the unique morphism from $Y$ to the terminal object. A tensorial strength $t$ of $(T, \eta, \mu)$ is the unique family of morphisms $t_{X,Y} : X \times TY \to TX \times TY$ of $\mathcal{C}$ such that:

$$\forall a : 1 \to X, b : 1 \to TY. t_{X,Y} \circ (a, b) = T((ac \lambda_1, Id_Y)) \circ b.$$ 

If $t$ is a tensorial strength of $(T, \eta, \mu)$ we say that $(T, \eta, \mu, t)$ is a strong monad.

**Proposition 3.11 (Tensorial Strength of $\mathcal{S}$)** The state monad $\mathcal{S}$ has a tensorial strength given by:

$$t_{X,Y}(x, \alpha) := \lambda s \in \mathcal{S}. (x, \alpha(s)),$$

where $(\_, \_)$ is just set theoretic pairing.

**Proof.** Set $\bar{x} : \{ * \} \to X$ to $\lambda_\_ \bar{x}$ and $\bar{\alpha} : \{ * \} \to SY$ to $\lambda_\_. \bar{\alpha}$ for $x \in X$ and $\alpha \in SX$, which are the points such that $(t_{X,Y} \circ (\bar{x}, \bar{\alpha}))(\alpha) = t_{X,Y}(x, \alpha)$. Now, by Remark 3.8 and since $!_Y = \lambda_\in Y \cdot s$:

$$(\mathcal{S}(\langle \bar{x} \circ \lambda_1, Id_Y \rangle) \circ \bar{\alpha})(s) = (\lambda \beta \in SY \ s \in \mathcal{S}. (\lambda_\in Y \cdot x, Id_Y))(\beta(s)) = \lambda \beta \in SY \ s \in \mathcal{S}. (x, \beta(s)),$$

which is of type $SY \to \mathcal{S}(X \times Y)$. Hence :

$$(\mathcal{S}(\langle \bar{x} \circ \lambda_1, Id_Y \rangle) \circ \bar{\alpha})(s) = (\lambda \beta \in SY \ s \in \mathcal{S}. (x, \beta(s)))(\alpha(s)) = s \in \mathcal{S}. (x, \alpha(s)) = t_{X,Y}(x, \alpha),$$

and therefore $t_{X,Y} \circ (\bar{x}, \bar{\alpha}) = \mathcal{S}(\langle \bar{x} \circ \lambda_1, Id_Y \rangle) \circ \bar{\alpha}$ as desired.

Putting $\psi_{X,Y} := (t_{X,Y} \circ c_{SY,X})^* \circ ts_{Y,X} \circ c_{SX,SY}$, where $c_{X,Y} : X \times Y \to Y \times X$ is the canonical exchange isomorphism, we have a function $\psi_{X,Y} : \mathcal{S}X \times \mathcal{S}Y \to \mathcal{S}(X \times Y)$ which is the component at $X, Y$ of a natural transformation (see [27], Remark 3.6). By definition unfolding we obtain:

$$\psi_{X,Y}(\alpha, \beta) = (\alpha, \beta) = \lambda s \in \mathcal{S}. (\alpha(s), \beta(s)).$$

This concludes the categorical detour about strong monads.
Let \( eq : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{B} \) be the equality map: \( eq(m, n) = \text{true} \) if \( m = n \), \( eq(m, n) = \text{false} \) else. If \([t_1]^\varphi_\xi\) and \([t_2]^\varphi_\xi\) have been defined then:

\[
(S(eq) \circ \psi_{\varphi,N})([t_1]^\psi_\xi, [t_2]^\psi_\xi) = S(eq)([t_1]^S_\xi, [t_2]^S_\xi) = \lambda s \in S. eq([t_1]^S_\xi(s), [t_2]^S_\xi(s)),
\]

which is natural to take as the interpretation of equality.

It is straightforward to generalize \( \psi \) to \( k \)-ary products:

\[
\psi_{X_1, \ldots, X_k} : S(X_1 \times \cdots \times X_k) \rightarrow S(X_1 \times \cdots \times X_k)
\]

where, if \( \alpha_1 \in S(X_1), \ldots, \alpha_k \in S(X_k) \) we have:

\[
\psi_{X_1, \ldots, X_k}(\alpha_1, \ldots, \alpha_k) = \langle \alpha_1(1), \ldots, \alpha_k(1) \rangle.
\]

If \( f : X_1 \times \cdots \times X_k \rightarrow Y \) then we abbreviate:

\[
f^S := S(f) \circ \psi_{X_1, \ldots, X_k} : S(X_1 \times \cdots \times X_k) \rightarrow SY,
\]

where if \( k = 1 \) then \( f^S := S(f) \).

In particular consider \( \psi_{\varphi_1, \ldots, \varphi_n} : (SN)^k \rightarrow S(N^k) \), with \( k \) occurrences of \( N \) in the subscript of \( \psi \), that we shall denote shortly by \( \psi_{\varphi_1} \). Then we eventually obtain, for the \( k \)-ary function symbol \( f \) and its semantics \( f : N^k \rightarrow N \) in the standard model:

\[
[f]^S := f^S = \lambda \alpha_1 \in SN \ldots \alpha_k \in SN s \in S. f(\alpha_1(s), \ldots, \alpha_k(s)),
\]

namely the semantics of \( f \) is the pointwise lifting of its meaning in the standard model. The same construction works for the \( k \)-ary predicates. Finally we abuse notation and write:

\[
[\varphi_p]^S := [\varphi_p]^* \circ \psi_{\varphi_1} \quad \text{and} \quad [\chi_p]^S := [\chi_p]^* \circ \psi_{\varphi_k},
\]

where \([\varphi_p]\) and \([\chi_p]\) have been defined in 3.2 for all \( k + 1 \)-ary predicates \( P \) of \( L_0 \), and \( ^* \) is the extension mapping of the monad \( S \).

In summary the interpretation of \( \text{PRA} + \text{EM}_1 \) atomic formulas is the following:

**Definition 3.12 (Terms and Atomic Formulas Interpretation)** Let \( \xi : \text{Var} \rightarrow SN \) be an environment for the individual variables, and \( t \) a term of the language \( L_1 \); then \([t]^S_\xi \in SN \) is inductively defined:

\[
[t]^S_\xi = \xi(x)
\]

\[
[0]^S_\xi = \eta_p(0)
\]

\[
[f(t_1, \ldots, t_k)]^S_\xi = [f]^S([t_1]^S_\xi, \ldots, [t_k]^S_\xi)
\]

\[
[\varphi_P(t_1, \ldots, t_k)]^S_\xi = [\varphi_P]^S([t_1]^S_\xi, \ldots, [t_k]^S_\xi)
\]

where \( P \) is a \( k + 1 \)-ary predicate symbol.

If \( A \) is an atomic formula in \( L_1 \) and \( \xi \) an environment then \([A]^S_\xi \in SB \) is defined:

\[
[Q(t_1, \ldots, t_k)]^S_\xi = [Q]^S([t_1]^S_\xi, \ldots, [t_k]^S_\xi)
\]

\[
[\chi_P(t_1, \ldots, t_k)]^S_\xi = [\chi_P]^S([t_1]^S_\xi, \ldots, [t_k]^S_\xi)
\]

where \( P \) is a \( k + 1 \)-ary predicate symbol.
Remark 3.13 By definition unfolding we have:

\[
\begin{align*}
\llbracket 0 \rrbracket^S_\xi &= \lambda\. 0 \\
\llbracket f(t_1, \ldots, t_k) \rrbracket^S_\xi &= \lambda s \in S. f(\llbracket t_1 \rrbracket^S_\xi(s), \ldots, \llbracket t_k \rrbracket^S_\xi(s)) \\
\llbracket \varphi(t_1, \ldots, t_k) \rrbracket^S_\xi &= \lambda s \in S. [\varphi_\xi][\llbracket t_1 \rrbracket^S_\xi(s), \ldots, \llbracket t_k \rrbracket^S_\xi(s), s]
\end{align*}
\]

where \( f : \mathbb{N}^k \rightarrow \mathbb{N} \) is the standard interpretation of \( f \), and

\[
\begin{align*}
[\llbracket t_1 = t_2 \rrbracket^S_\xi &= \lambda s \in S. eq(\llbracket t_1 \rrbracket^S_\xi(s), \llbracket t_2 \rrbracket^S_\xi(s)) \\
\llbracket Q(t_1, \ldots, t_k) \rrbracket^S_\xi &= \lambda s \in S. Q(\llbracket t_1 \rrbracket^S_\xi(s), \ldots, \llbracket t_k \rrbracket^S_\xi(s)) \\
\llbracket \chi_\xi(t_1, \ldots, t_k) \rrbracket^S_\xi &= \lambda s \in S. [\chi_\xi][\llbracket t_1 \rrbracket^S_\xi(s), \ldots, \llbracket t_k \rrbracket^S_\xi(s), s]
\end{align*}
\]

where \( Q : \mathbb{N}^k \rightarrow \mathbb{B} \) is the standard interpretation of \( Q \). Note that in Definition 3.12 we do not need to mention explicitly the interpretation of the equality, which is actually a primitive recursive predicate, and hence an instance of \( Q \).

Let \( \sim : \mathbb{B} \rightarrow \mathbb{B} \) be the boolean negation, and \( \wedge, \vee, \rightarrow : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B} \) be the respective binary boolean functions. Then set \( [\ast]^S := \ast^S \) for \( \ast = \sim, \wedge, \vee, \rightarrow \).

**Definition 3.14 (Non Atomic Formulas Interpretation)** If \( A \in \mathcal{L}_1 \) is a non atomic formula and \( \xi \) any environment, then \( [A]^S_\xi \in SB \) is defined by cases:

\[
\begin{align*}
[\neg A]^S_\xi &= [\neg]^S([A]^S_\xi), \\
\end{align*}
\]

for \( \ast = \wedge, \vee, \rightarrow \).

Remark 3.15 The interpretation of formulas considered above has some similarities with Kripke semantics of the intuitionistic predicate calculus: in both cases indeed the meaning of a formula is indexed over a partial order; more, the states of knowledge can be easily seen as (finite) possible worlds. However the monotonicity property of Kripke models fails in our case:

\[
[A]^S_\xi(s) = \text{true} \& s \sqsubseteq s' \neq [A]^S_\xi(s') = \text{true}.
\]

As a counterexample to the implication take \( A := \chi_\xi(x) \rightarrow (x = \text{succ}(x)) \), where \( P(x, y) := x < y \), \( s = \{(P, 1, 2)\} \), \( s' = \{(P, 1, 2), (P, 0, 1)\} \) and \( \xi(x) = \lambda\. 0 \).

The next lemma states a standard property of the interpretations. We write \( \xi[x \mapsto \alpha] \) for the environment whose domain is \( \text{dom}(\xi) \cup \{x\} \) and which is everywhere equal to \( \xi \) but in \( x \) where it holds \( \alpha \).

**Lemma 3.16 (Substitution Lemma)** For all \( t, t', A \in \mathcal{L}_1 \) and variable \( x \):

\[
[\llbracket t[t/x] \rrbracket^S_\xi] = [\llbracket t' \rrbracket^S_{\xi[x \mapsto [t]^S_\xi]}] \quad \text{and} \quad [A[t/x]]^S_\xi = [A]^{S_{\xi[x \mapsto [t]^S_\xi]}}.
\]

**Proof.** By induction over \( t' \) and \( A \).

**Proposition 3.17** Let \( A \in \mathcal{L}_1 \). If \( A \) is either a non logical axiom of \( \text{PRA} \), or a logical axiom of \( \text{PRA + EM}_1 \), or an instance of the \( (\varphi) \)-axiom, then \( [A]^S_\xi(s) = \text{true} \) for any environment \( \xi \) and state \( s \).
Proof. If \( t, A \in L_0 \) let us write \([t]_\rho^S\) and \([A]_\rho^\mathcal{B}\) for the respective interpretations of \( t \) and \( A \) in the standard model w.r.t. the standard environment \( \rho : \text{Var} \to \mathbb{N} \). Then an immediate consequence of the interpretation of symbols in \( L_0 \) by pointwise lifting of their standard interpretations is that for all environments \( \xi : \text{Var} \to SN \) and \( s \in S \):

\[
[t]_\xi^S(s) = [t]_{\rho_\xi}^N \quad \text{and} \quad [A]_\xi^S(s) = [A]_{\rho_\xi}^\mathcal{B}, \quad \text{where} \rho_\xi(x) = \xi(x, s),
\]

which can be formally established by an easy induction over \( t \) and \( A \).

Now if \( A \) is a non logical axiom of PRA then \( A \in L_0 \) and \([A]_\rho^\mathcal{B} = \text{true} \) for any \( \rho \), thus \([A]_\xi^S(s) = [A]_{\rho_\xi}^\mathcal{B} = \text{true} \) for any \( \xi \) and \( s \).

Let \( A \in L_1 \) be a logical axiom of PRA + EM1. Then there exists an axiom \( A' \) of IPC, the propositional variables \( p_1, \ldots, p_k \) and the formulas \( A_1, \ldots, A_k \in L_1 \) such that \( A = A'[A_1/p_1, \ldots, A_k/p_k] \).

