Non-deterministic untyped λ-calculus

A study about explicit non determinism in higher-order functional calculi

(Preliminary Version)

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Chapter 1

Introduction

1.1 Explicit non determinism

Non determinism is a natural concept in formal language theory and in complexity theory, where it is essential to increase expressive power or efficiency. It is however implicit in many aspects of practical programming. Think of any situation in which more processes interact each other, either directly, by sending and receiving messages, or indirectly, by using shared computing resources. In all this cases, if the whole environment is described both in its static and its dynamic aspects, then one can think of it as a deterministic system. But as soon as we abstract from such dependances or from execution speed and other unpredictable events, the behavior of the system must be considered as nondeterministic.

In this perspective many authors have considered explicit non deterministic control structures, and this is true not only in the case of calculi explicitly designed to model concurrent computation, such as CCS, CSP and Process Algebras, but also within the framework of term rewriting and λ-calculus, that is of formal systems whose inspiration source is not automata theory, but recursion theory.

In late 70's and early 80's, interest in explicit non determinism led to extensive studies of such constructs, with regard to operational semantics, algebraic semantics, and denotational semantics (see e.g. [Plo82, Apt-Plo, Nivat, Boud80, Abr83]; the latter essentially consisted in the introduction and development of powerdomains constructions [Plo76, Plo-Smy].

These researches coincided with a first period in which the adequacy of Scott's elegant theory for abstract modeling of computation was tested against more intentional aspects of computing, such as value passing mechanisms, and sequentilality of an important class of term rewriting systems [Huet-Levy] and λ-calculus [Berry].

The adequacy problem was understood in [Plo77, Mil77] as the full abstraction problem, namely as the problem of finding an abstract interpretation, usually an ordered structure, inducing the same partial ordering on terms as the evaluation mechanism under consideration does.

On the other hand the studies about modeling various kinds of value passing mechanisms were faced with problems whose denotational counterpart actually amounts to different ways of composing functions (see [Henn80]), and it is not difficult to see behind them a concurrency perspective in which, at the present time, they are rediscovered [Mil90]. In this setting non determinism was seen as a possible descriptive tool, while retaining a certain level of abstraction [Henn82].

Similar things could be said about the non deterministic analysis of parallelism e.g. in [Henn-Plo]; a similar philosophy inspires Milner and Hoare work about communicating systems based on interleaving semantics, although it is very hard to model communication abilities using powerdomains.

Once non determinism has been introduced as an explicit control structure in the setting of applicative programming, the problem of understanding the interaction with functional application becomes central, since it actually includes many of those we have just mentioned. As a matter of fact call-by-name and call-by-value reflect different definition of the functional spaces we are working with, as recent studies about laziness [Abr-Ong] and computational λ-calculus [Moggi] have shown. Similarly run-time-choice and call-time-choice (see chapter 2), introduced by [Henn80] in the setting of term rewriting, and rephrased by [Sharma] in that of pure λ-calculus, are likely to be mechanisms capturing properties of different functional spaces.

To the sake of our interest in this thesis, it is worth to mention [Ash-Henn], where a non deterministic extension of Plotkin's PCF where considered and the full abstraction problem for the resulting system was investigated. In this connection, problems arising from the value passing mechanism are even more difficult than in the first order case of term rewriting, as it will be discussed in the first chapter of the thesis. About the drawbacks of the construction of [Ash-Henn] was the fact that operational and denotational equality for the new system was not an extension of those of the deterministic subsystem; researchers, unsatisfied with the solution proposed by the authors, tried to get an abstract modeling of control techniques such as sharing of subcomputations, to preserve the conservativity of the new system with respect to the original one [Ast-Co].

PCF is a typed language, and the higher order systems mentioned above are typed as well: no work was carried out about non deterministic type free λ-calculus, with the only exception of the attempt by [Sharma].

The deveolpment of the theory of concurrency coincides with the eclipse of the studies about explicit non determinism. Reasons are to be found in the major ability of the automata paradigm in modeling communication and synchronization.

Recent studies in this area, to overcome limitations in the expressive power of process algebras, introduce in the calculi typical features of applicative languages, namely functional application and abstraction. We refer to Milner's π-calculus [Mil89], where this happens implicitly through the ability of passing also port names as values, and to Thomsen CHOCS [Thom], were functional abstraction and application are introduced explicitly. This fact has raised new interest in the old problem of combining functionality and non determinism; morever the natural way of looking at this calculi is that of type-free λ-calculus, instead of the typed calculus: this means that the interest is now closer to the neglected direction of [Sharma].

1.2 What this thesis is about

The major topic of investigation in this thesis is the interplay between functionality and non determinism. We are interested in the understanding of this phenomenon in the special case of untyped applicative calculi, where self application is the natural way to encode recursive objects. Suppose you are modeling non determinism by adding an explicit choice operator $\oplus$ to the syntax of the pure λ-calculus. The operational behaviour of this operator is simply decribed by the reduction rules

$$M \oplus N \longrightarrow M \quad \text{and} \quad M \oplus N \longrightarrow N.$$

It is immediate that the convertibility relation induced by these rules, when adding to usual β-reduction rules and closing under context formation, is the trivial one; so that the first question we ask is:

- Under what criteria should we consider two terms of the extended language equivalent?

May be we could consider our terms as standing for the set of their normal forms, where we have to use plural because of the lack of the Church-Rosser property. We know however that this is not a good criterion even for the classical λ-calculus, where (symmetrically) the theory identifying all terms without normal form is inconsistent. The classical way out is to identify all "unsolvable" terms and to distinguish those terms which, in at least one context, behave differently, getting an unsolvable and a solvable term respectively. To make use of the concepts of "solvable" and "unsolvable" in our setting we have to find a suitable extension of them, so that the next problem is:

- Is there a natural extension of the notion of solvability which possibly includes the classical one?

After some progress in the understanding of the syntax we have choosen, we must turn our attention to the abstract objects we are describing, and to investigate their structure. If the functional interpretation is the intended one, we could conider our system as a calculus of functions from sets to sets. In this perspective, because of the type free setting we are working with, we should think of an object as something being, at the same time, a function and a set, that is a possible argument of a function, even of itself. Now should we see such functions as sets of functions, so that the value of their application to an argument is the set of the values of each function in the set applied to the given argument? Should we consider "additive" functions w.r.t. the (semantic counterpart of) the operator $\oplus$? What about functional abstraction of an object which is actullay a set? Are these sets always finite? All these questions, and others naturally arising from any answer to them could be summarized as follows:

- What are the models of this calculus?

- Does the equality induced by the abstract interpretation in the model coincide with the operational semantics induced by the extension of the notion of solvability?

Our study would be defective if, once answered all previous questions, we couldn't give any information about the properties of the structures of interest and of their calculi. This typically amounts to ask for an axiomatization of the equality relation studied so far, that is to study the theory of the intended models. This would provide us with the technical machinery to compare the equality in the extended language with those known for the classical calculus. All this could be summarized:

- What looks like an axiomatization of the non deterministic calculus?
- Does the resulting theory extend the theory or the theories of the classical calculus?
- If yes, is this extension conservative?

1.3 Related work

Boudol's γ-calculus [Boud89] Goes in the opposite direction of Milner's π-calculus, that is from λ-calculus to higher order process algebras; a fragment of this calculus has been given a model by [Jag-Pan]. Altough the full calculus includes operators whose semantics is problematic even from an intuitive point of view, its relevance is due to the fact of hinting the other way around, that is to start with assessed theories and mathematics about recursion and higher order calculi, before embarking in higher order process calculi.

Non deterministic extensions of the pure λ-calculus calculus are of interest also in the analysis of lazy λ-calculus and of other reduction oriented λ-calculi: this is shown by recent work [Boud91] and by the interesting topic coding such calculi in the π-calculus [Mil90].

We finally recall the work of Moggi [Moggi] about the computational λ-calculus, proposing the categorical notion of monad as a general framework to capture intensional aspects of computing, including nondeterminism.

1.4 Summary and results

The thesis is organized as follows. In a first chapter (chapter 2) the topics quickly sketched in this introduction are discussed and the problem of the interplay between functionality and explicit non determinism in the type free setting is illustrated. The chapter surveys some of the works referred to above and introduces an extension af pure λ-calculus with an operator $\oplus$ meaning non deterministic choice.

Chapter 3 is devoted to the study of operational semantics, concentrating on a preorder we call "must-preorder", written $\sqsubseteq_{must}$, which is a possible generalization of the classical notion of "solvability"; it is also an adaptation to our framework of the homonym testing relation, introduced in [DeN-Henn] for CCS. As a first step in the understanding of the operational semantics, we prove that, in spite of the lack of the Church-Rosser property, a standardization theorem can be established. The main novelty in this chapter is the introduction of "non deterministic Böhm trees", allowing a concrete representation of the functional behaviour of a non deterministic term. This is proved by the semiseparability theorem at the end of the chapter (for classical results about semiseparability in the λ-calculus see [CDR, Wads, Bar]).

Chapter 4 investigates denotational semantics. The concepts of linear and semilinear applicative structures are introduced. An inverse limit construction in the category of non deterministic algebras gives a first example of a proper semilinear structure, which turns out to be extensional. In this chapter it is proved that the order in the model coincides with the operational order induced by quotienting the preorder $\sqsubseteq_{must}$, and the order defined by tree "inclusion" (full abstraction theorem). We finally investigate a semilinear structure built in the category of **CPO** using Moggi's construction. This turns out to be non extensional, thus proving the independence of the axiom stating that the abstraction distributes over choices.

Chapter 5 studies theories. We prove consistency of the full λ_c-calculus and λ_r-calculus introduced in [Sharma]; the former calculus is also proved to be conservative w.r.t. the theory λ. The theory $\mathcal{T}_{must}$, induced by the preorder $\sqsubseteq_{must}$, is shown to be a conservative extension of the theory $\mathcal{H}^*$ (see [Bar]) of the classical λ-calculus, providing by the way an indipendent (synctactical) proof of consistency of $\mathcal{T}_{must}$ itself, and of the theory λ_r, which is a subtheory of $\mathcal{T}_{must}$. This is a surprising result, distinguishing the pure calculus from its typed version (with constants) studied in [Ash-Henn].

Some of the results presented in this thesis will appear in [deL-Pip].

Chapter 2

Non-determinism and type-free λ-calculus

In this chapter we quickly recall the essential definitions and facts about domains and powerdomains. We illustrate the problem of introducing explicit non determinism in the (first order) algebraic setting of term rewriting systems.

After a short presentation of classical λ-calculus and of some of its variants introduced in [Plo75, Abr-Ong], we illustrate a nondeterministic extension of **PCF** and finally introduce the type-free non deterministic λ-calculus, we will study in the rest of the thesis.

2.1 Powerdomains

In this section we survey some facts about constructions modeling non determinism in the framework of usual denotational semantics. This exposition is intended to fix notation and to keep the subsequent treatment selfcontained as much as possible; consequently proofs will be omitted or simply sketched. A detailed account of these topics can be found in now standard literature (see e.g. [Gun-Sco]).

Definition 1 *Let $\langle P, \sqsubseteq \rangle$ be any (pre)-ordered set; define*

$$\mathrm{Fin}(P) = \{u \subseteq P \mid u \textit{ finite}, u \neq \emptyset\};$$

then for $u, v \in \mathrm{Fin}(P)$ define *i)* $u \sqsubseteq^\flat v \;\Leftrightarrow\; \forall x \in u\, \exists y \in v.\, x \sqsubseteq y$;

ii) $u \sqsubseteq^\sharp v \;\Leftrightarrow\; \forall y \in v\, \exists x \in u.\, x \sqsubseteq y$;

iii) $u \sqsubseteq^\natural v \;\Leftrightarrow\; u \sqsubseteq^\flat v \wedge u \sqsubseteq^\sharp v$.

Proposition 1 *Let $\langle P, \sqsubseteq \rangle$ be any (pre)-ordered set; if $S \subseteq P$ then abbreviate $\{x \in P \mid \exists y \in S.\, x \sqsubseteq y\}$ with $\downarrow S$ and similarly $\{x \in P \mid \exists y \in S.\, y \sqsubseteq x\}$ with $\uparrow S$. Then, for any $u, v \in \mathrm{Fin}(D)$:*

i) $\sqsubseteq^\flat, \sqsubseteq^\sharp$ *and* $\sqsubseteq^\natural$ *are preorders;*

ii) $u \sqsubseteq^\flat v \;\Leftrightarrow \downarrow u \subseteq \downarrow v$;

iii) $u \sqsubseteq^\sharp v \;\Leftrightarrow\; \uparrow u \supseteq \uparrow v$;

iv) $u \sqsubseteq^\flat u \cup v$;

v) $u \cup v \sqsubseteq^\sharp u$.

Definition 2 *Let $\langle P, \sqsubseteq \rangle$ be any (pre)-ordered set and $S \subseteq P$; then*

i) S is downward closed *iff $S = \downarrow S$;*

ii) S is directed *iff $S \neq \emptyset$ and*

$$\forall x, y \in S\, \exists z \in S.\, x \sqsubseteq z \wedge y \sqsubseteq z;$$

iii) S is an ideal *iff it is downward closed and directed.*

Finally $\mathrm{Idl}(P, \sqsubseteq) = \{I \subseteq P \mid I$ *is an ideal w.r.t.* $\sqsubseteq\}$.

If x is an element of a pre-ordered set P, it is easy to verify that $\downarrow\{x\}$ is an ideal: it is called the *principal ideal* generated by x.