If \( \eta : \text{PropVar} \to \mathcal{S} \mathcal{B} \) is an interpretation of the propositional letters in our model, then by an obvious extension of Lemma 3.16 to the propositional variables we have:

\[
[A'[A_1/p_1, \ldots, A_k/p_k]]_\xi^S = [A']_{\xi, \eta}^S \quad \text{where} \eta(p_i, s) = [A_i]_{\xi}^S(s).
\]

Then the thesis follows by the fact that \( A' \) is a tautology, since IPC is a subtheory of CPC, and by reasoning along the same pattern as above.

Eventually let \( A := \chi_P(\vec{x}) \to P(\vec{x}, \varphi_P(\vec{x})) \) be an instance of the \((\varphi)\)-axiom, where \( P \) is a primitive recursive predicate, and let \( \xi \) and \( s \) be an arbitrary environment and a state respectively. Then

\[
[A]_\xi^S(s) = [\to]^S([\chi_P(\vec{x})]_\xi^S, [P(\vec{x}, \varphi_P(\vec{x}))]_\xi^S(s)) = [\chi_P(\vec{x})]_\xi^S(s) \to [P(\vec{x}, \varphi_P(\vec{x}))]_\xi^S(s).
\]

Now if \( [\chi_P(\vec{x})]_\xi^S(s) = [\chi_P](\vec{m}, s) = \text{false} \), where \( \vec{m} = m_1, \ldots, m_k \) for some \( k \) and \( m_i = \xi(x_i, s) \) for all \( i = 1, \ldots, k \), then \([A]_\xi^S(s) = \text{true} \) vacuously. Otherwise \((P, \vec{m}, n) \in s \) for some \( n \in S \): this implies that \( P(\vec{m}, n) \) is true in the standard interpretation and that \([\varphi_P(\vec{x})]_\xi^S(s) = [\varphi_P](\vec{m}, s) = n \), so that \([P(\vec{x}, \varphi_P(\vec{x}))]_\xi^S(s) = [P(\vec{m}, n)] = \text{true} \).

More is actually true, namely that for any \( A \in L_1 \), if PRA + (\varphi) \vdash A then \([A]_\xi^S(s) = \text{true} \) for all environment \( \xi \) and state \( s \): we do not prove this fact here, since it is a consequence of Theorem 6.17 (see Corollary 6.18).

We also observe that Proposition 3.17 fails in case of the \((\chi)\)-axiom. Consider the instance \( P(\vec{x}, y) \to \chi_P(\vec{x}) \), where \( P(\vec{m}, n) \) is true in the standard model for some \( \vec{m}, n \in \mathbb{N} \). Then there exist infinitely many \( s \in S \) such that \([P(\vec{x}, y) \to \chi_P(\vec{x})]_{\xi, \lambda, n/\vec{x}, y}]_\xi^S(s) \neq \text{true} \), for which it suffices that \( \lambda(s) = \vec{m} \), but \((P, \vec{m}, n') \not\in s \) for any \( n' \in \mathbb{N} \). Indeed the \((\chi)\)-axiom is the essential difference between PRA and PRA + EM1.

4 Convergence, Individuals and Global Functions

In the previous section we have introduced a dynamic (or perhaps epistemic) concept of individual, which is a map from states of knowledge to individuals in the ordinary sense. In this section we select a subset of the maps in \( SN \) and \( S \mathcal{B} \) that will represent (dynamic) individuals and truth values in our model, and show that the denotation of any term and of any formula in the language of PRA + EM1 actually is such a kind of map.

Definition 4.1 (Sequences, Strong Convergence and Individuals) A weakly increasing sequence over \( S \), shortly a w.i. sequence, is some countable subset \( \{s_0, s_1, \ldots \} \subseteq S \) such that:

\[
s_0 \subseteq s_1 \subseteq s_2 \subseteq \cdots,
\]

that is it is a mapping \( \sigma : \mathbb{N} \to S \) such that if \( i \leq j \) then \( \sigma(i) \subseteq \sigma(j) \). Let \( \alpha \in SX \):

14
1. \( \alpha \circ \sigma \) is convergent and it has a limit point \( \lim(\alpha \circ \sigma) = x \) if
\[
\exists i \forall j. \ (\alpha \circ \sigma)(i) = (\alpha \circ \sigma)(i + j) = x;
\]

2. \( \alpha \) is strongly convergent if for all w.i. sequences \( \sigma \), \( \alpha \circ \sigma \) is convergent.

We call a strongly convergent \( \alpha \in SX \) an individual of \( X \).

When speaking of \( \alpha \in SX \), we use the terms individual, strongly convergent or just convergent as synonyms. For each \( \alpha \in SX \) a sequence of states \( \sigma \) induces a sequence \( \alpha \circ \sigma \) of values in \( X \); it has a limit if it is eventually constant (namely if it becomes stable), that is we consider the limit w.r.t. the discrete topology over \( X \). Individuals are intended to refer to their limits, although these are not necessarily unique: in fact the limit of \( \alpha \circ \sigma \) depends on \( \sigma \) in general, so that they can be different for different w.i. sequences.

**Definition 4.2 (Constant Individuals and Functions with Global State)**

1. \( \alpha \in SX \) is a constant individual (or just a constant) if \( \alpha = \lambda_x \), for some \( x \in X \);
2. \( f : SX \to SY \) has global state if \( f(\alpha, s) = f(\lambda_x \cdot \alpha(s), s) \), for all \( \alpha \in SX \) and \( s \in S \).

A constant individual is trivially convergent, hence it is an individual in the sense of Definition 4.1. Constant individuals correspond to Kripke’s rigid designators (see [22]), which are terms denoting the same object in all possible worlds.

Functions with global state, henceforth called *global functions* for short, can evaluate their functional argument \( \alpha \) in the second argument \( s \) only: that is they have essentially a unique global state, whence the name. In fact a non global \( f : SX \to SY \) is easily constructed by violating this constrain: let \( \alpha \in SX \) and \( h : S \to S \) be such that \( h(s) = s' \) and \( \alpha(s) \neq \alpha(s') \) for certain \( s, s' \in S \); then the function \( f := \lambda \beta. \beta \circ h \) is not global since \( f(\alpha, s) = \alpha(h(s)) = \alpha(s') \), while \( f(\lambda_x \cdot \alpha(s), s) = (\lambda_x \cdot \alpha(s))(h(s)) = \alpha(s) \).

Note that, if \( h \) is strongly convergent, then \( f \) sends individuals to individuals, so that the latter property is not sufficient for a function to be global.

**Lemma 4.3 (Retraction Lemma)** Let
\[
\Phi : SX \to SY \quad \Phi(f)(\alpha, s) := f(s)(\alpha(s)) \\
\Psi : SY \to SX \quad \Psi(g)(s, x) := g(\lambda_x, s)
\]

Then \( \Phi \circ \Psi \) is a retraction; moreover the image of \( \Phi \) is exactly the set of global functions.

**Proof.** Let \( f \in S(X \to Y) = S \to (X \to Y) \); then for all \( s \in S \) and \( x \in X \):
\[
(\Psi \circ \Phi)(f)(s, x) = \Phi(f)(\lambda_x, s) = f(s)((\lambda_x)(s)) = f(s, x),
\]
therefore \( \Psi \circ \Phi = \text{Id}_{S(X \to Y)} \), so that \( (\Phi \circ \Psi) \circ (\Phi \circ \Psi) = \Phi \circ \Psi \) follows. Now let \( \alpha \in SX = S \to X \) and \( s \in S \); then we have:
\[
\Phi(f)(\lambda_x \cdot \alpha(s), s) = f(s)((\lambda_x \cdot \alpha(s))(s)) = f(s)(\alpha(s)) = \Phi(f)(\alpha, s)
\]
that is \( \Phi(f) \) is global. On the other hand if \( g \in SX \to SY = (S \to X) \to (S \to Y) \) is global and \( f = \Psi(g) \) then:
\[
\Phi(f)(\alpha, s) = \Psi(g)(s, \alpha(s)) = g(\lambda_x \cdot \alpha(s), s) = g(\alpha, s)
\]
that is \( (\Phi \circ \Psi)(g) = g \), namely the image of \( \Phi \) is exactly the subset of the global functions in \( SX \to SY \).
As a corollary, global functions are a characterization of “lifted” morphisms:

**Corollary 4.4** A function $g : SX \to SY$ is global if and only if $g = f^*$ for some $f : X \to SY$. Thus $\text{Set}_S$ is the largest sub category of $\text{Set}_S$ whose arrows are exactly the global functions.

**Proof.** If $f : X \to SY$ then for all $\alpha \in SX$ and $s \in S$:

$$f^*(\alpha, s) = f(\alpha(s), s) = f((\lambda \alpha)(s), s) = f^*(\lambda \alpha(s), s),$$

hence $f^*$ is global. Vice versa by Lemma 4.3 if $g$ is global then $g = \Phi(h) = \lambda \alpha \in SX \ s \in S. h(s)(\alpha(s))$, for some $h : S \to (X \to Y)$. Set $f(x, s) = h(s, x)$, so that $f : X \to SY$; then:

$$g = \lambda \alpha \in SX \ s \in S. f(\alpha(s), s) = f^*. $$

**Remark 4.5** By Lemma 4.3 global functions in $SX \to SY$ are, up to an application of $\Phi$, families of maps in $X \to Y$ indexed over $S$. The corollary relates global functions to the extension map $\wedge^*$ of the state monad. To spell it out further, consider the following:

- $F : (X \to Y) \to (X \to SY)$ where $F(f) : = \lambda x \in X \lambda \in S.f(x)$
- $G : (X \to SY) \to S(X \to Y)$ where $G(g) : = \lambda s \in S \lambda x \in X.g(x, s)$
- $H : (X \to Y) \to S(X \to Y)$ where $H(f) : = \lambda \in S.f$

Then it is easy to see that the following diagram commutes:

Since $\Phi \circ G = \wedge^*$, by Corollary 4.4 the image of $G$ is the set of global maps, up to the embedding $\Phi$. By $\Psi \circ S = H$, we see that $H$ is just the action of the functor $S$ over arrows, which is nothing more than a pointwise lifting of functions in $X \to Y$ to functions in $SX \to SY$: in Remark 3.8 we called the image of $H$ (or more precisely of $\Phi \circ H$) the subset of *pointwise maps* in $SX \to SY$. The fact that $H = G \circ F$ makes it clear that the pointwise maps are a subset of the global ones.

The most relevant property of global functions is that their behaviour is determined by their values over constant individuals.

**Theorem 4.6 (Density of $\eta_X(X)$ in $SX$)**

1. If $f, g : SX \to SY$ are global and such that $f(\lambda \alpha x) = g(\lambda \alpha x)$ for all $x \in X$, then $f = g$;

16
2. If $f : SX \to SY$ is global and $f(\alpha)$ is an individual for all constant individuals $\alpha$, then $f(\beta)$ is an individual for all individuals $\beta$.

Proof.

(1): if $f$ and $g$ are global functions which coincide over constant individuals then

$$f(\alpha, s) = f(\lambda_\cdot \alpha(s), s) = g(\lambda_\cdot \alpha(s), s) = g(\alpha, s).$$

(2): let $\beta \in SX$ be an individual i.e. strongly convergent; then for any w.i. sequence of states $\sigma$ there exists $i_0 \in \mathbb{N}$ such that for all $j \geq i_0$, $\beta(\sigma(i_0)) = \beta(\sigma(j))$. Since $f$ is global, we know that $f(\beta, s) = f(\lambda_\cdot \beta(s), s)$ for all $s \in \mathbb{S}$; therefore

$$f(\beta, \sigma(j)) = f(\lambda_\cdot \beta(\sigma(j)), \sigma(j)) = f(\lambda_\cdot \beta(\sigma(i_0)), \sigma(j)),$$

for all $j \geq i_0$. By the hypothesis that $f(\alpha)$ is strongly convergent for all constant $\alpha$ it follows that $f(\lambda_\cdot \beta(\sigma(i_0)))$ is strongly convergent, so that there exists $i_1$ such that for all $k \geq i_1$,

$$f(\lambda_\cdot \beta(\sigma(i_0)), \sigma(k)) = f(\lambda_\cdot \beta(\sigma(i_0)), \sigma(i_1)).$$

Then for all $h \geq \max(i_0, i_1)$:

$$f(\beta, \sigma(h)) = f(\lambda_\cdot \beta(\sigma(i_0)), \sigma(h)) = f(\lambda_\cdot \beta(\sigma(i_0)), \sigma(i_1)).$$

We conclude that $f(\beta)$ is strongly convergent.

Note that $\psi_{X_1,\ldots,X_k}(\alpha_1,\ldots,\alpha_k, s) = (\alpha_1(s),\ldots,\alpha_k(s))$, so that, if all $\alpha_i$'s are constant then $\psi_{X_1,\ldots,X_k}(\alpha_1,\ldots,\alpha_k)$ is such. Strictly speaking (2) of Theorem 4.6 does not apply directly to $\psi$. However this can be proved by a similar and easier argument.

Corollary 4.7 Each component $\psi_{X,Y} : SX \times SY \to S(X \times Y)$ of the natural transformation $\psi$ sends individuals $\alpha, \beta$ into the strongly convergent $(\alpha, \beta) = \lambda s \in S.(\alpha(s), \beta(s))$. The same holds of $\psi_{X_1,\ldots,X_k}$ for all $X_1,\ldots,X_k$.

Proof. Given the individuals $\alpha \in SX$ and $\beta \in SY$ and a w.i. sequence $\sigma$ there exist $i_0, i_1 \in \mathbb{N}$ such that for all $j$:

$$x = (\alpha \circ \sigma)(i_0 + j) = (\alpha \circ \sigma)(i_0) \quad \text{and} \quad y = (\beta \circ \sigma)(i_1 + j) = (\beta \circ \sigma)(i_1).$$

Therefore for all $i \geq \max(i_0, i_1)$:

$$((\alpha, \beta) \circ \sigma)(i) = (\alpha(\sigma(i)), \beta(\sigma(i))) = (x, y).$$

The statement about $\psi_{X_1,\ldots,X_k}$ follows by induction.

To provide a sufficient condition for the convergence of the output of a map with $k$ arguments, consider the obvious generalisation of the notion of functions with global state to the case of $k$-ary functions $f : SX_1 \times \cdots \times SX_k \to SY$, that we call $k$-global if for all $\alpha_1 \in SX_1,\ldots,\alpha_k \in SX_k$ and $s \in \mathbb{S}$:

$$f(\alpha_1,\ldots,\alpha_k, s) = f(\lambda_\cdot \alpha_1(s),\ldots,\lambda_\cdot \alpha_k(s), s). \quad (6)$$
Lemma 4.8 If $f : S \times \cdots \times S \times \times S \times \times S \to SY$ then there exists a unique $\hat{f} : S(X_1 \times \cdots \times X_k) \to SY$ such that $f = \hat{f} \circ \psi_{X_1, \ldots, X_k}$ that is the following diagram commutes:

\[
\begin{array}{ccc}
SX_1 \times \cdots \times SX_k & \xrightarrow{f} & SY \\
\psi_{X_1, \ldots, X_k} \downarrow & & \downarrow \hat{f} \\
S(X_1 \times \cdots \times X_k) & & 
\end{array}
\]

Moreover $f$ is $k$-global if and only if $\hat{f}$ is global.

Proof. Define $\hat{f} := \lambda \gamma.f(\pi_1 \circ \gamma, \ldots, \pi_k \circ \gamma)$. The first part of the lemma follows by the universal property of the cartesian product. Indeed we first observe that $\psi_{X_1, \ldots, X_k}$ is a surjective map: if $\gamma \in S(X_1 \times \cdots \times X_k)$ and $\pi_i : X_1 \times \cdots \times X_k \to X_i$ is the $i$-th projection then:

$$
\gamma = \langle \pi_1 \circ \gamma, \ldots, \pi_k \circ \gamma \rangle = \psi_{X_1, \ldots, X_k}(\pi_1 \circ \gamma, \ldots, \pi_k \circ \gamma).
$$

Thus, writing $\vec{a} = a_1, \ldots, a_k$ and $\langle \vec{a} \rangle = \langle a_1, \ldots, a_k \rangle$ we know that if $\hat{f}$ exists then:

$$
(f \circ \psi_{X_1, \ldots, X_k})(\vec{a}) = \hat{f}((\vec{a})) = f(\vec{a}),
$$

establishing at the same time unicity and existence of $\hat{f}$.

Now if $f$ is $k$-global then for any $s \in S$:

$$
\hat{f}(\lambda_\gamma.\delta(s), s) = f(\pi_1 \circ (\lambda_\gamma.\delta(s)), \ldots, \pi_k \circ (\lambda_\gamma.\delta(s)), s)
= f(\pi_1 \circ \delta, \ldots, \pi_k \circ \delta, s)
= f(\delta, s)
$$

by the fact that $\pi_i \circ (\lambda_\gamma.\delta(s)) = \lambda_\gamma.(\pi_i \circ \delta)(s)$. Vice versa if $\hat{f}$ is global and $s \in S$ then:

$$
f(\vec{a}, s) = (\hat{f} \circ \psi)(\vec{a}, s)
= \hat{f}(\langle \vec{a} \rangle, s)
= \hat{f}(\lambda_\gamma.(\vec{a}), s)
= f(\pi_1 \circ \lambda_\gamma.(\vec{a}), \ldots, \pi_k \circ \lambda_\gamma.(\vec{a}), s)
= f(\lambda_\gamma.a_1(s), \ldots, \lambda_\gamma.a_k(s), s).
$$

Corollary 4.9 If $f : S \times \cdots \times S \times \times S \times \times S \to SY$ is $k$-global and it sends constant individuals to individuals, then it sends individuals to individuals.

Proof. Let $\hat{f}$ be the unique global function such that $f = \hat{f} \circ \psi_{X_1, \ldots, X_k}$, which exists by Lemma 4.8: since $\psi_{X_1, \ldots, X_k}$ and its inverse send constant individuals to constant individuals, $\hat{f}$ sends constant individuals to individuals by the hypothesis on $f$, so that it sends individuals to individuals by (2) of Theorem 4.6. By Corollary 4.7, $\psi_{X_1, \ldots, X_k}$ also sends individuals to individuals so that $f$ satisfies the same property.
Remark 4.10 If \( f : X \to Y \) then by Remark 3.8 we have:
\[
(\mathcal{S} f)(\lambda_\cdot x) = \lambda s \in \mathcal{S} f((\lambda_\cdot x)(s)) = \lambda_\cdot f(x),
\]
that is \( \mathcal{S} f \) sends constants in \( \mathcal{S} X \) into constants in \( \mathcal{S} Y \). On the other hand \( \mathcal{S} f = (\eta_X \circ f)^* \) is global by Corollary 4.4, so that by Theorem 4.6.2, \( \mathcal{S} f \) sends convergent elements into convergent ones.

Corollary 4.11 If \( f : X_1 \times \cdots \times X_k \to Y \) then \( f^S \) is \( k \)-global. Moreover \( f^S \) sends (constant) individuals to (constant) individuals.

Proof. The first part of the thesis is immediate by Corollary 4.4 and Lemma 4.8. The remaining part follows by \( f^S = \mathcal{S}(f) \circ \psi_{X_1, \ldots, X_k} \), the fact that \( \mathcal{S}(f) \) sends constant individuals to constant individuals by Remark 4.10 and that the components of the natural transformation \( \psi \) send constant individuals to constant individuals.

The next lemma relates the interpretation of terms and formulas to global and \( k \)-global functions and will be useful in Section 6.

Lemma 4.12 For any variable \( x \), term \( t \in \mathcal{L}_1 \) and formula \( A \in \mathcal{L}_1 \) and for any environment \( \xi \) the functions
\[
\lambda_\alpha \in \mathcal{S}_N[\mathcal{T}|_{\backslash x = \alpha}] \quad \text{and} \quad \lambda_\alpha \in \mathcal{S}_N[A|_{\backslash x = \alpha}]
\]
are global. In general the functions \( \lambda_\alpha[\mathcal{T}|_{\backslash x = \alpha}] \) and \( \lambda_\alpha[A|_{\backslash x = \alpha}] \) are \( k \)-global (for \( k \) equal to the length of the vectors \( \vec{a} \) and \( \vec{x} \)), provided that both \( \text{FV}(t) \) and \( \text{FV}(A) \) are included in \( \vec{x} \).

Proof. By an easy induction over \( t \) and \( A \). If \( t \equiv x \) then for any \( s \in \mathcal{S} \):
\[
[x]^S_{\xi|_{\backslash x = \alpha}}(s) = \alpha(s) = (\lambda_\cdot \alpha(s))(s) = [x]^S_{\xi|_{\backslash x = \alpha, \cdot \alpha(s)}}(s).
\]
The inductive cases are immediate consequences of the inductive hypothesis. E.g. let \( t \equiv \varphi_P(t_1, \ldots, t_k) \), then for any \( s \in \mathcal{S} \):
\[
[\varphi_P(t_1, \ldots, t_k)]^S_{\xi|_{\backslash x = \alpha}}(s) = [\varphi_P([t_1]^S_{\xi|_{\backslash x = \alpha}}, \ldots, [t_k]^S_{\xi|_{\backslash x = \alpha}}), s] \quad \text{by ind. hyp.}
\]
The rest is equally straightforward.

An immediate consequence of Remark 4.10 and Corollary 4.11 is that if \( t \) and \( A \) are a term and a formula in the language \( \mathcal{L}_0 \) respectively (hence not including any \( \varphi_P \) nor \( \chi_P \) symbol), then their interpretations \( [t]^S_{\xi} \in \mathcal{S}_N \) and \( [A]^S_{\xi} \in \mathcal{S}_B \) are constant, provided that each \( \xi(x) \) is such. We prove now that in general terms and formulas of the language \( \mathcal{L}_1 \) denote strongly convergent individuals, provided that the free variables occurring in them are interpreted by strongly convergent individuals.

Lemma 4.13 For all \( \vec{m} \) of the appropriate length, both \( \lambda s \in \mathcal{S}[\varphi_P](\vec{m}, s) \in \mathcal{S}_B \) and \( \lambda s \in \mathcal{S}[\varphi_P](\vec{m}, s) \in \mathcal{S}_N \) are strongly convergent.
Proof. Consider \( \alpha = \lambda s \in S. [\varphi_p](\bar{m}, s) \in S \mathcal{B} \), and let \( \sigma \) be any w.i. sequence. Now either \((P, \bar{m}, n) \notin \sigma(i)\) for all \( i \), so that \( \alpha \circ \sigma \) is the constantly false function; or there exists \( i_0 \) such that \((P, \bar{m}, n) \in \sigma(i_0)\): then, since \( \sigma \) is weakly increasing, \((P, \bar{m}, n) \in \sigma(j)\) and \( \alpha(\sigma(j)) = \text{true} \) for all \( j \geq i_0 \).

The case of \( \lambda s \in S. [\varphi_p](\bar{m}, s) \in S \mathcal{N} \) is similar.

\[ \blacksquare \]

**Theorem 4.14** If \( t \) is any term and \( A \) any formula of the language \( \mathcal{L}_1 \) and \( \xi \) an environment whose domain includes the free variables of \( t \) and \( A \), such that \( \xi(x) \) is strongly convergent for all \( x \), then both \( [t]_{\xi}^S \in S \mathcal{N} \) and \( [A]_{\xi}^S \in S \mathcal{B} \) are strongly convergent.

**Proof.** By induction over \( t \) and \( A \), using Lemma 4.13 for the cases \( \varphi_p(t_1, \ldots , t_k) \) and \( \chi_p(t_1, \ldots , t_k) \) respectively, and Corollary 4.11 for the remaining inductive steps.

\[ \blacksquare \]

5 The realizers monad

The compositional construction of the search of a state that makes true a certain formula, which we shall describe in Section 6, rests on the ability of merging pairs of states, even if incompatible. We give an abstract definition of a merge operation \( \otimes \), and show that there exists one such. In fact more concrete and non-equivalent definitions of merge are possible, as we suggest in a remark.

We then define a quadruple \((\mathcal{R}, R^T, \cdot^*, \otimes)\), parametric in the merge operation \( \otimes \), such that \( \mathcal{R} \mathcal{X} = \mathcal{S}(\mathcal{X}) \times \mathcal{S}(\mathcal{S}) \) is the type of pairs of an individual and a realizer (a concept defined in the next section) interacting each other to the extent of satisfying a formula, which is the goal of the interaction.

The monoidal structure of the merge is lifted to the maps in \( \mathcal{S}(\mathcal{S}) \), in order for to meet the requirement for the functor \( \mathcal{R} \) to be a monad.

**Definition 5.1 (Merge)** A merge is a mapping \( \otimes : S \times S \rightarrow S \) such that, for all \( s_1, s_2 \in S \):

1. \((S, \otimes, \bot)\) is a monoid;
2. if \( s_1 \otimes s_2 = \bot \) then \( s_1 = \bot = s_2 \);
3. \( s_1 \otimes s_2 \subseteq s_1 \cup s_2 \).

Note that in clause (3) above we cannot write \( s_1 \otimes s_2 \subseteq s_1 \cup s_2 \) since \( s_1 \cup s_2 \) might be an inconsistent set, so not in \( S \).

**Lemma 5.2** If \( \otimes \) is a merge then for all \( s, s_1, s_2 \in S \):

1. if \( s \uparrow s_1 \) and \( s \uparrow s_2 \) then \( s \uparrow (s_1 \otimes s_2) \);
2. if \( s \cap s_1 = s \cap s_2 = \bot \) then \( s \cap (s_1 \otimes s_2) = \bot \).