Definition 3 *A* **CPO** *D is a partially ordered set with a bottom and all lubs of directed subsets; furthermore*

i) $x \in D$ is a finite object *iff for all directed $S \subseteq D$*

$$x \sqsubseteq \bigsqcup S \Rightarrow \exists y \in S.\, x \sqsubseteq y;$$

ii) $\mathcal{K}(D) = \{x \in D \mid x \text{ is finite}\}$*;*

iii) D *is* algebraic *iff for all* $x \in D$

$$x = \bigsqcup\{y \in \mathcal{K}(D) \mid y \sqsubseteq x\};$$

iv) D *is* ω-algebraic *iff it is algebraic and* $\mathcal{K}(D)$ *is countable.*

CPOs are usually considered with Scott topology; it is known that functions between **CPO**s are continuous under this topology iff they preserve directed lubs.

Theorem 1 *Let* $\langle P, \sqsubseteq\rangle$ *be a (pre)-order with a bottom* $\bot$*, then* $\langle \mathrm{Idl}(P, \sqsubseteq), \subseteq\rangle$ *is an algebraic* **CPO** *with principal ideals as finite objects. If* P *is countable, then* $\mathrm{Idl}(P, \sqsubseteq)$ *is* ω*-algebraic.*

Definition 4 *Let* D *be an algebraic* **CPO***, then*

i) $\mathrm{M}(D) = \mathrm{Fin}(\mathcal{K}(D))$*;*

ii) $D^* = \mathrm{Idl}(\mathrm{M}(D), \sqsubseteq^*)$*, for* $* \in \{\flat, \sharp, \natural\}$.

Historically $D^\natural$ is *Plotkin powerdomain*, and $D^\sharp$ is *Smyth powerdomain* (see [Plo76, Smy]). $D^\flat$ is called *Hoare powerdomain*; it is actually the set of closed subsets of D w.r.t. the Scott topology.

Definition 5 *A partial order* $\langle P, \sqsubseteq\rangle$ *is* bounded complete *iff it has a bottom and every bounded subset of* P *has a lub. A* domain *is an* ω*-algebraic* **CPO** *which is bounded complete.*

Proposition 2 *If* D *is an algebraic* **CPO***, then* D^* *is an algebraic* **CPO** *for* $* \in \{\flat, \sharp, \natural\}$*; furthermore if* D *is a domain, then* $D^\flat$ *and* $D^\sharp$ *are domains.*

It is known that, even if D is a domain, in general $D^\natural$ is not bounded complete: this was the reason in [Plo76] for introducing the category **SFP**.

2.2 The non-determinism in term rewriting systems

Suppose to be given a signature $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$, where $\mathcal{S}$ is the set of sorts, and $\mathcal{F}$ the (non empty) set of operators, each one of a fixed aritity. For the sake of simplicity we choose $\mathcal{S}$ to be a singleton set (the reader is referred to the standard literature for the general case of multisorted signatures and algebras), and define:

Definition 6 *A* Σ-algebra *is a structure*

$$\mathcal{A} = \langle A, \{f^{\mathcal{A}} : A^n \to A \mid f \in \mathcal{F}, f \text{ of arity } n\} \rangle.$$

$\mathcal{A}$ *is a* monotonic (continuous) algebra *iff* A *is an ordered set with a least element (a* **CPO***), and each operation* $f^{\mathcal{A}}$ *is a monotonic (continuous, that is preserving directed limits) function.*

Σ-algebras form a category Σ-Alg, whose objects are Σ-algebras, and morphism are Σ-homomorphisms, where, if $\mathcal{A}$ and $\mathcal{B}$ are Σ-algebras, then a map $\varphi : A \to B$ is a Σ-*homomorphism* iff for all $f \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$

$$\varphi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n)).$$

The set of terms we form from the objects in $\mathcal{F}$ is a Σ-algebra $\mathcal{T}_\Sigma$, where $f^{\mathcal{T}_\Sigma} = f$ for all $f \in \mathcal{F}$. This algebra is the *initial algebra* in Σ-Alg, that is for every Σ-algebra $\mathcal{A}$ there exists a unique morphism $\varphi_{\mathcal{A}}$ from $\mathcal{T}_\Sigma$ to $\mathcal{A}$.

If X is any set of variables (disjoint from $\mathcal{F}$), the structure $\mathcal{T}_\Sigma(X)$ is the Σ-algebra freely generated from X.

Definition 7 *A* term rewriting system *in the given signature* Σ *is a finite subset* $\mathcal{R} \subseteq \mathcal{T}_\Sigma(X) \times \mathcal{T}_\Sigma(X)$ *such that, for all* $\langle t, t' \rangle \in \mathcal{R}$*:*

i) $t \not\equiv x$ *for any* $x \in X$*;*

ii) $\mathrm{Var}(t) \subseteq \mathrm{Var}(t')$*, where* $\mathrm{Var}(t)$ *is the set of variables occurring in* t*.*

A term rewriting system $\mathcal{R}$ induces a binary relation $\to_{\mathcal{R}}$ over $\mathcal{T}_\Sigma$; it is defined as follows:

Definition 8 *Call (ground) substitution the unique extension to* $\mathcal{T}_\Sigma$ *of a map* $\vartheta : X \to \mathcal{T}_\Sigma$*; then* $\to_\mathcal{R}$ *is the least binary relation over* $\mathcal{T}_\Sigma$ *such that:*

i) $\langle t, t' \rangle \in \mathcal{R} \land \exists \vartheta.\ s \equiv \vartheta(t) \Rightarrow s \to_\mathcal{R} \vartheta(t')$*;*

ii) $s \to_\mathcal{R} s' \Rightarrow \forall f \in \mathcal{F}, s_1, \ldots, s_{i-1}, s_{i+1} \ldots s_n \in \mathcal{T}_\Sigma.$

$$f(s_1, \ldots, s_{i-1}, s, s_{i+1} \ldots s_n) \to_\mathcal{R} f(s_1, \ldots, s_{i-1}, s', s_{i+1} \ldots s_n)$$

where n *is the arity of* f*.*

Writing $\to_\mathcal{R}$ *simply* $\to$*, we define* $\xrightarrow{+}$ *and* $\xrightarrow{*}$ *as its transitive, reflexive and transitive closure respectively.*

Intuitively a term rewriting system abstractly describes an evaluation mechanism for symbolic computation. A system satisfies the property of being *Church-Rosser* iff

$$(\mathbf{CR}) \qquad \forall t, t', t'' \in \mathcal{T}_\Sigma.\ t \xrightarrow{*} t' \land t \xrightarrow{*} t'' \Rightarrow \exists t''' \in \mathcal{T}_\Sigma.\ t' \xrightarrow{*} t''' \land t'' \xrightarrow{*} t'''.$$

Informally a *reduction* is any finite or infinite sequence of terms $t_0, t_1, \ldots$ such that $t_0 \to t_1 \to \cdots$ A system is *strongly normalizing* (**SN**) if there exists no infinite reduction.

In general term rewriting systems are not **SN**, but if they are **CR**, each term t has at most one normal form, we could consider as the "value" of t. We do not need, however, to consider all terms without normal form as undefined objects. Following the algebraic approach of [ADJ, Gue] terms can be interpreted in the initial algebra of the subcategory Σ-ConAlg of the continuous Σ-algebras with continuous Σ-homomorphisms.

Definition 9 *Let* Ω *be a nullary symbol not in* $\mathcal{F}$*, and call* $\Sigma + \Omega$ *the signature resulting by adding it to* $\mathcal{F}$*. Define the binary relation* $\preceq$ *over* $\mathcal{T}_{\Sigma+\Omega}$ *as the least preorder such that:*

i) $\Omega \preceq t$ *for all* $t \in \mathcal{T}_{\Sigma+\Omega}$*;*

ii) $t_1 \preceq t'_1, \ldots, t_n \preceq t'_n \Rightarrow f(t_1, \ldots, t_n) \preceq f(t'_1, \ldots, t'_n)$ *for all* $f \in \mathcal{F}$*.*

The relation $\preceq$ is actually a partial order; we call $\mathcal{T}_\Sigma^\infty$ the ideal completion of $\mathcal{T}_{\Sigma+\Omega}$ under $\preceq$.

Proposition 3 *$\mathcal{T}_\Sigma^\infty$ is the initial algebra in the category Σ-ConAlg.*

Proof. See [ADJ].

□

Now, fixed a rewriting system $\mathcal{R}$ we define a map $\omega_\mathcal{R} : \mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma+\Omega}$ by

$$\omega_\mathcal{R}(t) = \begin{cases} \Omega & \text{if } \exists \langle t', t'' \rangle \in \mathcal{R}\ \exists \vartheta. \\ & \quad f(\omega_\mathcal{R}(t_1), \ldots, \omega_\mathcal{R}(t_n)) \preceq \vartheta(t') \\ f(\omega_\mathcal{R}(t_1), \ldots, \omega_\mathcal{R}(t_n)) & \text{otherwise} \end{cases}$$

where $t \equiv f(t_1, \ldots, t_n)$, and we have the basis of the induction when $n = 0$.

The idea is that $\omega_\mathcal{R}$ takes the current "approximation" of its argument, modulo the subterms which have to be computed with respect to the rewriting system $\mathcal{R}$; these parts are mapped in Ω, which means "undefined", so that the relation $\omega_\mathcal{R}(t) \preceq \omega_\mathcal{R}(t')$ means that t is less defined, less explicit than t' w.r.t. $\mathcal{R}$.

Example 1 Let $\mathcal{F} = \{a^{(0)}, h^{(1)}, f^{(1)}, g^{(2)}\}$, where we use exponents to express arity (therafter they are omitted), and consider the rewriting system $\mathcal{R} = \{\langle f(x), g(x, f(h(x)))\rangle\}$. Now the term $t \equiv f(a)$ has an infinite reduction, namely:

$$\begin{aligned} f(a) &\to g(a, f(h(a))) \\ &\to g(a, g(h(a), f(h(h(a))))) \\ &\to \cdots \end{aligned}$$

Computing the approximations of the terms occurring in the reduction we have:

$$\begin{aligned} \omega_\mathcal{R}(f(a)) &= \Omega \\ \omega_\mathcal{R}(g(a, f(h(a)))) &= g(a, \Omega) \\ \omega_\mathcal{R}(g(a, g(h(a), f(h(h(a)))))) &= g(a, g(h(a), \Omega)), \end{aligned}$$

which is easily seen to be increasing w.r.t. $\preceq$.

What we have just seen in the example 1 is a general fact, as it is stated in the following proposition.

Proposition 4 *Given any rewriting system $\mathcal{R}$ we have*

i) $t \xrightarrow{*}_\mathcal{R} t' \Rightarrow \omega_\mathcal{R}(t) \preceq \omega_\mathcal{R}(t')$;

ii) $\mathcal{R} \models \mathbf{CR} \Rightarrow \forall t \in \mathcal{T}_\Sigma.\ \{\omega_\mathcal{R}(t') \mid t \xrightarrow{*}_\mathcal{R} t'\}$ *is directed wrt* $\preceq$.

Proof. Part (i) is proved by a straightforward induction on t. Part (ii) is then immediate.

□

Under the assumption that $\mathcal{R}$ is **CR**, this proposition allows us to assign to each term $t \in \mathcal{T}_\Sigma$ the object $\bigsqcup\{\omega_\mathcal{R}(t') \mid t \xrightarrow{*}_\mathcal{R} t'\}$ as its value, existing in $\mathcal{T}_\Sigma^\infty$ because of the propositions 3 and 4. Note that, if t has a normal form t', then the limit is t' itself (remember that there exists a natural injection of $\mathcal{T}_\Sigma$ into $\mathcal{T}_\Sigma^\infty$, namely $\varphi_{\mathcal{T}_\Sigma^\infty}$).

Add to the signature Σ a binary operator **or** expressing explicit nondeterminism; call $\mathcal{R} + \textbf{or}$ the system $\mathcal{R} \cup \{\langle x \,\textbf{or}\, y, x\rangle, \langle x \,\textbf{or}\, y, y\rangle\}$, using infix notation. No matter whether $\mathcal{R} \models \textbf{CR}$ or not, $\mathcal{R} + \textbf{or}$ is not Church-Rosser; consequently, for some $t \in \mathcal{T}_{\Sigma + \textbf{or}}$ we loose the directness of the set $\{\omega_{\mathcal{R}+\textbf{or}}(t') \mid t \xrightarrow{*}_{\mathcal{R}+\textbf{or}} t'\}$.

Example 2 Let Σ as in example 1, and consider $\Sigma + \textbf{or}$; let $\mathcal{R}$ be the system with the unique rule $\langle f(x), g(x, x \,\textbf{or}\, f(h(x)))\rangle$. We compute again $t \equiv f(a)$ w.r.t. $\mathcal{R} + \textbf{or}$:

$$\begin{array}{rcl} f(a) & \rightarrow & g(a, a \,\textbf{or}\, f(h(a))) \\ & \rightarrow & g(a, f(h(a))) \\ & \rightarrow & g(a, g(h(a), h(a) \,\textbf{or}\, f(h(h(a))))) \\ & \rightarrow & \cdots \end{array}$$

but also

$$\begin{array}{rcl} \cdots & \rightarrow & g(a, a \,\textbf{or}\, f(h(a))) \\ & \rightarrow & g(a, a). \end{array}$$

The map $\omega_{\mathcal{R}+\textbf{or}}$ is still increasing along each reduction, but for example

$$\omega_{\mathcal{R}+\textbf{or}}(g(a, a)) = g(a, a),$$

while

$$\omega_{\mathcal{R}+\textbf{or}}(g(a, g(h(a), h(a) \,\textbf{or}\, f(h(h(a)))))) = g(a, g(h(a), \Omega)),$$

and there is no upper bound of these objects wrt $\preceq$ in $\mathcal{T}_{\Sigma+\Omega}$.