**Proof.**

(1): \( s \nmid (s_1 \otimes s_2) \) implies that there exists \( a \in s_1 \otimes s_2 \) such that \( s \nmid \{a\} \). Since \( s_1 \otimes s_2 \subseteq s_1 \cup s_2 \), it is the case that \( a \in s_i \) for either \( i = 1 \) or \( i = 2 \), contradicting \( s \uparrow s_1 \) and \( s \uparrow s_2 \).

(2): we observe that \( s \cap (s_1 \otimes s_2) \subseteq s \cap (s_1 \cup s_2) = (s \cap s_1) \cup (s \cap s_2) \), hence if \( (s \cap s_1) \cup (s \cap s_2) = \bot = \emptyset \) then \( s \cap (s_1 \otimes s_2) = \bot \).
A very simple example of “merging” consists in dropping one of the merged states.

**Proposition 5.3** The following mapping is a merge:

\[
s_1 \otimes_0 s_2 = \begin{cases} 
  s_1 & \text{if } s_1 \neq \bot, \\
  s_2 & \text{otherwise.}
\end{cases}
\]

**Proof.** It is immediate that \( s_1 \otimes_0 s_2 \in S \) for all \( s_1, s_2 \in S \).

For all \( s \in S \), \( \bot \otimes_0 s = s \). If \( s = \bot \) then \( s \otimes_0 \bot = \bot = s \). On the other hand if \( s \neq \bot \) then \( s \otimes_0 \bot = s \); hence \( \bot \) is the unit of \( \otimes_0 \).

Let \( s_1 = \bot \), then

\[
(s_1 \otimes_0 s_2) \otimes_0 s_3 = s_2 \otimes_0 s_3 = s_1 \otimes_0 (s_2 \otimes_0 s_3).
\]

Else, if \( s_1 \neq \bot \) then

\[
(s_1 \otimes_0 s_2) \otimes_0 s_3 = s_1 \otimes_0 s_3 = s_1 = s_1 \otimes_0 (s_2 \otimes_0 s_3).
\]

Therefore (1) of Definition 5.1 holds.

If \( s_1 \neq \bot \) then \( s_1 \otimes_0 s_2 = s_1 \neq \bot \); similarly if \( s_1 = \bot \) and \( s_2 \neq \bot \) then \( s_1 \otimes_0 s_2 = s_2 \neq \bot \), so that condition (2) of Definition 5.1 follows by contraposition.

Finally \( s_1 \otimes_0 s_2 = s_i \) for either \( i = 1, 2 \), hence (3) of Definition 5.1 is satisfied.

**Remark 5.4** The map \( \otimes_0 \) is essentially a selector of non \( \bot \)-states, with a bias toward its first argument: it considers the second argument just in case the first one is not informative at all. In particular it is not commutative, while it is clearly idempotent: \( s \otimes_0 s = s \). It is a very simple, thought crude example of merge. Beside it and its symmetric \( s_1 \otimes_0 s_2 := s_2 \otimes_0 s_1 \), there exist other examples of merge that one could consider. We mention two of them omitting proofs.

- A “parallel” non-commutative merge. Define \( \bar{s} = \{(P, \bar{m}, n) \mid \exists n'.(P, \bar{m}, n') \in s\} \), and set

\[
s_1 \otimes_1 s_2 := s_1 \cup (s_2 \setminus s_1).
\]

This merge saves all of the information in \( s_2 \) which is consistent with \( s_1 \), while in case of inconsistency, the elements of \( s_1 \) prevail: hence it is not commutative, and its symmetric is a different merge. This is the merge operation used in [4].

- A “parallel” commutative merge. For any \( X \subseteq \bigcup S \) define \( \tilde{X} := \{(P, \bar{m}, n) \in X \mid \forall (P, \bar{m}, n') \in X.n \leq n'\} \). Then we set:

\[
s_1 \otimes_2 s_2 := s_1 \cup \tilde{s_2}.
\]

The effect of \( \tilde{X} \) is, for all predicate \( P \) and vector of numbers \( \bar{m} \), to select, among all possibly inconsistent tuples \( (P, \bar{m}, n_1), (P, \bar{m}, n_2), \ldots \) in \( X \), the tuple \( (P, \bar{m}, n_i) \), where \( n_i \) is the minimum among \( n_1, n_2, \ldots \). It follows that \( \tilde{X} \) is always consistent and, if \( X \subseteq S \) is finite, then it is an element of \( S \). Moreover \( \tilde{X} \subseteq X \) and \( \tilde{X} = \tilde{X} \), hence it is an interior operator. The remarkable property of \( \otimes_2 \) is commutativity. This merge appears in [6].

We observe that \( \otimes_0, \otimes_1 \) and \( \otimes_2 \) are all computable functions.

Since a merge is a function in \( \otimes : S \times S \rightarrow S \) it can be pointwise lifted to the mapping \( \otimes^S = S(\otimes) \circ \psi_S : S(S) \times S(S) \rightarrow S(S) \), where \((r \otimes^S r')(s) = r(s) \otimes r'(s)\). By means of \( \otimes^S \) we may define a new monad:
Definition 5.5 (The Realizer Monad) Let \( \otimes \) be a merge. Then we say that the tuple \( (\mathcal{R}, \eta^\mathcal{R}, *, \otimes) \) is a realizer monad if:

\[
\begin{align*}
\mathcal{R}X &= \mathcal{S}(X) \times \mathcal{S}(\emptyset) & \text{where } X \in |\text{Set}|, \\
\eta^\mathcal{R}_X(x) &= (\eta^\mathcal{S}_X(x), \eta^\mathcal{S}_\emptyset(\bot)) & \text{for } x \in X, \\
f^* (\alpha, r) &= (f^1* (\alpha), r \otimes^\mathcal{S} f^2* (\alpha)) & \text{for } f : X \to \mathcal{R}Y \in \text{Set} \text{ and } (\alpha, r) \in \mathcal{R}X,
\end{align*}
\]

where \( f_i = \pi_i \circ f \), for \( i = 1, 2 \).

Below we write \( \eta \) and \( \cdot \) for \( \eta^\mathcal{S}, \cdot^\mathcal{S} \) respectively, to simplify the notation, while keeping \( \eta^\mathcal{R}, \cdot^\mathcal{R} \) to distinguish the unit and the extension map of the monad \( \mathcal{R} \).

The set \( \mathcal{R}X = \mathcal{S}(X) \times \mathcal{S}(\emptyset) \) is larger than its part of interest: as shown in Section 4, the relevant part of \( \mathcal{S}(X) \) is the set of individuals; on the other hand, as we shall see in the next section, we concentrate on realizers which are individuals in \( \mathcal{S}(\emptyset) \) satisfying some further conditions. So that there is a slight abuse of terminology. However the monad provides an elegant way of pairing individuals and transformations over the states, which is at the basis of the forcing relation and the realizability interpretation we shall meet in Section 6.

A realizer monad is built on top of the state monad \( (\mathcal{S}, \eta^\mathcal{S}, \cdot^\mathcal{S}) \), and it is parametric in the merge \( \otimes \). By definition unfolding we have:

\[
\mathcal{R}X = (\mathcal{S} \to \mathcal{S}) \times (\mathcal{S} \to \emptyset), \quad \eta^\mathcal{R}_X(x) = (\lambda_x, \lambda_\bot),
\]

and

\[
r \otimes^\mathcal{S} f^*_1(\alpha) = \lambda s \in \mathcal{S}. r(s) \otimes f_2(\alpha(s), s).
\]

To better understand the definition of \( f^* \) observe that the function

\[
f : X \to [(\mathcal{S} \to Y) \times (\mathcal{S} \to \emptyset)]
\]

is identified with the pair \( (f_1, f_2) \) (as they are the same in any cartesian category), so that:

\[
f_1 : X \to (\mathcal{S} \to Y) \quad \text{and} \quad f_2 : X \to (\mathcal{S} \to \emptyset),
\]

and therefore

\[
f^*_1 : (\mathcal{S} \to X) \to (\mathcal{S} \to Y) \quad \text{and} \quad f^*_2 : (\mathcal{S} \to X) \to (\mathcal{S} \to \emptyset).
\]

Of these the component \( f^*_1 \) is intended to associate individuals over \( X \) to individuals over \( Y \); the second and more relevant component \( f^*_2 \) formalises how functions over \( \emptyset \), in particular realizers, can depend on individuals. In particular the importance of merging of \( r \) with \( f^*_2(\alpha) \) as the second component of \( f^* \) will be discussed in Remark 5.8.

Lemma 5.6 If \( \otimes \) is a merge, then \( (\mathcal{S}(\emptyset), \otimes^\mathcal{S}, \lambda_\bot) \) is a monoid.

Proof. Observe that, by Remark 3.8 and the definition of \( \psi \), \( (r \otimes^\mathcal{S} r')(s) = r(s) \otimes r'(s) \) for all \( s \in \mathcal{S} \), so that the fact that \( \otimes \) is a monoidal operation over \( \emptyset \) with unit \( \bot \) immediately implies that \( (\mathcal{S}(\emptyset), \otimes^\mathcal{S}, \lambda_\bot) \) is a monoid: for example \( (r \otimes^\mathcal{S} \lambda_\bot)(s) = r(s) \otimes \bot = r(s) \) for all \( s \), so that \( r \otimes^\mathcal{S} \lambda_\bot = r \).

\[\blacksquare\]

Theorem 5.7 If \( (\mathcal{R}, \eta^\mathcal{R}, *, \otimes) \) is a realizer monad then \( (\mathcal{R}, \eta^\mathcal{R}, *, \otimes) \) is a Kleisli triple, and hence a monad.
Proof. By checking that \((\mathcal{R}, \eta^\mathcal{R}, \cdot^\mathcal{R})\) satisfies the three equations of Definition 3.4, and using the fact that, by Lemma 5.6, if \(\otimes\) is a merge then \(\otimes^S\) is a monoidal operation over \(S(S)\) with unit \(\lambda_{-\bot}\).

\(f^* \circ \eta^\mathcal{R}_X = f\), that is the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta^\mathcal{R}_X} & (S \to X) \\
\downarrow{f} & & \downarrow{f^*} \\
(S \to Y) \times (S \to Y) & \xrightarrow{f^* \otimes f^*} & (S \to Y) \\
\end{array}
\]

Given \(x \in X\) we have:

\((f^* \circ \eta^\mathcal{R}_X)(x) = f^* (\lambda_{-\bot} x, \lambda_{-\bot}) = (f_1^* (\lambda_{-\bot} x), \lambda_{-\bot} \otimes^S f_2^* (\lambda_{-\bot} x)) = (f_1^* (\lambda_{-\bot} x), f_2^* (\lambda_{-\bot} x)),\)

since \(\lambda_{-\bot}\) is the unit of \(\otimes^S\). But for all \(s \in S\), \(f_1^* (\lambda_{-\bot} x, s) = f_1(x)\), and \(f_2^* (\lambda_{-\bot} x, s) = f_2(x)\), and therefore we get:

\((f^* \circ \eta^\mathcal{R}_X)(x) = (f_1(x), f_2(x)) = (f_1, f_2)(x) = f(x).\)

\((\eta^\mathcal{R}_Y)^* = 1d_{\mathcal{R}Y}.\) By definition \(\eta^\mathcal{R}_Y = (\eta_Y, \lambda \in Y, \eta_{\mathcal{R}}(\bot))\), where for any \(\alpha \in SY\) we have:

\((\lambda, \in Y, \eta_{\mathcal{R}}(\bot))^* (\alpha) = \lambda s \in S. (\lambda_{-\bot})(s) = \lambda_{-\bot}\)

Now for all \(r : S \to S\) we have:

\((\eta^\mathcal{R}_Y)^* (\alpha, r) = ((\eta_Y)^* (\alpha), r \otimes^S \lambda_{-\bot}) = (\alpha, r),\)

by the fact that \((\eta_Y)^* = 1d_{SY}\) (Proposition 3.7) and that \(\lambda_{-\bot}\) is the unit of \(\otimes^S\).

\(g^* \circ f^* = (g^* \circ f)^\mathcal{R},\) where \(g : Y \to \mathcal{R}Z\): let again \(f = (f_1, f_2)\) and \(g = (g_1, g_2)\); given \(\alpha : S \to X\) and \(r : S \to S\), by definition unfolding we have:

\[
(g^* \circ f^*)(\alpha, r) = \bigg(g^* (f_1^* (\alpha), r \otimes^S f_2^* (\alpha))\bigg) = \bigg((g_1^* \circ f_1^*(\alpha), (r \otimes^S f_2^* (\alpha)) \otimes^S (g_2^* \circ f_1^*(\alpha))\bigg)
\]

On the other hand let \(h = g^* \circ f = (h_1, h_2),\) where \(h_i = \pi_i \circ g^* \circ f.\) Then

\[
(g^* \circ f)^\mathcal{R} (\alpha, r) = (h_1^* (\alpha), r \otimes^S h_2^* (\alpha)).
\]

Now:

\[
h_1^* (\alpha) = \lambda s \in S. (\pi_1 \circ g^* \circ f)(\alpha(s), s)
= \lambda s \in S. \pi_1((g^* \circ f)(\alpha(s)))(s)
= \lambda s \in S. \pi_1(g^* (f_1(\alpha(s)), f_2(\alpha(s))))(s)
= \lambda s \in S. (g_1^* (f_1(\alpha(s))))(s)
= \lambda s \in S. g_1^* \circ f_1^*(\alpha).
\]

Similarly we have:

\[
h_2^* (\alpha) = \lambda s \in S. (\pi_2 \circ g^* \circ f)(\alpha(s), s)
= \lambda s \in S. \pi_2((g^* \circ f)(\alpha(s)))(s)
= \lambda s \in S. f_2^* (\alpha(s)) \otimes^S (g_2^* \circ f_1^*(\alpha))(s)
= f_2^* (\alpha) \otimes^S (g_2^* \circ f_1^*(\alpha))\]

by claim (7) below.
Remark 5.8 In any ccc it is the case that

\[(f^*(\alpha) \otimes^S g^*(\alpha))(s) = (f(\alpha(s)) \otimes^S g(\alpha(s)))(s)\]  

is easily checked by definition unfolding. Summing up:

\[(g^* \circ f)^{\ast K}(\alpha, r) = ((g^*_1 \circ f_1)^{\ast}(\alpha), r \otimes^S (f_2^* \circ f_1)^{\ast}(g^*_2 \circ f_1)^{\ast}(\alpha)),\]

and we conclude by noting that \((g^*_1 \circ f_1)^{\ast} = g^*_1 \circ f_1^\ast, (g^*_2 \circ f_1)^{\ast} = g^*_2 \circ f_1^\ast\) since \(\ast\) is the extension of a Kleisli triple by Proposition 3.7, and because \(\otimes^S\) is associative.

\[\blacksquare\]

**Remark 5.8** In any ccc it is the case that

\[(Z \to X) \times (Z \to Y) \simeq Z \to (X \times Y)\]

is a natural isomorphism, given by \((f, g) \mapsto (f, g)\). Therefore

\[\mathcal{R}X = (S \to X) \times (S \to S) \simeq S \to (X \times S),\]

which is equal to \(S(X \times S)\). \((\mathcal{R}, \eta^S, \ast^S)\) is similar to the side effect monad \((\mathcal{E}, \eta^E, \ast^E)\) in [27]:

\[
\begin{align*}
\mathcal{E}X &= S \to (X \times S) \\
\eta^E_x(x) &= \lambda s \in S. x(s) = \langle \eta_X(x), \text{Id}_S \rangle \\
f^\ast(\gamma) &= \lambda s \in S. f((\pi_1 \circ \gamma))(s), (\pi_2 \circ \gamma)(s))
\end{align*}
\]

where \(S\) is some set of states, \(f : X \to (S \to (Y \times S))\) and \(\gamma : S \to (X \times S)\). In case of \(S = S\) we have that if \((\alpha, r)\) is a pair of a convergent mappings, then \(\langle \alpha, r \rangle : S \to (X \times S)\) is such (see Corollary 4.7): therefore the isomorphism \(\mathcal{R}X \simeq S(X \times S)\) preserves convergence.

The computational idea behind \(\mathcal{E}\) and \(\mathcal{R}\) is however different. In the case of the side effects monad the function \(f^\ast(\gamma)(s)\), where \(\gamma = \langle \gamma_1, \gamma_2 \rangle\), first evaluates \(\gamma_1\) in the state \(s\), possibly leading to a new state \(s' = \gamma_2(s)\), intuitively because of side effects in the evaluation of \(\gamma_1(s)\); then \(f(\gamma_1(s)) : S \to (X \times S)\) is evaluated in the new state \(s'\). This is necessarily a sequential process.

In the case of the realizer monad the function \(f^{\ast K}(\alpha, r) = (f_1^*(\alpha), r \otimes^S f_2^*(\alpha))\) first computes a new dynamic object \(f_1^*(\alpha)\), then forces it to satisfy some property using the realizer \(f_2^*(\alpha)\) merged with the realizer \(r\), that is supposed to satisfy some other (possibly different) property. The reason is that the search procedure represented by \(f_2^*(\alpha)\) might change the state reached by some previous attempt by \(r\) to force \(\alpha\) into its own goal, destroying the work by \(r\). Hence both \(f_2^*(\alpha)\) and \(r\) have to be kept while evaluating the realizer obtained by their merge, and cannot be sequentialized.

### 6 Interactive realizers

This section introduces the central concepts of **interactive realizer** and of **interactive forcing**, which are the main contribution of our work. Realizers have been introduced by Kleene as an interpretation of Brouwer’s and Heyting’s concept of construction. In the case of constructive theories a realizer is a direct computation, possibly depending on some parameters. With a non constructive theory like PRA + EM\(_1\) the saving of such an idea involves the shift from recursiveness to recursiveness in the limit. In this perspective a realizer is not an algorithm (a recursive function), rather it is the recursive generator of a search procedure that, along a series of attempts and failures, eventually attains its goal.

**Definition 6.1 (Interactive Realizers)** An interactive realizer is a map \(r \in S(S) = S \to S\), such that:
1. \( r \) is strongly convergent;

2. \( r \) is compatible with its arguments that is: \( r(s) \uparrow s \) for all \( s \in S \);

3. \( r(s) \cap s = \bot \) for all \( s \in S \).

A state \( s \in S \) is a prefix point of \( r : S \to S \) if \( r(s) \subseteq s \); by \( \text{Prefix}(r) \) we denote the set of prefix points of \( r \).

Remark 6.2 By clause (1) above the realizers are individuals over \( S \). Note that identity over \( S \) is not convergent, and so it is not a realizer. Compatibility condition (2) is essential, together with convergence, for the existence of pre-fixed points: see Proposition 6.3 below. The function \( \lambda_\bot \bot \) is a (trivial) realizer and, because of clause (3), the only one among constant individuals.

By clause (3), if \( r \) is a realizer we have that \( s \in \text{Prefix}(r) \) if and only if \( r(s) = \bot \), because \( r(s) \subseteq s \) implies \( r(s) \cap s = \bot \). Namely \( \text{Prefix}(r) \) is the set of “roots” of \( r \). This clause is just intended to simplify the treatment of realizers, in the sense that if \( r \) is a realizer, then \( r(s) \) just adds “new” atoms to \( s \); hence if \( r(s) \subseteq s \) this means that there is actually nothing to add.

Proposition 6.3 (Cofinality of Realizers Prefix Points) If \( r : S \to S \) is a realizer, then for all \( s \in S \) there is \( s' \in \text{Prefix}(r) \) such that \( s \subseteq s' \), namely \( \text{Prefix}(r) \) is cofinal in \( S \) (in particular it is non empty).

Proof. Given \( s \in S \) define the mapping \( \sigma : \mathbb{N} \to S \) by \( \sigma(0) := s \) and \( \sigma(i+1) := \sigma(i) \cup r(\sigma(i)) \), which exists because of the compatibility of \( r \) with its argument. By construction \( \sigma \) is a w.i. sequence, hence by the convergence of \( r, r \circ \sigma \) has a limit \( \sigma(i_0) \) for some \( i_0 \), that is \( r(\sigma(i_0)) = r(\sigma(i_0) + 1) \). Then

\[
\sigma(i_0 + 1) = \sigma(i_0) \cup r(\sigma(i_0)) = \sigma(i_0) \cup r(\sigma(i_0) + 1),
\]

which implies \( r(\sigma(i_0) + 1) \subseteq \sigma(i_0 + 1) \); clearly \( s = \sigma(0) \subseteq \sigma(i_0 + 1) \in \text{Prefix}(r) \).

Remark 6.4 The proof of Proposition 6.3 describes a computation that, given an arbitrary \( s_0 \), produces the w.i. sequence \( \sigma(0) = s_0, \sigma(1) = r(s_0) \cup s_0, \sigma(2) = r(r(s_0) \cup s_0) \cup r(s_0) \cup s_0, \ldots \) until a prefix point \( \sigma(n) \) is found. Each time the sequence strictly increases, it is because \( r(\sigma(i)) \neq \bot \), that intuitively means that the realizer \( r \) has something to add to \( \sigma(i) \) to reach its own goal, abstractly constituted by the set of prefix points of \( r \). This search procedure, which is recursive in \( r \), is monotonic as the knowledge grows, but this happens because only positive information is stored in the state. As a matter of fact the growth of the sequence generated by \( r \) might redefine the values of some \( \chi_p \) and \( \varphi_p \) occurring in a formula \( A \), which is the actual goal of \( r \) when it is a realizer of \( A \): this is an implicit backtracking (more precisely 1-backtracking; see [7]), in the sense that we are retracting previous definitions of these symbols, and in particular of the Skolem functions \( \varphi_p \), until \( A \) becomes true.

After some \( i \) has been found such that \( r(\sigma(i)) = \bot \), the whole construction stops. However nothing prevents that, later, new atoms might be added to \( \sigma(i) \), producing some \( s' \sqsubseteq \sigma(i) \) not in \( \text{Prefix}(r) \). Now the cofinality of \( \text{Prefix}(r) \) in \( S \) implies that we can resume the search at \( s' \) and that it will eventually succeed in finding some other \( s'' \sqsubseteq s' \) which is in \( \text{Prefix}(r) \).

The reaching of a goal is represented by finding a prefix point of the relative realizer. The next proposition says that the prefix points of a merge are exactly the prefix points common to both the merged realizers.

Proposition 6.5 Suppose that \( \otimes \) is a merge: then for any pair of realizers \( r, r' \), \( r \otimes_S r' \) is a realizer. Moreover:

\[
\text{Prefix}(r \otimes_S r') = \text{Prefix}(r) \cap \text{Prefix}(r').
\]
Proof. In view of the definition $\otimes^S = \mathcal{S}(\otimes) \circ \psi_{S}$, we know that $r \otimes^S r'$ is strongly convergent by Corollary 4.7 and Remark 4.10, since $r$ and $r'$ are such.

For any $s \in S$ we know that $s \uparrow r(s)$ and $s \uparrow r'(s)$; now $(r \otimes^S r')(s) = r(s) \otimes r'(s)$ for all $s \in S$ and we conclude that $s \uparrow (r(s) \otimes r'(s))$ by (1) of Lemma 5.2.

By (1) of Definition 6.1, $r(s) \cap s = \bot = r'(s) \cap s$; by (2) of Lemma 5.2 this implies:

$$(r \otimes^S r')(s) \cap s = (r(s) \otimes r'(s)) \cap s = \bot.$$ 

This concludes the proof that $r \otimes^S r'$ is a realizer.

This last fact implies that $\text{Prefix}(r \otimes^S r') = \{ s \in S \mid r(s) \otimes r'(s) = \bot \}$: by (2) of Definition 5.1 we know that $r(s) \otimes r'(s) = \bot$ implies both $r(s) = \bot$ and $r'(s) = \bot$, namely that $	ext{Prefix}(r \otimes^S r') \subseteq \text{Prefix}(r) \cap \text{Prefix}(r')$.

Viceversa, if $s \in \text{Prefix}(r) \cap \text{Prefix}(r')$ then $r(s) = \bot = r'(s)$, so that, by $\bot \cap \bot = \bot$, we have that $r(s) \otimes r'(s) = \bot$, that is $s \in \text{Prefix}(r \otimes^S r')$.

We now relate formally interactive realizers to formulas of $\mathcal{L}_1$. First we define an abstract relation between realizers and families of sets indexed over $S$ which we call interactive forcing.

**Definition 6.6 (Interactive Forcing)** Let $r$ be a realizer, $\alpha \in SX$ and $Y = \{ Y_s \mid s \in S \}$ a family of subsets of $X$ indexed over $S$. Then $r$ interactively forces $\alpha$ into $Y$, written $r \models^\mathcal{L}_1 \alpha : Y$, if for all $s \in \text{Prefix}(r)$ it is the case that $\alpha(s) \in Y_s$.

Let us now consider the formulas. In the standard model the semantics of a formula $A$ with (free) variables included into $\vec{x} = x_1, \ldots, x_k$ is a $k$-ary relation over $\mathbb{N}$, which is the extension of the formula. In our model the extension of $A$ is the $S$-indexed family of sets $\text{ext}(A) := \{ \text{ext}(A)_s \mid s \in S \}$ where:

$$\text{ext}(A)_s := \{ \vec{m} \mid \text{FV}(A) \subseteq \vec{x} \ & \ |\vec{m}| = |\vec{x}| \ & \ [A]_{\lambda_x/m/\vec{x}}(s) = \text{true} \}.$$ 

Here $\vec{m} = m_1, \ldots, m_k$ is a $k$-ple of natural numbers, $|\vec{m}| = |\vec{x}| = k$, and $[\lambda_x/m/\vec{x}] = [\lambda, m_1/x_1, \ldots, \lambda, m_k/x_k]$ is the environment associating $\lambda, m_i$ to $x_i$ for each $i$. We now define the forcing of $A$ in terms of the extension of $A$.