A possible way out is proposed e.g. in [Boud80]; we use it, however, to sketch how powerdomains fit in the algebraic theory we are surveying: this is not the case of [Boud80], where this approach is criticized (see also [Smy] and [Abr83] for the limits of this construction).

First define a map $\psi : \mathcal{T}_{\Sigma+\textbf{or}} \rightarrow \mathrm{Fin}(\mathcal{T}_\Sigma)$ inductively as follows:

i) $\psi(t\,\mathbf{or}\,t') = \psi(t) \cup \psi(t')$;

ii) $\psi(f(t_1, \ldots, t_n)) = \{f(t'_1, \ldots, t'_n) \mid t'_i \in \psi(t_i); i = 1, \ldots, n\}$.

Suppose that $\mathcal{R} \models \mathbf{CR}$ and consider all possible reducts of a given term $t \in \mathcal{T}_{\Sigma + \mathbf{or}}$ by the relation $\rightarrow_{\mathcal{R}}$: this means that we evaluate t considering the operator **or** as a "constant" operator, not to be further computed. Now we take as the approximated value of each reduct t' of t the finite set $\psi \circ \omega_{\mathcal{R}}(t')$, which is a subset of $\mathcal{T}_{\Sigma+\Omega}$. Then by the very construction we have:

Proposition 5 *Identify $\mathcal{T}_{\Sigma+\Omega}$ with its homomorphic image into $\mathcal{T}_{\Sigma}^{\infty}$ via the $\Sigma + \Omega$-homomorphism $\varphi(s) = \downarrow s$; then, for all $t \in \mathcal{T}_{\Sigma + \mathbf{or}}$:*

i) $\psi \circ \omega_{\mathcal{R}}(t) \in \mathrm{M}(\mathcal{T}_{\Sigma}^{\infty})$;

ii) $\{\psi \circ \omega_{\mathcal{R}}(t') \mid t \xrightarrow{*}_{\mathcal{R}} t'\}$ is directed wrt $\sqsubseteq^{\flat}, \sqsubseteq^{\sharp}, \sqsubseteq^{\natural}$.

$\square$

A natural extension of the "algebraic" interpretation in the case of **CR** systems is now to assign as the value of each term $t \in \mathcal{T}_{\Sigma + \mathbf{or}}$, wrt the rewriting system $\mathcal{R} + \mathbf{or}$, the limit $\bigsqcup\{\psi \circ \omega_{\mathcal{R}}(t') \mid t \xrightarrow{*}_{\mathcal{R}} t'\}$, existing in the algebra $(\mathcal{T}_{\Sigma}^{\infty})^{*}$ for $* \in \{\flat, \sharp, \natural\}$ (note that in this algebra the operations are the "extensions" of the operations in $\mathcal{T}_{\Sigma}^{\infty}$: see e.g. [Abr83] and chapter 4 in this thesis).

2.3 The classical λ-calculus

Classical λ-calculus theory is mainly concerned with the study of β-convertibility. Reduction relations, although studied at length, seem to be only auxiliary, proof theoretic tools for proving consistency of equational theories, usually extending the theory λ. The primacy is made more evident by the fact that every known model of the λ-calculus is actually a model of conversion instead of reduction (however in recent time some people started to study possible notions of models for the reduction see [Jac-Mar-Zac, Plo91]). The theory λ and its basic properties are sketched below.

Definition 10 *Let X be an infinite denumerable set of variables, then the set Λ of terms is the least one such that:*

i) $X \subseteq \Lambda$;

ii) $M, N \in \Lambda \Rightarrow (MN) \in \Lambda$;

iii) $M \in \Lambda, x \in X \Rightarrow \lambda x.M \in \Lambda$.

In clause (iii) the λ is a binding operator, whose scope is M; an occurrence of x in a term M is free iff it is not in the scope of an abstraction of the form λx. Usually the set of variables having a free occurrence in M is denoted by $\mathrm{FV}(M)$. Members of the set $\Lambda^0 = \{M \in \Lambda \mid \mathrm{FV}(M) = \emptyset\}$ are called closed terms.

Some terms have, for historical reasons, a name; we list here some of them:

$$\begin{array}{lllllllll} \mathbf{I} & \equiv & \lambda x.x, & \mathbf{K} & \equiv & \lambda xy.x, & \mathbf{O} & \equiv & \lambda xy.y, \\ \mathbf{S} & \equiv & \lambda xyz.xz(yz), & \boldsymbol{\Delta} & \equiv & \lambda x.xx & \mathbf{U}_i^n & \equiv & \lambda x_1 \ldots x_n.x_i \end{array}$$

where for $\mathbf{U}_i^n$ it is required that $i \leq n$.

As usual with binding operators some care is needed when defining substitution. In the sequel $\equiv$ will mean syntactical equality.

Definition 11 *Let $M, N \in \Lambda$ and $x \in X$ then $M[N/x]$ is defined inductively:*

i) $x[N/x] \equiv N$;

ii) $y[N/x] \equiv y$ *for* $y \in X - \{x\}$;

iii) $(M_1 M_2)[N/x] \equiv (M_1[N/x])(M_2[N/x])$;

iv) $(\lambda x.M)[N/x] \equiv (\lambda x.M)$;

v) $(\lambda y.M)[N/x] \equiv \lambda z.(M[z/y])[N/z]$ *where* $z \in X - \mathrm{FV}(N)$.

In clause (v) the z, to be choosen in some fixed way, is replaced to the y to avoid variable clashes with possible free occurrences of the y itself in N.

Definition 12 *The theory λ is the set of closed equations derivable from the following axioms and rules:*

α) $\lambda x.M = \lambda y.M[y/x]$ *for* $y \notin \mathrm{FV}(M)$,

β) $(\lambda x.M)N = M[N/x]$,

ϱ) $M = M$,

σ) $M = N \Rightarrow N = M$,

τ) $M = N, N = L \Rightarrow M = L$,

μ) $M = N \Rightarrow LM = LN$,

ν) $M = N \Rightarrow ML = NL$,

ξ) $M = N \Rightarrow \lambda x.M = \lambda x.N$.

The relation of β-reduction is usually introduced substituting in the definition above $=$ with the symbol $\longrightarrow$, and deleting rule (σ). Equivalently one can define the binary relation

$$\beta = \{\langle(\lambda x.M)N, M[N/x]\rangle \mid M, N \in \Lambda\}$$

and take its "compatible closure" $\longrightarrow_\beta$ (see [Bar]), that is

Definition 13 $\longrightarrow_\beta$ *is the least binary relation on* Λ *such that*

i) $\beta \subseteq \longrightarrow_\beta$,

ii) $M \longrightarrow_\beta N \Rightarrow ML \longrightarrow_\beta NL, LM \longrightarrow_\beta LN, \lambda x.M \longrightarrow_\beta \lambda x.N$.

The relations $\stackrel{+}{\longrightarrow}_\beta$ *and* $\stackrel{*}{\longrightarrow}_\beta$ *are respectively the transitive and the transitive and reflexive closures of* $\longrightarrow_\beta$*;* $=_\beta$ *is the symmetric closure of* $\stackrel{*}{\longrightarrow}_\beta$*.*

Proposition 6

$$\forall M, N \in \Lambda.\ \lambda \vdash M = N \;\Leftrightarrow\; M =_\beta N.$$

Proof. Straightforward induction in both directions.

□

The main result about (β) is known as the Church-Rosser theorem:

Theorem 2 (**Church-Rosser**)

$$\forall M, N, L \in \Lambda.\ M \stackrel{*}{\longrightarrow}_\beta N \wedge M \stackrel{*}{\longrightarrow}_\beta L \Rightarrow \exists P \in \Lambda.\ N \stackrel{*}{\longrightarrow}_\beta P \wedge L \stackrel{*}{\longrightarrow}_\beta P.$$

Proof. See [Bar].

□

Corollary 1 *For any $M, N \in \Lambda$*

i) $M =_\beta N \;\Leftrightarrow\; \exists L \in \Lambda.\ M \stackrel{*}{\longrightarrow}_\beta L \wedge N \stackrel{*}{\longrightarrow}_\beta L$;

ii) the theory λ is consistent.

Even if consistent, the theory λ is not Hilbert-Post complete, that is we can add to it new equations without loosing consistency. A classical example is the (scheme of) axiom

$$(\eta) \qquad \lambda x.Mx = M \qquad \text{if } x \notin \mathrm{FV}(M),$$

yielding a theory equivalent to that obtained from λ adding

$$(ext) \qquad Mx = Nx \Rightarrow M = N \qquad \text{if } x \notin \mathrm{FV}(M) \cup \mathrm{FV}(N).$$

A fundamental limitative result about consistent extensions of the theory λ is Böhm's theorem.

Definition 14 *For any $M \in \Lambda$,*

i) M is a normal form, *iff* $\neg\exists N \in \Lambda.\ M \longrightarrow_\beta N$; *write* **NF** *for the set of these terms;*

ii) M has a normal form *iff* $\exists N \in \mathbf{NF}.\ M \stackrel{*}{\longrightarrow}_\beta N$.

It is easily seen that any term in normal form has the shape

$$\lambda x_1 \dots x_n.\xi M_1 \dots M_m,$$

where the M_i are in normal form too; it is also easy to see that there are terms that doesn't have normal form: tipically $\Delta\Delta$.

Definition 15 *The set of* contexts $\Lambda[\,]$ *is recursively defined as follows:*

i) $[\,] \in \Lambda[\,]$ *the* hole*;*

ii) $X \subseteq \Lambda[\,]$*;*

iii) $C[\,], C'[\,] \in \Lambda[\,] \Rightarrow C[\,]C'[\,] \in \Lambda[\,]$;

iv) $C[\,] \in \Lambda[\,] \Rightarrow \lambda x.C[\,] \in \Lambda[\,]$.

In the sequel contexts are assumed having just one hole. If $M \in \Lambda$ and $C[\,] \in \Lambda[\,]$, then $C[M] \in \Lambda$ is the term resulting by replacing (in a sense filling) the hole in $C[\,]$ by M.

Theorem 3 (**Böhm**) *Let $M, N \in \Lambda$ be two distinct $\beta\eta$-normal forms, then there exists a context $C[\,]$ and two distinct variables x, y such that*

$$\lambda \vdash C[M] = x \quad \textit{and} \quad \lambda \vdash C[N] = y.$$

It follows from Böhm's theorem that any theory equating two terms having distinct $\beta\eta$-normal forms is inconsistent. In view of this result it is tempting to subdivide λ-terms into two classes: those having a normal form and those without normal form. The distinction should coincide with that between meaningfull and meaningless terms, so that the latter could be all equated. This leads however to inconsistency. Consider for example the terms

$$P \equiv \lambda xyz.xy(\Delta\Delta) \quad \text{and} \quad Q \equiv \lambda xyz.xz(\Delta\Delta).$$

Both of them are without normal form; now if $P = Q$ is postulated, then $P\mathbf{K}MN = Q\mathbf{K}MN$ is derivable for any $M, N \in \Lambda$; it follows that $M = N$ is derivable as well.

The notion of being meaningless seems to be better captured by the following definition.

Definition 16 *$M \in \Lambda$ is* fully undefined *iff*

$$\forall N \in \Lambda\ \forall C[\,] \in \Lambda[\,].\ C[M] \textit{ has a normal form} \Rightarrow C[N] \textit{ has a normal form.}$$

This means that being meaningless consists in having no influence in any ending computation.

Among the main achievements of the classical studies about λ-calculus is the characterization of these terms by means of the notion of *(un)solvability*: this is why the classical theory could be called the theory of solvability.

Definition 17

i) $M \in \Lambda^0$ *is* solvable *iff*

$$\exists n \in \omega \, \exists N_1, \ldots, N_n. \, \lambda \vdash M N_1 \ldots N_n = \mathbf{I};$$

ii) $M \in \Lambda$ *is* solvable *iff its closure* $\lambda\vec{x}.M$ *is solvable;*

where $\lambda x_1 \ldots x_n.M$ *is the closure of* M *iff* $\mathrm{FV}(M) = \{x_1, \ldots, x_n\}$ *. The set of solvable terms is denoted by* **SOL**.

Definition 18

i) $M \in \Lambda$ is in head normal form *iff it is of the shape*

$$\lambda x_1 \ldots x_n.\xi M_1 \ldots M_m,$$

no matter what the M_i *are; the set of such terms is denoted by* **HNF***;*

ii) $M \in \Lambda$ has a head normal form *iff* $\exists N \in \mathbf{HNF}. \, M \xrightarrow{*}_\beta N$.

In a sense being in head normal form means to be a normal form at least at the first level.