**Definition 6.7 (Interactive Forcing of a Formula)** Let $r$ be a realizer, $A \in \mathcal{L}_1$ with $\text{FV}(A) \subseteq \vec{x} = x_1, \ldots, x_k$, and $\vec{\alpha} = \alpha_1, \ldots, \alpha_k \in SN$. Then we say that $r$ interactively forces $\vec{\alpha}$ into $A$, written $r \models^\mathcal{L}_1 \vec{\alpha} : A(\vec{x})$, if $r \models (\alpha_1, \ldots, \alpha_k) : \text{ext}(A)$.

To each formula $A \in \mathcal{L}_1$ Definition 6.7 associates the relation $\{(r, \vec{\alpha}) \mid r \models^\mathcal{L}_1 \vec{\alpha} : A(\vec{x}) \} \subseteq \mathcal{S}(S) \times \mathcal{S}(\mathbb{N}^k) \simeq \mathcal{R}(\mathbb{N}^k)$, where $k$ is the length of $\vec{\alpha}$ and $\vec{x}$, making apparent the connection between forcing and the realizer monad $\mathcal{R}$.

In view of Proposition 6.5 and of Remark 6.4, the intuitive idea of the forcing relation $r \models^\mathcal{L}_1 \vec{\alpha} : A(\vec{x})$ is that, whenever the variables $\vec{x}$ including all the free variables of $A$ are interpreted by the individuals $\vec{\alpha}$, the sequence generated by $r$ out of an arbitrary $s_0$ will eventually reach (in a finite number of steps) some state $s \in \text{Prefix}(r)$ making true that $\vec{\alpha}(s) \in \text{ext}(A)_s$. This is however a subtly complex task: the action of $r$ is to direct $\vec{\alpha}$ into $\text{ext}(A)$ by extending the given state; but we must keep in mind that such a search aiming at the target $\text{ext}(A)_s$ for some $s$, moves the target itself as a side effect. Note also that:

$$(\alpha_1, \ldots, \alpha_k)(s) = (\alpha_1(s), \ldots, \alpha_k(s)) \in \text{ext}(A)_s \iff [A]_{\lambda_x/m/\vec{x}}^S(s) = [A]_{\lambda_x/m/\vec{x}}^S(\vec{x}) = \text{true}.$$ 

By the fact that we do not ask that the free variables of $A$ are exactly $\vec{x}$, but only included among them, the sets $\text{ext}(A)_s$ contain tuples of different length (thought there is a minimum length which is the
cardinality of \( \text{FV}(A) \), which implies that if \( r \models \vec{\alpha} : A(\vec{x}) \) then \( r \models \vec{\alpha}, \vec{\beta} : A(\vec{x}, \vec{y}) \) for all vectors \( \vec{y} \) and \( \vec{\beta} \) such that \( |\vec{y}| = |\vec{\beta}| \).

Toward the proof of the claim that any theorem of \( \text{PRA} + \text{EM}_1 \) is interactively realizable, we begin with the logical and arithmetic axioms. Together we consider also the \( \phi \)-axioms, since in all these cases the realizer turns out to be the trivial one.

**Lemma 6.8 (Logical, Arithmetic or \( \phi \)-Axioms)** If \( A \) is either a non logical axiom of \( \text{PRA} \), or an axiom of IPC, or an instance of the \( \phi \)-axiom, then \( \lambda_\perp \models \lambda_\perp \models A \).

**Proof.** It follows by Proposition 3.17, since \( \lambda_\perp \) is a realizer and \( \text{Prefix}(\lambda_\perp) = S \).

Now we come to the study of the \( \chi \)-axioms. For any \( k + 1 \)-ary primitive recursive predicate \( P \) (we abuse notation below, writing ambiguously \( P \) for the symbol and for its standard interpretation) let us define \( r_P : \mathbb{N}^{k+1} \times S \rightarrow S \) as follows:

\[
r_P(\vec{m}, n, s) = \begin{cases} 
\{(P, \vec{m}, n)\} & \text{if } P(\vec{m}, n) \text{ and } \forall n'. (P, \vec{m}, n') \not\in s \\
\bot & \text{else.}
\end{cases}
\]

**Lemma 6.9** For all \( \vec{m}, n \in \mathbb{N}, \lambda s \in S \), \( r_P(\vec{m}, n, s) \) is a realizer.

**Proof.** That \( r_P(\vec{m}, n, s) \cap s = \bot \) for all \( s \in S \) is immediate by definition. It remains to prove that \( \lambda s \in S \). \( r_P(\vec{m}, n, s) \) is strongly convergent and consistent with its arguments.

Let \( \sigma \) be any w.i. sequence. If \( \neg P(\vec{m}, n) \) then \( r_P(\vec{m}, n, \sigma(i)) = \bot \) for all \( i \). Suppose instead that \( P(\vec{m}, n) \) is true. If \( \langle P, \vec{m}, n' \rangle \not\in \sigma(i) \) for all \( n' \) and \( i \), then \( r_P(\vec{m}, n, \sigma(i)) = \{(P, \vec{m}, n)\} \) for all \( i \); otherwise there exist \( i \) and \( n' \) such that for all \( j \geq i \), \( \langle P, \vec{m}, n' \rangle \in \sigma(j) \), as \( \sigma \) is weakly increasing. Then \( r_P(\vec{m}, n, \sigma(j)) = \bot \) for all \( j \geq i \).

If \( r_P(\vec{m}, n, s) = \{(P, \vec{m}, n)\} \) then \( P(\vec{m}, n) \) is true so that \( \{(P, \vec{m}, n)\} \in S \). Moreover \( \langle P, \vec{m}, n' \rangle \not\in s \) for all \( n' \in \mathbb{N} \), hence \( \{(P, \vec{m}, n)\} \uparrow s \). If instead \( r_P(\vec{m}, n, s) = \bot \) then the thesis holds trivially since \( \bot \uparrow s \).

Recall that \( r_P^S = r_P \circ \psi_{n+1} \), so that for any \( \vec{\alpha}, \vec{\beta} \in \mathbb{SN} \) we have:

\[
r_P^S(\vec{\alpha}, \vec{\beta}) = \lambda s \in S. r_P(\vec{\alpha}(s), \vec{\beta}(s), s).
\]

The next lemma states that, under the condition that \( \vec{\alpha}, \vec{\beta} \) are individuals (that is strongly convergent), \( r_P^S(\vec{\alpha}, \vec{\beta}) \) is a realizer of the \( \chi \)-axiom instance relative to \( P \).

**Lemma 6.10 (\( \chi \)-Axiom)** If \( P \) is a \( k + 1 \)-ary primitive recursive predicate, and \( \vec{\alpha}, \vec{\beta} \in \mathbb{SN} \) are strongly convergent then \( r_P^S(\vec{\alpha}, \vec{\beta}) \) is a realizer, and it is such that:

\[
r_P^S(\vec{\alpha}, \vec{\beta}) \models \vec{\alpha}, \vec{\beta} : P(\vec{x}, \vec{y}) \rightarrow \chi_P(\vec{x}).
\]

**Proof.** By definition, \( r_P^S(\vec{\alpha}, \vec{\beta}, s) = r_P(\vec{\alpha}(s), \vec{\beta}(s), s) \), so that it is consistent with \( s \) and the intersection \( r_P^S(\vec{\alpha}, \vec{\beta}, s) \cap s = \bot \) because \( r_P(\vec{m}, n, s) \cap s = \bot \) for all \( \vec{m}, n \). Since \( r_P^s \) is global by Corollary 4.4, we know that \( r_P^s \) is \( k + 1 \)-global by Lemma 4.8. Now

\[
r_P^S(\lambda_\perp \vec{m}, \lambda_\perp \vec{n}, s) = r_P(\vec{m}, n, s),
\]
and the latter is an individual by Lemma 6.9. It follows that \( r^S_0(\alpha, \beta) \) is an individual if \( \alpha \) and \( \beta \) are such, by Corollary 4.9. We conclude that \( r^S_0(\alpha, \beta) \) is a realizer.

If \( s \in \text{Prefix}(r^S_0(\alpha, \beta)) \) then \( r_p(\alpha(s), \beta(s)) = \perp \). It follows that either \( \neg P(\alpha(s), \beta(s)) \) or \((P, \alpha(s), n) \in s\) for some \( n \in \mathbb{N} \) (not necessarily equal to \( \beta(s) \)): this implies that \( [\chi P]^{S}_{[\alpha, \beta/z, y]}(s) = [\chi P](\alpha(s), s) = \perp \).

In both cases we have \( [P(x, y) \rightarrow \chi P(x, y)]^{S}_{[\alpha, \beta/z, y]}(s) = \perp \), that is \( \alpha(s), \beta(s) \in \text{ext}(P(x, y) \rightarrow \chi P(x))_s \).

\[ \square \]

**Remark 6.11** By reading Lemma 6.10 together with Remark 6.4 we see one reason for naming “interactive realizer” the map \( r = r^S_0(\alpha, \beta) \). In fact we have seen that each time the sequence generated by \( r \) strictly increases it is because \( r(\sigma(i)) \neq \perp \), and this happens whenever \( P(\alpha(\sigma(i)), \beta(\sigma(i))) \) holds but \( (P, \alpha(\sigma(i)), n) \not\in \sigma(i) \) for any \( n \in \mathbb{N} \). In such a case the next state in the sequence is \( \sigma(i + 1) = r(\sigma(i)) \cup \sigma(i) = \{ (P, \alpha(\sigma(i)), \beta(\sigma(i))) \} \cup \sigma(i) \) so that the newly found tuple \( (P, \alpha(\sigma(i)), \beta(\sigma(i))) \) is added to \( \sigma(i) \). In particular if a prefix point of \( r \) is reached, that is \( r(s) = \perp \), no more information is needed to make the realized formula true w.r.t. such a state.

If we further consider the implication \( P(x, y) \rightarrow P(x, \varphi P(x)) \), which follows in PRA + EM by the (\( \varphi \))-axiom and the (\( \chi \))-axiom, we see that whenever

\[
[P(x, y) \rightarrow P(x, \varphi P(x))]^{S}_{[\alpha, \beta/z, y]}(s) = \perp,
\]

the tuple \((P, \alpha(s), \beta(s))\) is the witness of the fact that the implication fails at \( s = \sigma(i) \). This is used at the next step in the attempt to make it true at \( s' = r(s) \), i.e., by means of the fact that \([\varphi P(x)]^{S}_{[\alpha, \beta/z, y]}(s') = \beta(s)\). By this we see how the counterexample to the implication at a previous stage is used to redefine the value of the Skolem function \( \varphi P \) in the point \( \alpha(s') \), which is the result of the interaction between the realizer \( r \) and the “nature”, that is the standard model.

However it is not necessarily the case that \( \alpha(s') = \alpha(s) \), which implies that the value of \( [P(x, y) \rightarrow P(x, \varphi P(x))]^{S}_{[\alpha, \beta/z, y]}(s') \) could still be false. It is here that the hypothesis that \( \alpha \) is convergent is crucial, since \( s \subseteq s' \) and in general the value of \( \alpha \circ \sigma \) is eventually constant.

According to our interpretation, logical rules are realized by the merging of the realizers of the premises. Let us first consider the case of modus ponens.

**Lemma 6.12 (Modus Ponens Rule)** If \( r \vdash \alpha : A \) and \( r' \vdash \alpha : A \rightarrow B \) then \( r \otimes^S r' \vdash \alpha : B \).

**Proof.** Let \( \alpha = \alpha_1, \ldots, \alpha_k \): then \( \text{ext}(A \rightarrow B), \alpha \cap \text{ext}(B) \) are subsets of the universe \( \mathbb{N}^k \), so that in particular we can take the complement \( \text{ext}(A) = \mathbb{N}^k \setminus \text{ext}(A) \). Then let us observe that for all \( s \in \mathbb{S} \):

\[
\text{ext}(A \rightarrow B) = \overline{\text{ext}(A)} \cup \text{ext}(B).
\]

By Proposition 6.5 we know that \( r \otimes^S r' \) is a realizer such that \( \text{Prefix}(r \otimes^S r') = \text{Prefix}(r) \cap \text{Prefix}(r') \). Therefore, by the hypotheses, if \( s \in \text{Prefix}(r \otimes^S r') \) then

\[
\overline{\alpha}(s) \in \text{ext}(A) \cap \text{ext}(A \rightarrow B) = \text{ext}(A) \cap \overline{\text{ext}(A)} \cup \text{ext}(B) = \text{ext}(A) \cap \text{ext}(B),
\]

hence \( \alpha(s) \in \text{ext}(B) \), as desired.

\[ \square \]

**Remark 6.13** In Remark 6.11 we have stressed that even in the case of the (\( \chi \))-axiom the realizer might reach its prefix point in several steps, and after several tests against the standard model of arithmetic: this is a first form of interaction. The case of modus ponens rule MP, as well as the more complex one of IND treated below, illustrates a second form of interaction between two or more realizers. While
searching a prefix point of \( r \otimes S r' \) the given realizers \( r \) and \( r' \) do not necessarily move to the same states, not even to compatible ones. The realizer \( r \otimes S r' \) let \( r \) and \( r' \) to dialogue via the state by merging of the respective sequences they generate. This process depends on the choice of the merge: with \( \otimes_0 \), for example, it is a rigid interleaving of the searches generated by \( r \) and \( r' \), giving precedence to \( r \), while with \( \otimes_1 \) and \( \otimes_2 \) the resulting sequence is generated by parallel process.