Proposition 7 *For any* $M \in \Lambda$,

$$M \in \mathbf{SOL} \;\Leftrightarrow\; M \text{ has a head normal form.}$$

Proof. See [Bar]. □

Theorem 4 *For any* $M \in \Lambda$,

$$M \text{ is fully undefined} \;\Leftrightarrow\; M \notin \mathbf{SOL}.$$

Proof. For ($\Leftarrow$) see [Bar] theorem 14.3.24; for ($\Rightarrow$) simply observe that if $M \in \mathbf{SOL}$ then by proposition 7 $\lambda \vdash M = \lambda\vec{x}.\xi M_1 \ldots M_n$ and we can suppose wlog that for some $x_i \in \vec{x}$, $x_i \equiv \xi$; then

$$\lambda \vdash M x_1 \ldots x_{i-1} \mathbf{U}^{n+1}_{n+1} x_{i+1} \ldots x_n \mathbf{I} = \mathbf{I},$$

while $\Omega x_1 \ldots x_{i-1} \mathbf{U}^{n+1}_{n+1} x_{i+1} \ldots x_n \mathbf{I}$ has no normal form.

□

2.4 Lazy and Call-by-value λ-calculi

Machines are hardly assimilable to theories; on the contrary they are better seen as evaluation devices converting a term, representing a program applied to its input, into its value, that is eventually to a normal form. This amounts to stress reduction instead of convertibility, while the equational theory, to be thought of as the theory of program equivalence, has to be reconstructed from the concept of a suitable observable property, usually a convergency predicate.

Among the first studies in this direction is Plotkin's [Plo75], recently rediscovered and further investigated by Abramsky in [Abr-Ong], and thereafter in a series of papers (e.g. [Egi-Hon-Ron]).

Definition 19 *Over the set Λ^0 of closed λ-terms the following family of binary relations $\downarrow_l^k$ is defined:*

i) $\lambda x.M \downarrow_l^0 \lambda x.M$;

ii) $M \downarrow_l^h \lambda x.M', M'[N/x] \downarrow_l^k L \Rightarrow MN \downarrow_l^{h+k+1} L$;

iii) $M \downarrow_l N \;\Leftrightarrow\; \exists k.\, M \downarrow_l^k N$;

iv) $M\downarrow_l \;\Leftrightarrow\; \exists N.\, M \downarrow_l N$.

Call the predicate $\downarrow_l$ convergency predicate*.*

Intuitively $M \downarrow^k N$ means that M converges to N within k steps. This notion of convergency is not equivalent to that of having a normal form w.r.t. $\longrightarrow_\beta$, nor it means having a head normal form: it means to reach, actually in a unique way, the normal form w.r.t. the following reduction relation:

Definition 20 *The relation $\longrightarrow_l \subseteq \Lambda \times \Lambda$ is the least one such that*

(β) $(\lambda x.M)N \longrightarrow_l M[N/x]$;

(ν) $M \longrightarrow_l M' \Rightarrow MN \longrightarrow_l M'N$.

This reduction relation, clearly included into $\longrightarrow_\beta$, captures the idea of leftmost outhermost reduction strategy, up to a variable in head position or an abstraction: what is usually called a *weak normal from.* Morever

Proposition 8 *For all $M \in \Lambda^0$*

$$M \Downarrow_l^k N \;\Leftrightarrow\; M \xrightarrow{*}_l N \not\longrightarrow_l,$$

and $M \xrightarrow{*}_l N$ has length k; it follows that $\Downarrow_l$ is the convergency predicate for $\langle \Lambda^0, \longrightarrow_l \rangle$.

Proof.($\Leftarrow$): by induction on the length k of the reduction $M \xrightarrow{*}_l N$; if $k = 0$ then $M \equiv N \equiv \lambda x.M'$ for some M', hence $M \Downarrow_l^0 M'$; if $k > 0$ then the reduction has the form

$$M \equiv M_0 \longrightarrow_l M_1 \longrightarrow_l \ldots \longrightarrow_l M_r \longrightarrow_l M_{r+1} \longrightarrow_l \ldots M_k \equiv N$$

where, for some $r < k$, $M_r \equiv (\lambda x.P)Q \equiv P_r Q$, $M_{r+1} \equiv P[Q/x]$, while $M_i \equiv P_i Q$ and $P_i \longrightarrow_l P_{i+1}$ for each $i < r$. Now $P_0 \xrightarrow{*}_l P_r \equiv \lambda x.P \not\longrightarrow_l$, which is a reduction of length $r < k$, then, by inductive hypothesis, $P_0 \Downarrow_l^r \lambda x.P$; on the other hand $P[Q/x] \xrightarrow{*}_l M_k \equiv N \not\longrightarrow_l$ with a reduction of length $k - r - 1$; it follows, by inductive hypothesis, that $P[Q/x] \Downarrow_l^{k-r-1} N$, so that $M \Downarrow_l^k N$.
($\Rightarrow$): again by induction on k. In case $k = 0$ we have $M \equiv \lambda x.M'$ for some M', hence the thesis is trivial since $\lambda x.M' \not\longrightarrow_l$. If $k > 0$, then $M \equiv PQ$ for some P and Q such that $P \Downarrow_l^r \lambda x.P'$, $P'[Q/x] \Downarrow_l^s N$, and $r + s + 1 = k$. By inductive hypothesis $P \xrightarrow{*}_l \lambda x.P'$ in r steps so that

$$\begin{array}{rcll} PQ & \xrightarrow{*}_l & (\lambda x.P')Q & \text{in } r \text{ steps} \\ & \longrightarrow_l & P'[Q/x] & \\ & \xrightarrow{*}_l & N & \text{in } s \text{ steps.} \end{array}$$

□

The equational theory called in [Abr-Ong] the lazy λ-calculus is defined as follows:

Definition 21 *For $M, N \in \Lambda^0$:*

i) $M \sqsubseteq^B N \;\Leftrightarrow\; \forall \vec{P} \in \Lambda^0.\, M\vec{P}\Downarrow \Rightarrow N\vec{P}\Downarrow$;

ii) $M \sim^B N \;\Leftrightarrow\; M \sqsubseteq^B N \sqsubseteq^B M$;

iii) $\lambda_l \vdash M = N \;\Leftrightarrow\; M \sim^B N$.

To see that the theory defined above is consistent we recall the notion of *Morris theory* (see [Bar]).

Definition 22 *Let $\mathcal{P} \subseteq \Lambda^0$ be such that $\mathcal{P} \neq \emptyset, \Lambda^0$; then define*

i) $M \sqsubseteq^{\mathcal{P}} N \;\Leftrightarrow\; \forall C[\,] \in \Lambda^0[\,].\ C[M] \in \mathcal{P} \Rightarrow C[N] \in \mathcal{P}$;

ii) $M \sim^{\mathcal{P}} N \;\Leftrightarrow\; M \sqsubseteq^{\mathcal{P}} N \sqsubseteq^{\mathcal{P}} M$;

iii) $\mathcal{T}_{\mathcal{P}} = \{M = N \mid M, N \in \Lambda^0, M \sim^{\mathcal{P}} N\}$.

$\mathcal{T}_{\mathcal{P}}$ *is a* Morris theory *iff* $\mathcal{P}$ *is closed under* β *conversion.*

Proposition 9

i) if $\mathcal{T}_{\mathcal{P}}$ is a Morris theory, then it is a λ-theory, that is a consistent extension of the set $\{M = N \mid M, N \in \Lambda^0, \lambda \vdash M = N\}$, closed under derivability in λ;

ii) for all $M, N \in \Lambda^0$, $M \sqsubseteq^B N \;\Leftrightarrow\; \forall C[\,] \in \Lambda^0[\,].\ C[M]\!\downarrow\; \Rightarrow C[N]\!\downarrow$;

iii) for all $M, N \in \Lambda^0$, $M \sim^B N \;\Leftrightarrow\; M \sim^{\mathcal{F}} N$, *where* $\mathcal{F} = \{M \in \Lambda^0 \mid \exists M' \in \Lambda.\ M =_\beta \lambda x.M'\}$.

Since $\mathcal{F}$ is closed under β-conversion, it follows that λ_l is a Morris theory, hence a λ-theory.

Proof. See [Abr-Ong]

□

The idea of Morris theory allows to see the classical theory of solvability in a reduction oriented perspective. First we introduce formally the notion of head reduction.

Definition 23

(β) $(\lambda x.M)N \longrightarrow_h M[N/x]$;

(ν) $M \longrightarrow_h M' \Rightarrow MN \longrightarrow_h M'N$;

(ξ) $M \longrightarrow_h N \Rightarrow \lambda x.M \longrightarrow_h \lambda x.N$.

Definition 24

i) $xM_1 \dots M_n \downarrow_h^0 xM_1 \dots M_n$;

ii) $M \downarrow_h^r N \Rightarrow \lambda x.M \downarrow_h^r \lambda x.N$;

iii) $M \downarrow_l^r \lambda x.M', M'[N/x]\downarrow_h^s L \Rightarrow MN \downarrow_h^{r+s+1} L$;

iv) $M \downarrow_h N \;\Leftrightarrow\; \exists k.\, M \downarrow_h^k N$;

v) $M{\downarrow_h} \;\Leftrightarrow\; \exists N.\, M \downarrow_h N$.

Remark 1 In clause (iii) of the above definition the relation $\downarrow_l$ is used instead of $\downarrow_h$: this is due to the fact that, when reducing a term of the form MN by head reduction, if M reduces to an abstraction $\lambda x.M'$, it is in general not true that this is in head normal form; however the head reduction will not go on reducing M'; it will proceed instedad with the step $(\lambda x.M')N \longrightarrow_h M'[N/x]$, exactly as with the lazy reduction relation.

Arguing in a similar way as for the proposition 8, one can prove:

Proposition 10 *For all* $M \in \Lambda^0$

$$M \downarrow_h^k N \;\Leftrightarrow\; M \xrightarrow{*}_h N \not\longrightarrow_h,$$

and $M \xrightarrow{*}_h N$ *has length* k. *It follows that* $M{\downarrow_h} \;\Leftrightarrow\; M \in \mathbf{SOL}$.

Now the set $\mathbf{SOL}^0 = \mathbf{SOL} \cap \Lambda^0$ is a proper, non empty subset of Λ^0 which is closed under β-conversion. This determines the Morris theory $\mathcal{T}_{\mathbf{SOL}}$ usually called $\mathcal{H}^*$, which is w.r.t the reduction relation $\longrightarrow_h$ exactly the same as the theory λ_l w.r.t the relation $\longrightarrow_l$. We shall see in the sequel how this theory can be consistently extended to cope with a non deterministic choice operator.

Finally we sketch yet another λ-calculus strongly related with the notion of a reduction relation, namely Plotkin's *call-by-value* λ-calculus.

Definition 25 *Let* **Val** *be the set of variables and abstractions; then define* $\longrightarrow_v$ *as the least binary relation on* Λ *such that:*

(β) $(\lambda x.M)N \longrightarrow_v M[N/x]$ *if* $N \in$ **Val**;

(ν) $M \longrightarrow_v M' \Rightarrow MN \longrightarrow_v M'N$;

(μ) $M \longrightarrow_v M' \Rightarrow NM \longrightarrow_v NM'$.

Definition 26 *Over the set Λ^0 of closed λ-terms the following family of binary relations $\downarrow_v^k$ is defined:*

i) $\lambda x.M \downarrow_v^0 \lambda x.M$;

ii) $M \downarrow_v^h \lambda x.M', N \downarrow_v^k N', M'[N'/x]\downarrow_v^i L \Rightarrow MN \downarrow_v^{h+k+i+1} L$;

iii) $M \downarrow_v N \;\Leftrightarrow\; \exists k.\, M \downarrow_v^k N$;

iv) $M\downarrow_v \;\Leftrightarrow\; \exists N.\, M \downarrow_v N$.

The predicate $\downarrow_v$ is the convergency predicate w.r.t. $\longrightarrow_v$. By defining the theory λ_v in the same way as λ_l we get an equivalence among terms called $\approx_v$ in [Plo75]. It should be noted that it is not a λ-theory, althought it is consistent: this can be immediately seen considernig the equation $\mathbf{KI}(\Delta\Delta) = \mathbf{I}$ which is in λ but not in λ_v.

2.5 The non deterministic λ-calculus

To model in the setting of typed and untyped λ-calculus explicit non-determinism we need an enriched language, including some new operator representing the choice control structure. Now we have to define a (non Church-Rosser) reduction relation formalizing the evaluation mechanism, and to give an axiomatization allowing reasoning about non-deterministic terms; to this aim we are faced with the problem of finding an intuitive operational semantics justifying our choice of the new axioms, and, at the same time, with the problem of relating the new theory with the underlying theory of the original "deterministic system".

2.5.1 Typed non-deterministic λ-calculus

In [Ash-Henn] a nondeterministic control operator **or** is added to Plotkin's system **PCF** (see[Plo77]), and the denotational semantics of the resulting system, we are going to call **NPCF**, is studied with the help of techniques from the powerdomain theory. It should be noted, however, that this is only partly true, since the powerdomains involved are only the simpler ones, namely those of flat domains, because they are used only in defining the interpretation of ground types.

Definition 27 *The set Type is inductively defined:*

i) $o, \iota \in Type$ *(ground types);*

ii) $\sigma, \tau \in Type \Rightarrow (\sigma \to \tau) \in Type$.

As usual parentesis in type expressions associate to the right. Now the set of **NPCF** *terms* $Term = \bigcup_{\sigma \in Type} Term_\sigma$ *is defined:*

iii) $x_i^\sigma \in Type_\sigma$ *for all* $i \in \omega$*;*

iv) $tt, ff \in Term_o$*;* $k_n \in Term_\iota$*, for all* $n \in \omega$*;* $\mathbf{S}, \mathbf{P} \in Term_{\iota \to \iota}$*;* $\mathbf{Z} \in Term_{\iota \to o}$*;*

v) for each $\sigma \in Type$*,* $\mathbf{if}_\sigma \in Term_{o \to \sigma \to \sigma}$*, and* $\mathbf{Y} \in Term_{(\sigma \to \sigma) \to \sigma}$*;*

vi) $M \in Term_{\sigma \to \tau}, N \in Term_\sigma \Rightarrow (MN) \in Term_\tau$*;*

vii) $M \in Term_\tau \Rightarrow \lambda x^\sigma . M \in Term_{\sigma \to \tau}$*;*

viii) $M, N \in Term_\sigma \Rightarrow (M \mathbf{or} N) \in Term_\sigma$.

The reduction relation of the calculus **NPCF** is defined by the following axioms and rules:

$$
\begin{aligned}
&\mathbf{S}k_n \to k_{n+1} \\
&\mathbf{P}k_{n+1} \to k_n \\
&\mathbf{Z}k_0 \to tt \\
&\mathbf{Z}k_{n+1} \to ff \\
&\mathbf{if}\, tt MN \to M \\
&\mathbf{if}\, ff MN \to N \\
&(\lambda x.M)N \to M[N/x] \\
&\mathbf{Y}M \to M(\mathbf{Y}M)
\end{aligned}
$$

$$M\mathbf{or}N \to M$$

$$M\mathbf{or}N \to N$$

$$\frac{M \to M'}{MN \to M'N}$$

$$\frac{M \to M'}{cM \to cM'} \text{ if } c \in \{\mathbf{S}, \mathbf{P}, \mathbf{Z}, \mathbf{if}\}$$

The operational semantics of a "program" in this calculus, that is of a closed term of ground type, is defined as the set of constants it reduces to, plus a special value $\bot$ meaning the possibility of diverging (when there exists an infinite reduction starting with it), or deadlocking ("blocking"), that is when reducing to a normal form which is not a constant (eg. $\mathbf{P}k_0$, or xM for any M). Call $Eval(M)$ such a set when M is a program. Now

Definition 28 *Given $M, N \in Type_\sigma$, for some σ, define $M \sqsubseteq_{op} N$ iff for all context $C[\,]$ of ground type closing both M and N, we have $Eval(C[M]) \subseteq Eval(C[N])$. Define $\simeq_{op} = \sqsubseteq_{op} \cap \sqsubseteq_{op}^{-1}$.*

Observe that a program M can diverge iff either it is blocked, or it reduces to an "unsolvable" deterministic term, that is a term that cannot reduce to a constant, or, by a straightforward use of König lemma, there exists a reduction sequence starting with it, in which the rules of **or** are used infinitely often.

The matematical model of this language is constructed in [Ash-Henn] in the category of **NDA** (see chapter 4) interpreting o and ι in the Plotkin powerdomain of flat domains of respectively booleans and natural numbers, and functional types $\sigma \to \tau$ in the space of linear functions from the interpretation of σ to the interpretation of τ. To get the interpretation map for the terms, a clause dealing with the operator **or** has to be added to the interpretaion map of [Plo77], namely:

$$[\![M^\sigma \mathbf{or} N^\sigma]\!]_\rho = [\![M]\!]_\rho + [\![N]\!]_\rho$$

where $+$ is the continuous idempotent, commutative and associative operation existing in $[\![\sigma]\!]$.

The main result about this construction is a full abstraction theorem for "procedures" that is:

Theorem 5 *Given $M, N \in Type_\sigma$ where $\sigma = \alpha_1 \rightarrow \ldots \rightarrow \alpha_n$ and the α_i are all ground types (in which case M, N are called* procedures*), it holds that*

$$M \simeq_{op} N \;\Leftrightarrow\; \forall \rho.\, [\![M]\!]_\rho = [\![N]\!]_\rho.$$

This however doesn't hold in the general case.

In [Ast-Co] it was observed that the operational equivalence $\simeq_{op}$, when restricted to the sublanguage of **PCF**, doesn't coincides with the analogue notion introduced by Plotkin in [Plo77]; more precisely let $M, N \in$ **PCF** be closed terms of the same type; suppose that for each closing ground context $C[\,]$ not containing any occurrence of **or**, $Eval(C[M]) = Eval(C[N])$: but it is not true that the same holds allowing some occurrence of **or** in $C[\,]$.

Example 3 Let F and G be combinators of type $o \rightarrow o$ such that:

$$\begin{array}{rcl} F & \equiv & \lambda x.\mathbf{if}(\mathbf{Z}x)k_0k_0 \\ G & \equiv & \lambda x.\mathbf{if}(\mathbf{Z}x)xk_0. \end{array}$$

Since any closed term of type o either has an infinite reduction, or it reachs a "blocked" term (in both cases its value is undefined) or a numeric constant, these combinators are operationally equivalent in **PCF**.

Now consider the context $C[\,] \equiv [\,](k_0\, \mathbf{or}\, k_1)$, which is a **NPCF** context: then

$$\begin{array}{rcl} Eval(F(k_0\, \mathbf{or}\, k_1)) & = & \{k_0\} \\ & \neq & Eval(G(k_0\, \mathbf{or}\, k_1) \\ & = & \{k_0, k_1\}. \end{array}$$

From Example 3 it is clear that problems arise because the reduction of a term can make many copies of a subterm containing **or**, in such a way that each copy can behave differently. The solution proposed in [Ast-Co], that is introducing a sharing mechanism, leads to a strong restriction of the calculus, which seems a serious one.

Example 4 Consider the combinator H of type $o \rightarrow o$ which satisfies:

$$Hx \xrightarrow{*} x\, \mathbf{or}\, H(\mathbf{S}x).$$

This combinator exists, and is definable using the paradoxical combinator **Y** putting

$$H \equiv \mathbf{Y}(\lambda hx.x\, \mathbf{or}\, h(\mathbf{S}x)).$$

A simple computation shows that, if we do not constrain different copies (technically speaking residuals) of **or** to behave the same, then $Eval(Hk_0) = \{\perp, k_0, k_1, \ldots\}$, where $\perp$ means the possibility of diverging; on the other hand, if we add the constraint, we get simply $\{\perp, k_0\}$.

However this is not a typical problem of the higher order calculus **NPCF**; we can reproduce it within the framework of term rewriting systems. We cannot adopt here the same strategy to solve the problem, that is to delay all choices. Consider for example the term $(\mathbf{S}\,\mathbf{or}\,\mathbf{P})k_1$: then we connot delay the **or** reduction, since we would have in any case terms not in normal form, whose "approximated meaning", in the sense of the section 2.2, would be the undefined object.

2.5.2 Type-free non deterministic λ-calculus

Let us start by intoducing the syntax of the calculus. To rule out any possible confusion with the "parallel or", we shall use an entirely different symbol for explicit non-determinism.

Definition 29 *The set $\Lambda_\oplus$ of the terms of the non-deterministic λ-calculus is the least set s.t.*

i) $\Lambda \subseteq \Lambda_\oplus$,

ii) $M, N \in \Lambda_\oplus \Rightarrow M \oplus N \in \Lambda_\oplus$.

The set of closed terms is denoted by $\Lambda^0_\oplus$.

The terms are considered modulo α-congruence; clearly $\Lambda \subset \Lambda_\oplus$.

In the spirit of the reduction-oriented λ-calculi we shall consider the reduction relation as the primitive one; the equational theory we will discuss in the last chapter is the final product of the whole study, and not its starting point.

Following [Henn80, Sharma], we distinguish two parameter-passing mechanisms, namely two β-rules. Run-time-choice and call-time-choice are both call-by-name value passing mechanisms, although there is a similarity between call-time-choice and call-by-value, because in both cases there is a restriction to a proper subset of the whole set of terms: those terms that can be considered, in a sense, values.

Definition 30 *(Rules)*

(i) *Run-time choice*

(β_r) $(\lambda x.M)N \to M[N/x]$,
where $(M \oplus N)[L/x] \equiv M[L/x] \oplus N[L/x]$;

(ii) *Call-time choice*

(β_c) $(\lambda x.M)N \to M[N/x]$ if $N \in \Lambda$;

(iii)

(μ) $N \to N' \Rightarrow MN \to MN'$,
(ν) $M \to M' \Rightarrow MN \to M'N$,
(ξ) $M \to M' \Rightarrow \lambda x.M \to \lambda x.M'$,
(η) $\lambda x.Mx \to M$ *if* $x \notin FV(M)$,

$(\oplus.1)$ $M \to M' \Rightarrow M \oplus N \to M' \oplus N$,
$(\oplus.2)$ $N \to N' \Rightarrow M \oplus N \to M \oplus N'$,
$(\oplus.3)$ $M \oplus N \to M$,
$(\oplus.4)$ $M \oplus N \to N$.

The rules (β_c) and (β_r) yield two different reduction relations, namely $\longrightarrow_c$ and $\longrightarrow_r$. The constraint to disallow different behaviour of residuals of the same non-deterministic subterm, that is containing at least an occurrence of $\oplus$, is actually equivalent to force the choice before the substitution caused by some β-contraction. Hence it seems to us that the same effect of the (rather complex) construction in [Ast-Co], is simply catched by the β_c rule.

We do not insist about this constrained calculus, since we prove in the last chapter, that one gets an extension, and a conservative one, even for the case of the run-time-choice calculus. This is surprising, and is essentially due to the absence of types and constants in the pure calculus we are interested in. By the $\longrightarrow$ symbol we will mean the run-time-choice reduction relation.

Definition 31 *Consider the terms*

$$\lambda x_1 \ldots x_n.\underline{(\lambda y.P)Q}M_1 \ldots M_m, \quad \lambda x_1 \ldots x_n.\underline{(P \oplus Q)}M_1 \ldots M_m :$$

the underlined subterms are called head redexes*; when a head redex is contracted the reduction is called a* head reduction *and it is written* $\longrightarrow_h$*; it is called* internal reduction *otherwise, written* $\longrightarrow_i$.

Anticipating a result to be proved in the next chapter we claim:

Theorem 6 *For any $M, N \in \Lambda_\oplus$, if $M \xrightarrow{*} N$, then there exists an $L \in \Lambda_\oplus$ such that*

$$M \xrightarrow{*}_h L \xrightarrow{*}_i N.$$

As we have just seen in a previous section, in the classical λ-calculus a term has no meaning when it doesn't reduce to a head normal form, that is when it is *unsolvable*. In the present extension, two possible generalizations suggest themselves: the first one says that a term is solvable iff it reduces to a head normal form (we would call it *may convergency*): a study using a similar notion is e.g. [Boud91]. The second one defines a term to be solvable iff it has no infinite head reduction: we call this *must convergency* and write $M{\downarrow}_{must}$ or simply $M{\downarrow}$. Beside any other justification for choosing the latter notion as a research topic, we will show that it naturally leads to a conservative extension of the well known sensible theory $\mathcal{H}^*$.

Inspired by the extensional equivalence of Morris [Morris] and its analogue by Wadsworth [Wads] and by the idea of testing given by De Nicola and Hennessy [DeN-Henn] for process algebras, we define the following notions (see also [Jag-Pan]):

Definition 32 *For $M \in \Lambda_\oplus$ define*

i) $M{\downarrow}_{must} \Leftrightarrow M$ *has no infinite head reduction,*

ii) $M \sqsubseteq_{must} N \Leftrightarrow \forall C[\,].\ C[M]{\downarrow}_{must} \Rightarrow C[N]{\downarrow}_{must}$,

iii) $M \simeq_{must} N \Leftrightarrow M \sqsubseteq_{must} N \sqsubseteq_{must} M$.

We write $M{\uparrow}$ to mean not $M{\downarrow}$.

It should be noted that $\sqsubseteq_{must}$ is a preorder, so that, taking the quotient under $\simeq_{must}$, we get an order which is a precongruence.

Furthermore, such order is sensitive to the choice structure of a term, as the following example shows.

Example 5 let $M \equiv \lambda x.x(y \oplus z)$ and $N \equiv (\lambda x.xy) \oplus (\lambda x.xz)$, and consider the context $C[\,] \equiv (\lambda yz.[\,])\Delta H_0 H_1$, with $H_0 \equiv \lambda x.x\mathbf{U}^3_3\Delta$ and $H_1 \equiv \lambda xy.yy$, $\Delta \equiv \lambda x.xx$, $\mathbf{U}^3_3 \equiv \lambda x_1x_2x_3.x_3$; a simple computation shows that $C[M]{\uparrow}$ while $C[N]{\downarrow}$; hence $N \not\sqsubseteq_{must} M$: it will be proved that $M \sqsubseteq_{must} N$.

We observed in a previous section, that constraining the calculus to have uniform behaviour of residuals of the same subterm, may result in a calculus where each term has at most a finite set of normal forms. One may wonder whether the same problem doesn't arise in the case of our must convergency notion. We show that this is not the case in the following example.

Example 6 We can reproduce in pure calculus, using any numeric system, the example 4. It is easily seen that, since $H\mathbf{0}$ has an infinite head reduction, it is diverging from the point of view of must convergency. On the other hand this is not the case for the combinator satisfying

$$H'x \xrightarrow{*} x \oplus \mathbf{Succ}(Hx)$$

for which it can be shown that $H'\mathbf{0}$ has the same set of normal forms that $H\mathbf{0}$.