We observe that the merging of realizers is the meaning of any inference rule with more than one premise.

Recall the convention that the writing \( A(x) \) means that \( x \) might occur free in \( A \), and \( A(t) \) is informal for the substitution \( A[t/x] \) of \( t \) for \( x \) in \( A \).

**Lemma 6.14 (Substitution Rule)** If \( r \models \vec{a}, \beta : A(\vec{x}, y) \) for all convergent \( \vec{a}, \beta \), then for any \( t \in \mathcal{L}_1 \) such that \( \text{FV}(t) \subseteq \vec{x} \), \( r \models \vec{a}, \beta : A(\vec{x}, t) \).

**Proof.** By the hypothesis and the fact that \( [t]^{S}_{\vec{a}/\vec{x}} \) is convergent by Theorem 4.14, we have that \( r \models \vec{a}, [t]^{S}_{\vec{a}/\vec{x}} = A(\vec{x}, y) \), where we note that the environment \( [\vec{a}/\vec{x}] \) is not defined over \( y \), which however does not occur in \( t \). By Lemma 3.16

\[
[A(\vec{x}, y)]^{S}_{[\vec{a}/\vec{x}, t]} = [A(\vec{x}, t)]^{S}_{[\vec{a}/\vec{x}]},
\]

so that \( r \models \vec{a} : A(\vec{x}, t) \) and, since \( y \notin \text{FV}(A(\vec{x}, t)) \), also \( r \models \vec{a}, \beta : A(\vec{x}, t) \).

**Lemma 6.15 (Induction Rule)** Suppose that for all convergent \( \vec{a} \) and \( \beta \):

\[
r(\vec{a}) \models \vec{a} : A(x, 0) \quad \text{and} \quad r'(\vec{a}, \beta) \models \vec{a}, \beta : A(x, y) \rightarrow A(\vec{x}, \text{succ}(y)).
\]

For all \( \vec{a} \) let \( f(\vec{a}) : \mathbb{N} \rightarrow S(S) \) be defined by (primitive) recursion: \( f(\vec{a}, 0) = \lambda_\cdot \perp \) and \( f(\vec{a}, n+1) = f(\vec{a}, n) \otimes S r'(\vec{a}, \lambda_\cdot, n) \). Then for all convergent \( \vec{a} \) and \( \beta \), \( f(\vec{a})^*(\beta) \) is a realizer and:

\[
r \otimes S (f(\vec{a})^*(\beta)) \models \vec{a}, \beta : A(x, y).
\]

**Proof.** To simplify the notation, we fix the vector \( \vec{a} \) and write just \( r \) for \( r(\vec{a}), r'(\vec{a}, \beta) \) for \( r'(\vec{a}, \beta), f(n) \) for \( f(\vec{a}, n) \) and hence \( f^*(\beta) \) for \( f(\vec{a})^*(\beta) \).

First we have to check that \( f^*(\beta) \) is a realizer. Note that for any \( n \in \mathbb{N} \) we have \( f^*(\lambda_\cdot, n) = r'(\lambda_\cdot, 0) \otimes S \cdots \otimes S r'(\lambda_\cdot, n-1) \) (or just \( \lambda_\cdot \perp \) when \( n = 0 \)), which is a realizer by Proposition 6.5. The function \( f^*(\beta) \) is global (or \( k \)-global to take the \( \vec{a} \) into account) by Corollary 4.4 and, as we have just seen, it sends constant individuals into realizers which are individuals of \( S \); hence \( f^*(\beta) \) is an individual for any individual \( \beta \) by (2) of Theorem 4.6. The remaining conditions (2) and (3) of Definition 6.1 are immediately seen to hold by observing that for all \( s \in S \), \( f^*(\beta, s) = r'(\lambda_\cdot, 0, s) \otimes \cdots \otimes r'(\lambda_\cdot, \beta(s) - 1, s) \).

In order to prove the thesis, we establish by induction over \( n \) that:

\[
\forall n \in \mathbb{N}. \; r \otimes S f^*(\lambda_\cdot, n) \models \vec{a}, \lambda_\cdot, n : A(x, y). \tag{8}
\]

For the base case we have \( r \otimes S f^*(\lambda_\cdot, 0) = r \otimes S \lambda_\cdot \perp = r \), and we know that \( r \models \vec{a} : A(x, 0) \), which implies \( r \models \vec{a}, \lambda_\cdot, 0 : A(x, 0) \) vacuously as \( y \notin \text{FV}(A) \).

For the step case we have \( r \otimes S f^*(\lambda_\cdot, n+1) = r \otimes S f^*(\lambda_\cdot, n) \otimes S r'(\lambda_\cdot, n) \), but:

\[
r'(\lambda_\cdot, n) \models \vec{a}, \lambda_\cdot, n : A(x, y) \rightarrow A(\vec{x}, \text{succ}(y)) \quad \text{by the hypothesis of the lemma, and}
\]

\[
r \otimes S f^*(\lambda_\cdot, n) \models \vec{a}, \lambda_\cdot, n : A(x, y) \quad \text{by induction hypothesis.}
\]
We then obtain that \( r \otimes^S f^*(\lambda_{-} n + 1) \vdash \vec{a}, \lambda_{-} n : A(x, \text{suc}(y)) \), by Lemma 6.12. By the Substitution Lemma 3.16. \([A(x, \text{suc}(y))]|^S_{[\vec{a}, \lambda_{-} n / \vec{x}, y]} = [A(x, y)]|^S_{[\vec{a}, \lambda_{-} n + 1 / \vec{x}, y]}\), and therefore we conclude that \( r \otimes^S f^*(\lambda_{-} n + 1) \vdash \vec{a}, \lambda_{-} n + 1 : A(x, y) \).

Now for any \( \beta \in \mathbb{SN} \) and \( s \in S \):

\[
(r \otimes^S f^*(\beta))(s) = r(s) \otimes f^*(\lambda_{-} \beta(s), s),
\]

because \( f^* \) is global, and \( r \otimes^S f^*(\lambda_{-} \beta(s)) \vdash \vec{a}, \lambda_{-} \beta(s) : A(x, y) \) by (8) above since \( \beta(s) \in \mathbb{N} \). It follows that if \( s \in \text{Prefix}(r \otimes^S f^*(\beta)) \) then

\[
(r \otimes^S f^*(\beta))(s) = \bot = (r \otimes^S f^*(\lambda_{-} \beta(s)))(s),
\]

so that \( s \in \text{Prefix}(r \otimes^S f^*(\lambda_{-} \beta(s))) \). This implies that

\[
[A(x, y)]|^S_{[\vec{a}, \lambda_{-} \beta(s) / \vec{x}, y]}(s) = [A(x, y)]|^S_{[\vec{a}, \lambda_{-} \beta(s) / \vec{x}, y]}(s) = \text{true},
\]

by Lemma 4.12.

\[\blacksquare\]

**Remark 6.16** The point of the the proof of Lemma 6.15 is the use of Density Theorem 4.6. In fact the interpretation of induction via primitive recursion implies that we are proving some statement about numbers in \( \mathbb{N} \) and that we are able to compute with them, while the realizability interpretation deals with individuals in \( \mathbb{SN} \). The import of density is that, given that the interpretation of formulas is a global function of the individuals interpreting their variables, everything lifts uniformly from \( \mathbb{N} \) to \( \mathbb{SN} \).

**Theorem 6.17 (Interactive Realizability Theorem)** Suppose that \( \text{PRA} + \text{EM}_1 \vdash A \), for some \( A \in \mathcal{L}_1 \) with \( \text{FV}(A) \subseteq \vec{x} = x_1, \ldots, x_k \). Then for all \( \vec{a} = \alpha_1, \ldots, \alpha_k \) of individuals in \( \mathbb{SN} \) there exists a realizer \( r(\vec{a}) \) which is recursive in \( \vec{a} \), such that \( r(\vec{a}) \vdash A(x) \). Moreover the form of \( r(\vec{a}) \) depends on the proof of \( A \) in \( \text{PRA} + \text{EM}_1 \).

**Proof.** The existence of \( r(\vec{a}) \) follows by the lemmas 6.8, 6.10, 6.12, 6.14 and 6.15, and by the remark that (possibly after renaming) the length \( k \) of \( \vec{x} \) and \( \vec{a} \) can be taken to be large enough to include all variables occurring in the proof. That \( r \) is a recursive functional of \( \vec{a} \) follows by the fact that all realizers constructed in the lemmas above are \( \lambda \)-definable. Finally that \( r(\vec{a}) \) (and hence \( r \) itself) actually reflects the structure of the proof of \( A \) is clear by construction.

\[\blacksquare\]

By choosing as the convergent \( \vec{a} \) the vector of constant individuals \( \lambda_{-} \) one can use the realizer associated to the proof of a \( \text{PRA} + \text{EM}_1 \) theorem to compute the witness \( \varphi(\vec{t}) \) of \( A(\varphi(\vec{t})) \) in the standard input \( \vec{m} \); the general case of convergent \( \vec{a} \) is however needed in the proof of the theorem and for the compositionality of its construction.

The computational content of the proof, which we identify with the realizer, is however trivial in case no instance of the \( (\chi) \)-axiom occurs in it.

**Corollary 6.18** If \( \text{PRA} + (\varphi) \vdash A \) for \( A \in \mathcal{L}_1 \), then the realizer \( r(\vec{a}) \) of Theorem 6.17 is just \( \lambda_{-} \perp \). Hence \([A]_{\xi}^s(s) = \text{true} \) for all environment \( \xi \) and state \( s \).

**Proof.** By inspection of the lemmas used in the proof of the theorem, the realizer \( r(\vec{a}) \) is a composition of the realizers used in the axioms by the operator \( \otimes^S \): this is immediate in all cases but for induction rule (Lemma 6.15), which is however easily checked by induction over \( \mathbb{N} \), and then lifted to \( \mathbb{SN} \) by the very same argument we used in the proof of the lemma. But in case the proof of \( A \) does not use any instance of the \( (\chi) \)-axiom, they are all \( \lambda_{-} \perp \), which is the unit of \( \otimes^S \).

Now the second part of the thesis follows by Theorem 6.17, and by remembering that \( \text{Prefix}(\lambda_{-} \perp) = S \).

\[\blacksquare\]
7 Related works

The spectrum of research ideas that have been influential on our work and of those that, even a posteriori, reveal to be connected to our results is too wide to be exhaustively treated. Therefore we limit ourself to a sketch of the approaches we feel closer to ours, either because we are building over them, or because we want to underline similarities and differences.

Coquand’s game semantics of classical arithmetic. A primary source of the present research is Coquand’s semantics of evidence for classical arithmetic [11]. Non constructive principles like excluded middle are treated there by means of backtracking and learning, and rely on the fact that in each play only a finite amount of information about them is actually needed. The concept of 1-backtracking games appearing in [7], which can be seen as a restricted form of games in [11] but with plays of possibly infinite length, is closely related to the present work. Given a game $G$, the 1-backtracking game $\textsf{bck}(G)$ allows the player to come back to some previous move in a play, by undoing and forgetting all the intermediate moves made by either players. In this modified game the player does not lose if a loosing position is reached, rather the player looses only if forced to backtrack to the same position infinitely many times. This backtracking procedure, in which a player’s strategy over $\textsf{bck}(G)$ consists, can be interpreted as a learning procedure, in the sense that the player learns from her own trials and errors, and is allowed to follow a different path in a tree of plays using the experience made so far. A winning strategy over $\textsf{bck}(G)$ is effective, and the truth of any $\textsf{PRA} + \textsf{EM}_1$ theorem can be learned in this way: this is not true, however, if non constructive principles are admitted of logical complexity which is higher than $\textsf{EM}_1$.

We might interpret our construction as an implementation of the same idea. This has been explained in the text, and especially in the remarks of Section 6. With respect to [7] we provide the needed machinery, together with a language, to denote learning strategies which is tailored for extracting them out of proofs.

Gold’s theory of learning in the limit. The concept of dynamic individuals comes from Gold’s theory of learning in the limit, exposed in [17, 18]. A $k$-ary numerical function $f$ is computable in the limit if there exists a recursive (total) $k+1$-function $g$ such that for all $\vec{m} \in \mathbb{N}^k$ the sequence $g(\vec{m},0), g(\vec{m},1), \ldots$ is eventually constant and equal to $f(\vec{m})$, i.e. $f(\vec{m}) = \lim_{n \in \mathbb{N}} g(\vec{m},n)$. Then $g$ is a guessing function for $f$, that makes the values of $f$ learnable. This is equivalent to say that $f(\vec{m})$ is an individual in our sense. Viceversa the functions $\varphi_P(\vec{m})$ and the predicates $\chi_P(\vec{m})$ are computable in the limit (with the $(k+1)$-th argument in $\mathbb{S}$, but note that states are concrete and finite objects, hence encodable into $\mathbb{N}$) by their guessing functions $[\varphi_P]$ and $[\chi_P]$ (see Definition 3.2 above), with the minor difference that we take the limit w.r.t. w.i. sequences over $\mathbb{S}$ (but note that states are concrete and finite objects and that $\mathbb{S}$ is a decidable set, hence encodable into a decidable subset of $\mathbb{N}$). More importantly, the ordering of $\mathbb{S}$ is state extension rather than the arbitrary ordering of (code) numbers. Moreover infinitely many incompatible w.i. sequences exist in $\mathbb{S}$, hence the limit of an individual depends on the choice of the sequence, in general.