Finally, we note that, if $M \xrightarrow{*} N$ then $M \sqsubseteq_{must} N$, i.e. the order increases under reduction.

Chapter 3

Operational semantics

3.1 The standardization theorem

In the theory of reduction of the classical λ-calculus there are two fundamental theorems: the Church-Rosser theorem and the standardization theorem. The first one is the main tool for establishing consistency of β-conversion relation, which coincides with the equality in the theory λ; the second one plays a crucial role in the study of the algebraic semantics of the calculus.

In the case of the extensions of the λ-calculus studied in this thesis Church-Rosser theorem clearly fails. It will be shown in this section, however, that a standardization theorem still holds. This will provide the basis for the development of the operational semantics.

3.1.1 Residuals, Developments and Standard Reductions

To define the notion of standard reduction some machinery is needed, basically to keep track of the redexes and of the order in which they are contracted.

Given $M \in \Lambda_{\oplus}$, $\Delta \in M$ means that Δ is a redex occurrence in M; similarly, if $\mathcal{F} = \{\Delta_1, \ldots, \Delta_n\}$, then $\mathcal{F} \subseteq M$ means that $\Delta_i \in M$, for all $1 \leq i \leq n$. Finally, suppose the (binary) syntactical tree of each term $M \in \Lambda_{\oplus}$ labelled with strings in $\{0, 1\}$ in the usual way: then by M/u, for $u \in \{0, 1\}^*$, is meant the subterm of M rooted at u.

To be precise a redex occurrence is a couple $\langle \Delta, u \rangle$ formed by a redex and by the label of the node where it occurrs in a term M: this will be understood without any special notation.

Definition 33 *If $\Delta_1, \Delta_2 \in M$, with $\Delta_1 \equiv M/u$ and $\Delta_2 \equiv M/v$; then define*

$$\Delta_1 \leq \Delta_2 \;\Leftrightarrow\; u \leq_{lex} v,$$

where $\leq_{lex}$ is the lexicographic ordering. Now Δ_1/Δ_2 is the set of residuals *of Δ_1 after contracting Δ_2, defined as the following set of redex occurrences in M:*

i) $u = v \Rightarrow \Delta_1/\Delta_2 = \emptyset$;

ii) $\Delta_1 < \Delta_2$ *or* $\Delta_2 < \Delta_1$ *and* $\Delta_1 \not\in \Delta_2$ *and* $\Delta_2 \not\in \Delta_1 \;\Rightarrow\; \Delta_1/\Delta_2 = \{\Delta_1\}$;

iii) $\Delta_2 \in \Delta_1 \;\Rightarrow\; \Delta_1/\Delta_2 = \{\Delta'_1\}$, *where Δ'_1 is obtained from Δ_1 replacing Δ_2 with its contractum;*

iv) $\Delta_1 \in \Delta_2$, *then there are three subcases:*

a) $\Delta_2 \equiv (\lambda x.P)Q,\ \Delta_1 \in P \;\Rightarrow\; \Delta_1/\Delta_2 = \{\Delta_1[Q/x]\}$;

b) $\Delta_2 \equiv (\lambda x.P)Q,\ \Delta_1 \in Q \;\Rightarrow\; \Delta_1/\Delta_2 = \{\Delta^1, \ldots \Delta^r\}$ *where each Δ^i is a copy of Δ_1 and r is the number of occurrences of x in P;*

c) $\Delta_2 \equiv P \oplus Q,\ \Delta_1 \in P \;\Rightarrow\; \Delta_1/\Delta_2 = \{\Delta_1\}$, *if Δ_2 reduces to P; $\Delta_1/\Delta_2 = \emptyset$ otherwise; the case $\Delta_1 \in Q$ is similar.*

The concept of residuals, introduced above in the case of one step reductions, can be extended to any reduction sequence σ.

Notation: suppose that $\sigma : M \stackrel{*}{\longrightarrow} N$ is any finite reduction sequence; then it has the form:

$$\sigma : M \equiv M_0 \stackrel{\Delta_1}{\longrightarrow} M_1 \stackrel{\Delta_2}{\longrightarrow} M_2 \cdots \stackrel{\Delta_n}{\longrightarrow} M_n \equiv N,$$

for $n \geq 0$; write $\sigma = \Delta_1 + \cdots + \Delta_n$; furthermore with $\sigma_{i,j} : M_i \stackrel{*}{\longrightarrow} M_j$ is meant the subreduction of σ from M_i to M_j. Finally $|\sigma| = n$ is the length of σ.

Definition 34 *Suppose* $\Delta \in M$ *and* $\sigma = \Delta_1 + \cdots + \Delta_n$*, then the set of* residuals of Δ modulo σ*, written* Δ/σ*, is defined inductively:*

$$\begin{array}{rcl} \sigma = \Delta_1 & \Rightarrow & \Delta/\sigma = \Delta/\Delta_1, \\ \sigma = \Delta_1 + \sigma' & \Rightarrow & \Delta/\sigma = \bigcup\{\Delta'/\sigma' \mid \Delta' \in \Delta/\Delta_1\}. \end{array}$$

The idea behind residuals is that if $\Delta \in \Delta'/\sigma$, then Δ has not been created by σ. Now given any $\mathcal{F} \subseteq M$, it will be useful to consider reductions, starting in M, never contracting redexes not in $\mathcal{F}$ or among their residuals.

Definition 35 *Let* $\mathcal{F} \subseteq M$ *and* σ *a (finite or infinite) reduction starting in* M*; then:*

i) σ *is a* development *of* $\mathcal{F}$ *iff*

$$\forall i <|\sigma| \ \exists \Delta \in \mathcal{F}.\ \Delta_{i+1} \in \Delta/\sigma_{0,i};$$

ii) σ *is a* complete development *of* $\mathcal{F}$ *iff it is a finite development of* $\mathcal{F}$*, and*

$$\mathcal{F}/\sigma =_{def} \bigcup\{\Delta/\sigma \mid \Delta \in \mathcal{F}\} = \emptyset.$$

Notation: To mean that $\sigma : M \xrightarrow{*} N$ is a development or a complete development of $\mathcal{F}$ it will be written

$$\sigma : M \xrightarrow[dev]{\mathcal{F}} N, \qquad \sigma : M \xrightarrow[dev]{\mathcal{F}}_{cpl} N$$

respectively.

It is now possible to introduce the central notion of standard reduction.

Definition 36 *Suppose* $\sigma : M \xrightarrow{*} N$ *be any reduction, then* σ *is* standard *iff*

$$\forall i, j \leq|\sigma| \ .\ i < j \ \Rightarrow\ \neg\exists\Delta \in M_i.\ \Delta \leq \Delta_i \ \wedge\ \Delta_j \in \Delta/\sigma_{i,j}.$$

We write $\sigma : M \longrightarrow_s N$ *to mean that* σ *is standard.*

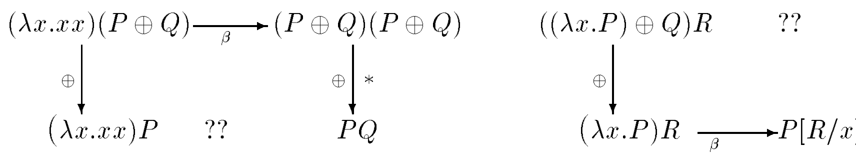
In words a reduction is standard iff a residual of a redex whose main operator occurs to the left of the main operator of a contracted redex is never contracted thereafter.

3.1.2 Proving the Standardization Theorem

The theorem to be proved is as follows:

Standardization Theorem
For all $M, N \in \Lambda_{\oplus}$. $M \xrightarrow{*} N \Rightarrow M \longrightarrow_s N$.

To achieve this result usually one looks for the possibility of performing the "same" contractions, although in a different order. In the present case some difficulties arise: first, as noted above, the Church-Rosser theorem doesn't hold; second in general one cannot permute β-reductions with $\oplus$-reductions:



indeed, the figure above shows that a β-contraction can multiply $\oplus$-redexes (left), hence the number of possible choices, and that $\oplus$-contractions may create new β-redexes (right). However, if the choice does not delete a β-redex, whose residual is contracted thereafter, then these reduction steps commute.

Formally, let c be a new constant, and $\Lambda(c)$ the language obtained from Λ by adding this constant and closing under term formation rules; define a map $\nu : \Lambda_{\oplus} \rightarrow \Lambda(c)$ by

$$\begin{array}{rcl} \nu(x) & \equiv & x, \\ \nu(MN) & \equiv & \nu(M)\nu(N), \\ \nu(\lambda x.M) & \equiv & \lambda x.\,\nu(M), \\ \nu(M \oplus N) & \equiv & c\,\nu(M)\nu(N). \end{array}$$

Clearly ν is one-to-one.

Lemma 1

i) $\forall M, N \in \Lambda_{\oplus}$. $\nu(M[N/x]) \equiv \nu(M)[\nu(N)/x]$;

ii) $\forall M, N \in \Lambda_{\oplus}$. $M \xrightarrow{\Delta}_{\beta} N \Leftrightarrow \nu(M) \xrightarrow{\nu(\Delta)}_{\beta} \nu(N)$;

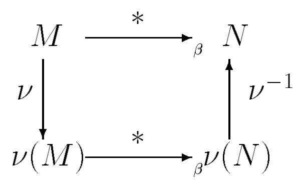


Figure 3.1:

iii) $M \xrightarrow{*}_\beta N \;\Leftrightarrow\; \nu(M) \xrightarrow{*}_\beta \nu(N)$.

Proof. Part (i) follows by a straightforward induction on M; to see (ii) just note that

$$\begin{array}{rcll} \nu((\lambda x.P)Q) & \equiv & (\lambda x.\nu(P))\nu(Q) & \text{by definition} \\ & \rightarrow_\beta & \nu(P)[\nu(Q)/x] & \\ & \equiv & \nu(P[Q/x]) & \text{by part (i)} \end{array}$$

The rest follows from the closure of $\rightarrow_\beta$ with respect to contexts. Finally part (iii) is immediate consequence of the first two: it is illustrated in the figure 3.1.

□

Corollary 2

i) $\forall M, M_1, M_2 \in \Lambda_\oplus.\ M_1 \ {}_\beta\!\xleftarrow{*}\ M \xrightarrow{*}_\beta M_2 \Rightarrow \exists M_3.\ M_2 \xrightarrow{*}_\beta M_3 \ {}_\beta\!\xleftarrow{*}\ M_2$, *i.e.* β *is* CR*;*

ii) if $\mathcal{F}$ *is a set of* β*-redexes of* $M \in \Lambda_\oplus$*, then every development of* $\mathcal{F}$ *is finite, and it ends in the same term.*

Proof. By lemma 1 and CR and FD! theorems of classical λ-calculus.

□

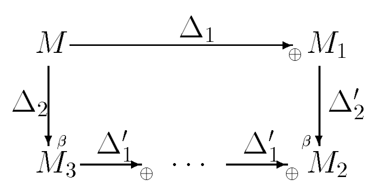
What the above corollary says is that the "inactive" presence of the choice operator preserves properties of the reduction in the classical case. Now this fact, toghether with the observation about the possibility of reordering contractions under certain conditions, will be used to prove that each development can be standardized. This will be done along the following steps:

- first it is established that each development of an $\mathcal{F}$ can be transformed into the development of the β-redexes in $\mathcal{F}$ followed by the development of the $\oplus$-redexes in $\mathcal{F}$;
- second, the devolopment of the β-redexes can be standardized using the classical theorem, while the development of the $\oplus$-redexes can easily reordered to get a leftmost outhermost reduction;
- finally these two parts will be rearranged to get the desired standard development.

Lemma 2 *Given $M \in \Lambda_\oplus$, for any $\Delta_1, \Delta_2 \in M$, with Δ_1 a $\oplus$-redex, and Δ_2 a β-redex, if $M \xrightarrow{\Delta_1} M_1$ and $M_1 \xrightarrow{\Delta_2'} M_2$, where $\Delta_2' \in \Delta_2/\Delta_1$, then there exists an M_3 such that:*

$$M \xrightarrow{\Delta_2}_\beta M_3 \quad \textit{and} \quad M_3 \xrightarrow{\Delta_1'}_\oplus \cdots \xrightarrow{\Delta_1'}_\oplus M_2,$$

where $\Delta_1' \in \Delta_1/\Delta_2$, that is the following diagram commutes:



Proof. Let $\Delta_1 \equiv P \oplus Q$ and $\Delta_2 \equiv (\lambda x.R)T$. Then there are three relevant cases:

Case 1: Δ_1 contains Δ_2; then say $Q \equiv D[\Delta_2]$ and $M \equiv C[P \oplus Q]$. It follows that

$$M_1 \equiv C[D[(\lambda x.R)T]] \text{ and } M_2 \equiv C[D[R[T/x]]]$$

then

$$M_3 \equiv C[P \oplus D[R[T/x]]] \text{ and } M_3 \to_\oplus M_2 \text{ in one step.}$$

Case 2: Δ_2 contains Δ_1 in the body of its functional subterm, that is $R \equiv D[\Delta_1]$; now $M \equiv C[(\lambda x.D[P \oplus Q])T]$ so that say

$$M_1 \equiv C[(\lambda x.D[P])T] \text{ and } M_2 \equiv C[D[P][T/x]]$$

hence

$$M_3 \equiv C[D[P \oplus Q][T/x]] \text{ and } M_3 \to_\oplus M_2 \text{ in one step.}$$

Case 3: Δ_2 contains Δ_1 in its argument subterm, that is $T \equiv D[\Delta_1]$; in this case $M \equiv C[(\lambda x.R)D[P \oplus Q]]$ and

$$M_1 \equiv C[(\lambda x.R)D[P]] \text{ and } M_2 \equiv C[R[D[P]/x]],$$

then, taking $M_3 \equiv C[R[D[P \oplus Q]/x]]$, it follows that $M_3 \xrightarrow{*}_{\oplus} M_2$ in k steps, where k is the number of occurrences of x in R.

□

Proposition 11 *Let $\mathcal{F} \subseteq M$ be a set of redex occurrences in M, with $\mathcal{F} = \mathcal{F}_\beta \cup \mathcal{F}_\oplus$ where $\mathcal{F}_\beta$ and $\mathcal{F}_\oplus$ are respectively the β- and $\oplus$-redexes in $\mathcal{F}$; then every development of $\mathcal{F}$ is finite; moreover*

$$M \xrightarrow[dev]{\mathcal{F}} N \Rightarrow \exists M'.\ M \xrightarrow[dev]{\mathcal{F}_\beta} M' \xrightarrow[dev]{\mathcal{F}_\oplus} N.$$

The term in which each development ends is in general not unique.

Proof. Let

$$\sigma : M \xrightarrow{\Delta_1} M_1 \xrightarrow{\Delta_2} M_2 \xrightarrow{\Delta_3} \cdots$$

be an infinite development of $\mathcal{F}$; clearly if a term contains k occurrences of the symbol $\oplus$, then at most k $\oplus$-reductions are possible without contracting some β-redex; so σ contains an infinite number of β-reductions. Let i be the least index such that Δ_i is a β-redex: by repeated applications of lemma 2, we know that there exists an M' such that

$$M \xrightarrow{\Delta_i}_\beta M' \xrightarrow{\{\Delta_1, \ldots, \Delta_{i-1}\}/\Delta_i}_\oplus M_i$$

It follows that we can build an infinite development of $\mathcal{F}_\beta$, contradicting corollary 2.

□

Lemma 3 *Given $M \in \Lambda_\oplus$, for any $\Delta_1, \Delta_2 \in M$, with Δ_1 a β-redex and Δ_2 a $\oplus$-redex, if $\Delta_2 < \Delta_1$ and*

$$M \xrightarrow{\Delta_1}_\beta M_1 \xrightarrow{\Delta_2'}_\oplus M_2$$

for some M_1 and M_2, where $\Delta_2' \in \Delta_2/\Delta_1$ then there is M_3 s.t.

$$M \xrightarrow{\Delta_2}_{\oplus} M_3 \xrightarrow{\Delta_1'}_{\beta} M_2$$

where $\Delta_1' \in \Delta_1/\Delta_2$; more precisely $M_3 \xrightarrow{\Delta_1'}_{\beta} M_2$ either in one or in zero steps (i.e. $M_3 \equiv M_2$).

Proof. By cases: just note that the condition $\Delta_2 < \Delta_1$ rules out the situation pictured in figure at the beginning of this subsection, left.

□

Lemma 4 *Given $M \in \Lambda_{\oplus}$, for any $\Delta_1, \Delta_2, \Delta_3 \in M$, if $\Delta_1 < \Delta_2 < \Delta_3$ and $M \xrightarrow{\Delta_2} N$ for some N, then $\Delta_3/\Delta_2 \neq \emptyset$ implies that Δ_1/Δ_2 has just one element, say Δ, and for any $\Delta' \in \Delta_3/\Delta_2$, $\Delta < \Delta'$.*

Proof. Immediate: the only possibility for Δ_2 to alter the relative order of Δ_1 and Δ_3 is when Δ_2 is a β-redex of the form $(\lambda x.D[\Delta_1])E[\Delta_3]$, and $M \equiv C[\Delta_2]$: but this contradicts the hypothesis.

□

Lemma 5 *If $\sigma : M \to_s N$ is a β-reduction and $N \to_{\oplus} N'$ contracting a redex which is a residual Δ' of a redex $\Delta \in M$ to the left of the first redex contracted in σ, then there is an M' s.t.*

$$M \xrightarrow{\Delta}_{\oplus} M' \xrightarrow{*}_{\beta\oplus} N'$$

is standard and contracts the redexes in σ in the same order, possibly omitting some of them.

Proof. By induction on the length $|\sigma|$ of σ. If $|\sigma| = 0$ the lemma is trivial; otherwise if $|\sigma| = n + 1$, then σ has the form

$$\sigma : M \xrightarrow{\Delta_1}_{\beta} M_1 \xrightarrow{\Delta_2}_{\beta} M_2 \cdots \xrightarrow{\Delta_n}_{\beta} M_n \xrightarrow{\Delta_{n+1}}_{\beta} M_{n+1} \equiv N \xrightarrow{\Delta'}_{\oplus} N'$$

Using inductively lemma 4 one sees that Δ_{n+1} is a residual of a $\Delta_{n+1}' \in M$ such that $\Delta < \Delta_{n+1}'$; it follows that there is a $\Delta'' < \Delta_{n+1} \in M_n$ of which Δ' is the (unique) residual. Now, by lemma 3 one gets

$$\sigma' : M \xrightarrow{\Delta_1}_\beta M_1 \cdots \xrightarrow{\Delta_n}_\beta M_n \xrightarrow{\Delta''}_\oplus M'_{n+1} \xrightarrow{\Delta_{n+1}/\Delta''}_\beta N'$$

But the reduction up to M_n is still a standard β-reduction, satisfing all the hypothesis in the lemma: hence the inductive hypothesis applies.

□

Lemma 6 (**Main lemma**) *Given $M \in \Lambda_\oplus$ and a subset $\mathcal{F}$ of its redexes,*

$$M \xrightarrow[dev]{\mathcal{F}} N \Rightarrow M \to_s N$$

Proof. By proposition 11 there is an M' s.t.

$$M \xrightarrow[\text{dev}]{\mathcal{F}_\beta} M' \xrightarrow[\text{dev}]{\mathcal{F}_\oplus} N;$$

the standardization theorem of the classical λ-calculus and corollary 2 imply that there exists an M' s.t. $M \to_s M'$: call this reduction σ_1. On the other hand it easily seen that any $\oplus$-reduction can be done in leftmost outhermost order (actually, when only $\oplus$-redexes are involved any change in the order in the reduction can only add some useless contraction): so suppose that $\sigma_2 : M' \xrightarrow{*}_\oplus N$ is such. Now let $\sigma = \sigma_1 + \sigma_2$ be the reduction

$$\underbrace{M \xrightarrow{\Delta_1}_\beta \cdots \xrightarrow{\Delta_n}_\beta M_n}_{\sigma_1} \equiv M' \equiv \underbrace{M'_0 \xrightarrow{\Delta'_1}_\oplus \cdots \xrightarrow{\Delta'_m}_\oplus M'_m}_{\sigma_2} \equiv N.$$

If for some $j \leq n$ and some $k \leq m$, Δ'_k is a residual of a redex Δ s.t. $\Delta < \Delta_j \in M_j$, then the occurrence of Δ is to the left of any accurrence touched by the β-reduction after M_j; but σ_2 is leftmost outermost, hence the occurrence of Δ'_1 in M' is to the left of that of say Δ''_k, of which Δ'_k is the (unique) residual; it follows that it is a residual of a redex to the left of Δ_j in M_j. Now if σ is not standard, then there is such a Δ_j occurring in M_j to the left of a redex Δ of which Δ'_1 is a residual; by lemma 5 we can perform the first step in σ_2 before the j-th step in σ_1, getting a reduction σ', possibly shorter than σ, which is standard up to the first step of the remaining part of σ_2. The result follows repeating exhaustively this transformation.

□

Using the previous lemmas, the standardization theorem can be proved along the pattern of Mitschke's proof of the same theorem in the classical case (see [Bar]).

Theorem 7 *For all* $M, N \in \Lambda_{\oplus}$. $M \xrightarrow{*} N \Rightarrow M \longrightarrow_s N$

Proof. Firstly, if $M \longrightarrow_i M' \longrightarrow_h N$ then, for some M'', $M \xrightarrow{*}_h M'' \xrightarrow{*}_i N$: this follows from classical results of λ-calculus and corollary 2 if both the contracted redexes are β-redexes, from lemma 2 and lemma 3, if they are a β and a $\oplus$-redex, and it is trivially true if they are both $\oplus$-redexes. Using inductively this fact one concludes that

$$M \longrightarrow_i M' \xrightarrow{*}_h N \;\Rightarrow\; \exists M''.\, M \xrightarrow{*}_h M'' \xrightarrow{*}_i N. \tag{3.1}$$

Secondly, if $M \xrightarrow{*} N$, then this reduction is of the form (say)

$$M \longrightarrow_i M_1 \longrightarrow_h M_2 \longrightarrow_h M_3 \cdots \longrightarrow_i M_n \equiv N;$$

now applying repeatedly 3.1 one sees that

$$M \xrightarrow{*} N \;\Rightarrow\; \exists M'.\, M \xrightarrow{*}_h M' \xrightarrow{*}_i N. \tag{3.2}$$

Finally, suppose that $M \xrightarrow{*}_h M' \xrightarrow{*}_i N$, then, by induction on the length of N, if $N \equiv x$ (base case), then there is noting to prove; otherwise $N \equiv \lambda x_1 \ldots x_n.N_1 \ldots N_m$; since the reduction from M' to N' is internal, it follows that $M' \equiv \lambda x_1 \ldots x_n.M_1 \ldots M_m$ and that $M_i' \xrightarrow{*} N_i$ for all $1 \leq i \leq m$; by inductive hypothesis there exist the standard reductions $\sigma_i : M_i' \xrightarrow{*}_s N_i$, for each i: calling $\sigma : M \xrightarrow{*}_h M'$, one concludes that the reduction $\sigma + \sigma_1 + \cdots + \sigma_m$ is standard.

□

The important consequence of this theorem is actually 3.2, i.e. that any reduction can be transformed into another one consisting of some head reductions followed by internal reductions only.

3.2 Non deterministic Böhm trees for Must-semantics

In order to represent the functional behaviour of each term, we introduce a kind of unfolding trees, generalizing the notion of Böhm trees of the classical λ-calculus (for this notion see [Bar]), and a suitable notion of approximation, to be considered as a cut of the tree.

The difficulty here is that, since the Church-Rosser property doesn't hold, we cannot consider our trees as a representation of the directed set of "approximated" reducts of a term: instead, we have to take into account all possible reductions, without making the choices, but simply representing all of them in the tree.

Definition 37 *We define, by mutual induction, two sets $\mathcal{S}_0$ and $\mathcal{S}_1$:*

i) $\Omega \in \mathcal{S}_0$, *where* Ω *is a new constant representing divergency;*

ii) $M \in \mathcal{S}_0 - \{\Omega\} \Rightarrow \lambda x.M \in \mathcal{S}_0$*;*

iii) $\{\Omega\} \in \mathcal{S}_1$*;*

iv) $M_1, ..., M_n \in \mathcal{S}_0 - \{\Omega\} \Rightarrow \{M_1, ..., M_n\} \in \mathcal{S}_1$ *for* $n > 0$*;*

v) $\mathcal{M}_1, ..., \mathcal{M}_m \in \mathcal{S}_1, x \in Var \Rightarrow x\mathcal{M}_1...\mathcal{M}_m \in \mathcal{S}_0$ *for* $m \geq 0$*.*

The idea behind this notation is that an approximation is a finite set in $\mathcal{S}_1$ of elements in $\mathcal{S}_0$; this comes out from the fact that, if $M\downarrow_{must}$, then there is a finite set of hnf we derive from it: this the first level, and we go on recursively with the bodies of these terms.

We consider in clause (v) the application of a variable rather than a set since we have in mind head normal forms: a set is a sum, and any term with a sum in head position is not an hnf. Similar remarks apply to clause (ii).

For every approximant, there is a term in $\Lambda_\oplus\Omega$ (i.e., $\Lambda_\oplus$ extended with the constant Ω) which corresponds to it in a natural way; we define, by mutual induction, $\vartheta_0 : \mathcal{S}_0 \to \Lambda_\oplus\Omega$ and $\vartheta_1 : \mathcal{S}_1 \to \Lambda_\oplus\Omega$:

$$\vartheta_0(\Omega) = \vartheta_1(\{\Omega\}) = \Omega, \qquad \vartheta_0(\lambda x.M) = \lambda x.\vartheta_0(M),$$
$$\begin{aligned} \vartheta_0(x\mathcal{M}_1 \dots \mathcal{M}_m) &= x\vartheta_1(\mathcal{M}_1)\dots\vartheta_1(\mathcal{M}_m); \\ \vartheta_1(\{M_1, \dots, M_n\}) &= \vartheta_0(M_1) \oplus \dots \oplus \vartheta_0(M_n). \end{aligned}$$

To simplify the notation, we will identify $\mathcal{M} \in \mathcal{S}_1$ with $\vartheta_1(\mathcal{M}) \in \Lambda_\oplus\Omega$.