The $\varepsilon$-substitution method. This is a method to eliminate quantifiers, by replacing them with $\varepsilon$-terms, which has been introduced in [21] (but see the exposition and improvement of the method in [26]). It consists in the introduction of the new term $\varepsilon x A$ for each formula $A(x)$, whose meaning is: “the least $x$ such that $A(x)$”, that makes quantifiers definable by adding the new axioms (called critical formulas): $A(t) \rightarrow A(\varepsilon x A)$. Any classical proof of arithmetic can be transformed into a proof without quantifiers and using as axioms a finite set of critical formulas instead; now Hilbert suggested and Ackermann proved that one can effectively find a solving substitution $S$ of $\varepsilon$-terms by numerals validating all the formulas in the proof (for which it suffices to validate the critical formulas). This is achieved by arranging a sequence $S_0, S_1, \ldots$ of substitutions such that $S_0$ is the identically 0 substitution, and $S_{i+1}$ is obtained from $S_i$ as follows: choose an axiom $A(t) \rightarrow A(\varepsilon x A)$ in the proof such that $S_i(A(t)) = \text{true}$ and $S_i(A(\varepsilon x A)) = \text{false}$ if any (in the negative case $S = S_i$ and we are done). Then put $S_{i+1}(\varepsilon x A) = \text{the least } n \leq S_i(t) \text{ such that } A(n)$.
This is strikingly similar to the action of the realizers of the \((\chi)\)-axioms, for which we refer to Remark 6.11. But this also reveals a key difference with our construction: in fact to avoid circularities, after redefining the value of \(\varepsilon x A\) one has to reset to 0 the values of all the \(\varepsilon\)-terms of greater rank than \(\varepsilon x A\) (a measure of the nesting of \(\varepsilon\)-terms). This indiscriminate form of backtracking, which is not very different from blind search, is the consequence of the limited use of information from the the proof in the \(\varepsilon\)-substitution method, that contributes only to determine the set of critical formulas that have to be satisfied. On the contrary the compositional nature of our realizability interpretation allows for an essential use of the proof structure, so that the nature and efficiency of the resulting algorithm strictly depends on the the proof itself, often embodying clever computational ideas.

**Friedman A-translation.** Friedman’s famous result in [15] is an extension of Gödel proof of conservativity of \(\text{PA}\) over \(\text{HA}\) for \(\Pi^0_2\)-statements, which is suitable for program extraction from classical proofs. As such, it has been developed into a system to synthesise programs from classically provable \(\Pi^0_2\)-statements, or equivalently \(\Sigma^0_2\)-formulas possibly with parameters: see e.g. [10] and the MINLOG project. The extraction process runs as follows: given a classical proof \(p\) of the arithmetic formula \(A = \exists y P(x, y)\), with \(P\) quantifier free (or equivalently primitive recursive) it is translated into a proof \(p'\) in minimal arithmetic \(\text{MA}\) (i.e. \(\text{HA}\) without the axiom schema \(\bot \rightarrow B\)), of the formula \(A^{\neg\neg}\), which is obtained from \(A\) by double negation of atomic subformulas and interpreting \(\forall, \exists\) by \(\neg \land \neg\) and \(\neg \forall\) respectively.

The translation is possible because, under such interpretation, the excluded middle law is trivially derivable, and in fact it is an instance of the identity rule. Since \(\neg B = B \rightarrow \bot\), we have \(A^{\neg\neg} = \forall y (P(x, y) \rightarrow \bot) \rightarrow \bot\). On the other hand by the absence of the ex-falso quodlibet law from \(\text{MA}\), \(\bot\) can be replaced by an arbitrary formula, so that in particular we have a proof \(p'' = p[A/\bot]\) of the formula \(A[A/\bot] = \forall y (P(x, y) \rightarrow A) \rightarrow A\). Now \(p''\) is a constructive proof, and \(A[A/\bot]\) is constructively equivalent to \(A\): hence we extract a program (a \(\lambda\)-term) from \(p''\) realizing \(\exists y P(x, y)\) which, for any \(x\), actually computes a \(y\) s.t. \(P(x, y)\).

Apparently Friedman’s interpretation bears no relation with our construction. However, if we consider only proofs of \(A\) in \(\text{EM}_1\) and analyse the reduction paths from \(p''\) we see that it computes the witness in a way conceptually similar to ours. For any term \(t\) if \(p''\) includes a subproof \(q = \lambda \xi q'[t/x]\) of \(\forall y (P(t, y) \rightarrow A)\), then \(p''\) corresponds to the assumption that \(\forall y \neg P(t, y)\) holds, and essentially stores the current state of the program. The term \(q\) is what is called a “continuation” in functional programming. In fact, if for some \(n\) a proof \(r\) of \(P(t, n)\) is found, then \(p''\) yields \(q(r)\), that reduces to \(q'[t/x, r/\xi]\), which actually restarts the computation from the point in which the wrong assumption \(\forall y \neg P(t, y)\) was made, and, at the same time, produces a proof of \(\exists y P(t, y)\) and a witness \(n\). According to our construction the same effect is achieved by adding \(P(t, n)\) to the state.

We observe that, beside being semantically more perspicuous, the method proposed in the present paper is even more general: the algorithm obtained via the functional interpretation of the Friedman’s translation is essentially sequential and deterministic, the latter being an accidental and arbitrary feature; on the other hand with the present construction realizers, and in particular the merge operation that on the other hand with the present construction realizers, and in particular the merge operation that

**Realizability of classical logic and theories.** The extension of Kleene’s realizability and of the Curry-Howard correspondence between proofs and programs to classical logic and theories began with Griffin’s discovery, illustrated in [19], that control operators and continuations can be typed by the classical law: \(\neg \neg A \rightarrow A\). Since then this idea has been pursued by several authors: see e.g. [28] dealing however with Friedman’s translation, and [23], which is directly inspired to the Curry-Howard correspondence. A connected development has been by introducing calculi that are to classical proofs what the \(\lambda\)-calculus is for intuitionistic ones: see Parigot’s \(\lambda\mu\)-calculus in [29], and its development into a symmetric calculus of classical proofs called \(\lambda\mu\tilde{\mu}\)-calculus in [14], but also the former “symmetric \(\lambda\)-calculus” in [5], exploiting similar ideas. With respect to all such works we have departed here because of the concept of realizer we have used, which is not based on the idea of continuations nor on that of control operators, but on
searching and learning. However, and a posteriori, we think that similar remarks could be done as in the case of Friedman translation, namely that, on the one hand, we might expect to obtain similar algorithms on particular examples; but on the other we have a better explanation of how and why the interpretation works, even if, to the time, the present results are limited to proofs of low logical complexity, as repeatedly explained in the previous sections, while the above mentioned systems deal with full classical logic and arithmetic.

In [20] a notion of realizability is introduced for a subclassical arithmetic, called Limit Computable Arithmetic (LCM). The theory combines Kleene realizability with Gold learning in the limit, which is achieved by asking a realizer to learn the evidence of the realized formula, instead of computing such an evidence. The essential departure from Kleene’s realizability is of course in the cases of achieved by asking a realizer to learn the evidence of the realized formula, instead of computing such an evidence. The essential departure from Kleene’s realizability is of course in the cases of 

\[ \exists x. A(x) \]: according to Hayashi a realizer of \( A \lor B \) is pair \( g(n) = (g_1(n), g_2(n)) \) such that \( \lim_n g(n) \) exists and if \( \lim_n g_1(n) = 0 \) then \( g_2 \) is a guessing function for \( A \), while it is a guessing function of \( B \) in case \( \lim_n g_1(n) \neq 0 \). Similarly a realizer of \( \exists x. A(x) \) is a function \( g(n) = (g_1(n), g_2(n)) \) that always converges and \( g_2 \) is guessing function for \( A(\lim_n g_1(n)) \).

Hayashi’s concept of realizer is similar to ours under relevant respects: it is constructed along the proof (and by this it is called the proof “animator”), and it is convergent. The fact that LCM uses quantified formulas while \( \text{PRA} + \text{EM}_1 \) is a quantifier free theory is a minor difference, and not a true limitation of our approach: see [4]. Rather the essential difference among the learning realizability of LCM and the model we present here lies in the use of the proof, and hence of the realizer itself. In the case of LCM a realizer is a guessing function, hence a tool for testing guesses which have to be provided by a “user” interacting with the proof; in the absence of the user the only strategy to learn the truth of the conclusion of the proof is exhaustive blind search. On the contrary an interactive realizer in our sense is the basic block of a learning strategy, capable to produce and test hypothesis against the “nature”, namely that part of the standard model that can be learned within a finite number of steps.

In [4] essentially the same model that we have studied here is combined with Kleene’s realizability obtaining an interactive realizability interpretation of \( \text{HA} + \text{EM}_1 \). This extension of the interactive realizability model makes the terming “realizability” even more acceptable for the construction we are proposing.

However with respect to that work, we take here a different research direction: first we isolate and investigate on their own the concepts of individuals, global functions, interactive realizers and merge of realizers. These concepts, that are at the hearth of the construction, are somewhat hidden in the presence of nested quantifiers, that for example enforce the interpretation of a formula to be of different type depending on the formula itself; as a matter of fact in [4] the type of \( [A] \) is \( S \to B \) only in the case if atomic formulas; consequently also the type of realizers gets arbitrarily complex. More, we think that the framing of our model in the theory of strong monads, which is a major contribution of the present paper, allows a more general view of the construction and hints to its possible extensions to cope with non constructive principles of higher complexity.

Monads and the interpretation non-constructive proofs. Monads come from category theory, and strong monads have been introduced into the world of typed \( \lambda \)-calculi and of the foundation of programming languages in [27], where the reader will find the definitions of side-effects and continuations monads, but not of the monads we use here: for that reason we described the monad \( S \) in some detail, even if it is a quite simple example of strong monad.

As it should be clear from the text, we do not make essential use of categorical techniques in our work, and base the exposition on simply typed \( \lambda \)-calculus. This is coherent with Moggi’s original presentation of monads as type constructors of a computational \( \lambda \)-calculus, and with the similar treatment of this topic e.g. in the book [2], chapter 8.

Coquand pointed out in [13] a suggestive connection between the constructive interpretation of classical principles and monads, in which monads play for non constructive features the same role that they have for simulating imperative aspects into functional programming languages. Indeed the monad \( S \) is
here the main tool for defining formulas interpretation in a non ad-hoc fashion, providing a nice characterization of global functions in terms of morphisms of the Kleisli category $\text{Set}_S$. It is while attempting to devise the right definition of the monad $R$ that the monoidal structure of the merge has been realized and its basic properties analyzed.

Beside the theoretical motivations, monads have became a powerful tool to implement non functional aspects into functional programming languages, thanks to Moggi’s original idea and to the work by Wadler (see e.g. [31] and a series of papers thereafter) and many others. It is now a day a common practice to model imperative features into functional languages by means of monads, especially by the community of Haskel programmers. This relation to the programming practice is not by chance: among the basic motivations of the research field we are about here is the desire of methods for using efficiently classical logic principles to develop programs whose adequacy to the specification can be formally certified.

We observe in the main part of the paper that $RX$ is isomorphic to $S \rightarrow (X \times S)$, that is isomorphic as a type (though not as a monad) to the side-effect monad. But the most striking connections with the theme of our work, is of course Moggi’s monad of continuations, which after Griffin’s intuition, is used to type control operators by Gödel-Gentzen doubly negated types. The fact that we do not find the continuation monad at the basis of our construction is easily explained by the limitation to 1-backtracking we have put forward: we think that the use of the full strength of continuations would give the possibility of interpreting unbounded backtracking, but at the price of losing any intuition about the relation among classical proofs and the interactive algorithms we could derive from them.

8 Conclusions

We have interpreted non-constructive proofs of arithmetical statements which can be obtained by using excluded middle over $\Sigma_0^1$ formulas as procedures that learn about their truth by redefining the value of Skolem functions. This process is at the same time an instance of two interpretations of classical logic: learning in the limit and 1-backtracking. The structure of proofs is reflected by their realizers, which are compositional, and parametric in the composition operation we call “merge”. Realizers inhabit a computational type, hence a particular monad; actually monads are the structuring principle on which our construction relies.

As further steps of the presented research, we envisage the recasting of the (existing) extension of interactive realizers to $\text{HA} + \text{EM}_1$ in the framework of monads and, more importantly, the generalisation of interactive realizers to encompass $\text{EM}_n$ axiom schemata.

References