Definition 38 *Let* $\mathcal{M} \in \mathcal{S}_1$*; rather informally, we define*

$$\mathrm{NBT}(\mathcal{M}) = \mathrm{NBT}_1(\mathcal{M}),$$

where $\mathrm{NBT}_0(\Omega) = \Omega$ *and*

$$\mathrm{NBT}_0(\lambda\vec{x}.\xi\mathcal{M}_1...\mathcal{M}_m) = \lambda\vec{x}.\xi \qquad \mathrm{NBT}_1(\{M_1,...,M_n\}) = \oplus$$

$$\mathrm{NBT}_1(\mathcal{M}_1) \cdots \mathrm{NBT}_1(\mathcal{M}_m) \qquad \mathrm{NBT}_0(M_1) \cdots \mathrm{NBT}_0(M_n)$$

Remark 2 Note that the order of the subnodes of a node labelled by $\lambda\vec{x}.\xi$ is relevant, as well as multiple occurrences of the same subtree; this is not the case for sons of nodes labelled by $\oplus$.

NBT's may be seen as infinite $\mathcal{S}_1$-terms, that is as elements of the completion of $\mathcal{S}_1$ under the order induced on $\mathcal{S}_1$ by the relation freely generated by $\Omega \preceq M$, for all M, on $\mathcal{S}_0$. More precisely, they are the limits of those directed subsets of $\mathcal{S}_1$ generated by terms in $\Lambda_\oplus$ with the following family of maps:

Definition 39 *For each natural number k, we define a map $\omega^k \colon \Lambda_\oplus \to \mathcal{S}_1$ by:*

i) $\omega^0(M) = \{\Omega\}$,

ii) *a)* $\omega^{k+1}(M) = \{\Omega\}$, *if M diverges; otherwise:*

b) $\omega^{k+1}(M) = \{\lambda\vec{x}.\xi\, \omega^k(M_1)...\omega^k(M_m) \mid \lambda\vec{x}.\xi M_1...M_m$ *is a principal head normal form of* $M\}$,

where an hnf is principal *when it is recahed by head reductions only.*

Furthermore, we denote

$$M^{[k]} = \vartheta_1 \circ \omega^k(M).$$

Remark 3 We note that $M^{[k]}$ is always a β-Ω-normal form (see [Bar]), where each $\oplus$-redex cannot create new β-Ω-redexes. We denote by $\mathrm{N}^\Omega_\oplus$ the set of such terms.

Example 7 Let $M \equiv \lambda x.x(yx) \oplus \lambda x.xz \ \in \ \Lambda_\oplus$. Then we have:

$$\begin{aligned}
\omega^1(M) &= \{\lambda x.x\ \omega^0(yx)\ ,\ \lambda x.x\ \omega^0(z)\} = \{\lambda x.x\{\Omega\}\},\\
\omega^2(M) &= \{\lambda x.x\ \omega^1(yx)\ ,\ \lambda x.x\ \omega^1(z)\} = \{\lambda x.x\{y\ \omega^0(x)\}\ ,\ \lambda x.x\{z\}\},\\
&= \{\lambda x.x\{y\{\Omega\}\}\ ,\ \lambda x.x\{z\}\},\\
M^{[1]} &= \lambda x.x\Omega,\\
M^{[2]} &= \lambda x.x(y\Omega) \oplus \lambda x.xz.
\end{aligned}$$

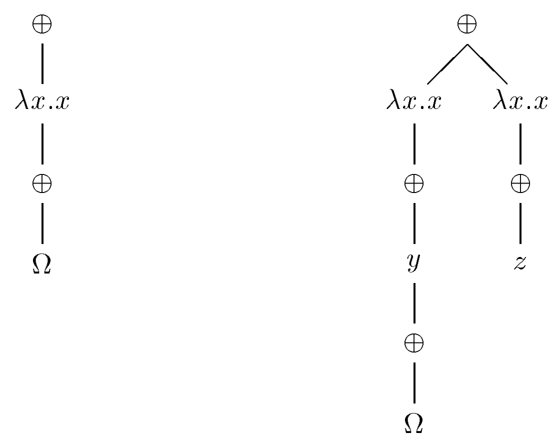


Figure 3.2: Non-deterministic Böhm trees

The trees $\mathrm{NBT}(\omega^1(M))$ and $\mathrm{NBT}(\omega^2(M))$ are shown in Figure 7.

To compare two terms, that is their trees, simple inclusion doesn't suffice even in the classical λ-calculus: what we need is a generalization of the relation ${}^{\eta}\sqsubseteq^{\eta}$ in [Bar], or, equivalently, of $<_k^s$ in [Hyl]: this will be achieved in several steps.

We first recall the notion of *equivalence* ($\sim$) for head normal forms [Böhm] (called *similarity* in [Hyl]).

Definition 40 *Given two classical hnf's*

$$M \equiv \lambda x_1 \ldots x_n.\xi M_1 \ldots M_m \quad \textit{and} \quad N \equiv \lambda x_1 \ldots x_{n'}.\zeta N_1 \ldots N_{m'}$$

(where m and m' are called degrees *of M and N)*

$$M \sim N \;\Leftrightarrow\; \xi \equiv \zeta \quad \textit{and} \quad n - m = n' - m'.$$

Since the relation $\sim$ is preserved by η-reduction and expansion, we note that, if $M \sim N$, then we can η-expand M and N in such a way that they have the same degree. The role of this relation in separating terms is illustrated in [Böhm, CDR]. This will suffice in the case that we find a set of terms pairwise not equivalent under the relation $\sim$.

In the general case, however, while comparing two terms, it is necessary to analyze their internal structure, which, in our case, encapsulates other sets of terms.

Definition 41 *Given* $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$*, define* $\mathsf{Pair}(\mathcal{M},\mathcal{N})$ *as the least set such that, if* $\mathcal{M}_{/\sim} = \{\mathcal{E}\} \cup \mathcal{M}'$ *and* $\mathcal{N}_{/\sim} = \{\mathcal{F}\} \cup \mathcal{N}'$*, where* $\mathcal{E} = \{M^1, \ldots, M^m\}$ *and* $\mathcal{F} = \{N^1, \ldots, N^n\}$ *and* $(\mathcal{E} \cup \mathcal{F})_{/\sim}$ *is a singleton, assuming*

$$\begin{array}{ll} M^i \equiv \lambda\vec{y}.x\mathcal{M}_1^i...\mathcal{M}_l^i, & \text{for } 1 \leq i \leq n, \\ N^j \equiv \lambda\vec{y}.x\mathcal{N}_1^j...\mathcal{N}_l^j, & \text{for } 1 \leq j \leq m, \end{array}$$

then, for each $p \leq l$*,*

$$\langle\{\mathcal{M}_p^1, ..., \mathcal{M}_p^n\}, \{\mathcal{N}_p^1, ..., \mathcal{N}_p^m\}\rangle \in \mathsf{Pair}(\mathcal{M},\mathcal{N}).$$

Given $\mathcal{M},\mathcal{N} \in \mathcal{S}_1$, $\mathsf{Pair}(\mathcal{M},\mathcal{N})$ selects the subterms to be compared during the first step of the analysis of the internal structure of $\mathcal{M}$ and $\mathcal{N}$. As in [Hyl], this notion has to be generalized to each level of the tree.

Definition 42 *Given* $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$*, define* $\mathsf{Pair}_k(\mathcal{M},\mathcal{N})$*, for each natural number* k*, as follows:*

i) $\mathsf{Pair}_1(\mathcal{M},\mathcal{N}) = \mathsf{Pair}(\mathcal{M},\mathcal{N})$.

ii) Let $\mathsf{Pair}(\mathcal{M},\mathcal{N}) = \{\langle\mathcal{U}_1, \mathcal{V}_1\rangle, \ldots, \langle\mathcal{U}_n, \mathcal{V}_n\rangle\}$ *and* $\mathcal{U}_i' = \bigcup\mathcal{U}_i$*,* $\mathcal{V}_i' = \bigcup\mathcal{V}_i$*, where* $1 \leq i \leq n$*: then*

$$\mathsf{Pair}_{k+1}(\mathcal{M}, \mathcal{N}) = \{\langle\mathcal{A}, \mathcal{B}\rangle \mid \exists i \leq n.\, \langle\mathcal{A}, \mathcal{B}\rangle \in \mathsf{Pair}_k(\mathcal{U}_i', \mathcal{V}_i')\}.$$

Remark 4 Note that in (ii), for $1 \leq i \leq n$, $\mathcal{U}_i$ is a finite non empty set of objects in $\mathcal{S}_1$, hence a family of finite non empty sets of objects in $\mathcal{S}_0$; it follows that its union $\mathcal{U}_i'$ is again an element of $\mathcal{S}_1$.

Example 8 Let $M \equiv x(y(a \oplus b)) \oplus xcd$ and $N \equiv x(ya \oplus yb) \oplus xcd$ (see next figure); we have

$$\begin{array}{lcl} \omega^3(M) & = & \{x\{y\{a,b\}\} \ , \ x\{c\}\{d\}\} \\ \omega^3(N) & = & \{x\{y\{a\},y\{b\}\} \ , \ x\{c\}\{d\}\}. \end{array}$$

From this we compute:

$$\begin{array}{lll} \mathsf{Pair}_1(\omega^3(M),\omega^3(N)) & = \{ & \langle\{\{y\{a,b\}\}\},\{\{y\{a\},y\{b\}\}\}\rangle, \\ & & \langle\{\{c\}\},\{\{c\}\}\rangle \ , \ \langle\{\{d\}\},\{\{d\}\}\rangle \quad \}; \\ \mathsf{Pair}_2(\omega^3(M),\omega^3(N)) & = & \mathsf{Pair}_1(\{y\{a,b\}\},\{y\{a\},y\{b\}\}) \cup \\ & & \mathsf{Pair}_1(\{c\},\{c\}) \cup \mathsf{Pair}_1(\{d\},\{d\}) \\ & = \{ & \langle\{\{a,b\}\},\{\{a\},\{b\}\}\rangle \quad \}. \end{array}$$

We are now ready to introduce the ordering relation $\leq$ over trees.

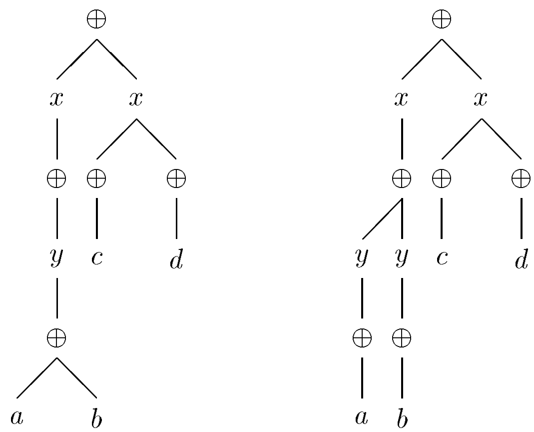


Figure 3.3: Respectively the trees of $\omega^3(M)$ and $\omega^3(N)$

Definition 43 *Given $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$ we define, for each k, a relation $\leq_k$ by:*

$$\begin{aligned} \mathcal{M} \leq_1 \mathcal{N} &\Leftrightarrow \mathcal{M} = \{\Omega\} \ \vee \ \forall N \in \mathcal{N}\ \exists M \in \mathcal{M}.\ M \sim N, \\ \mathcal{M} \leq_{k+1} \mathcal{N} &\Leftrightarrow \mathcal{M} \leq_k \mathcal{N} \wedge \\ &\quad\ \forall \langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_k(\mathcal{M}, \mathcal{N}). \mathcal{U} \sqsubseteq^\sharp \mathcal{V}, \end{aligned}$$

where

$$\mathcal{U} \sqsubseteq^\sharp \mathcal{V} \Leftrightarrow \forall \mathcal{N}_j \in \mathcal{V}\ \exists \mathcal{M}_i \in \mathcal{U}.\ \mathcal{M}_i \leq_1 \mathcal{N}_j.$$

From this we can define

$$\mathcal{M} \leq \mathcal{N} \Leftrightarrow \forall k.\ \ \mathcal{M} \leq_k \mathcal{N}.$$

Finally, for any M, $N \in \Lambda_\oplus$,

$$\begin{aligned} M \leq_k N &\Leftrightarrow \omega^k(M) \leq_k \omega^k(N), \\ M \leq N &\Leftrightarrow \forall k.\ \ M \leq_k N. \end{aligned}$$

Example 9 Consider the M and N of example 8. Now the following relations hold:

$$M \leq_i N \leq_i M \quad \text{for } i = 0, 1, 2 \quad .$$

Indeed

$$\omega^1(M) = \{x\{\Omega\}, x\{\Omega\}\{\Omega\}\} = \omega^1(N);$$

and

$$\omega^2(M) = \{x\{y\{\Omega\}\}, x\{c\}\{d\}\} = \omega^2(N).$$

But, looking at example 8, we see that in $\mathsf{Pair}_2(\omega^3(M).\omega^3(N))$ we have a (unique) pair formed by an $\mathcal{U} = \{\{a, b\}\}$ and by a $\mathcal{V} = \{\{a,\}, \{b\}\}$, such that

$$\mathcal{U} \sqsubseteq^\sharp \mathcal{V} \quad \text{and} \quad \mathcal{V} \not\sqsubseteq^\sharp \mathcal{U},$$

hence $M \leq_3 N \not\leq_3 M$.

3.3 The semiseparability theorem

To study head reduction and separability problems the following concepts are needed.

Definition 44

i) A context is a head context *iff it is of the form*

$$C[\,] \equiv (\lambda x_1 \dots x_n.[\,])X_1 \dots X_n U_1 \dots U_m;$$

ii) abbreviate $x_1 \dots x_n$ with $\vec{x}$, $X_1 \dots X_n$ with $\vec{X}$ and $U_1 \dots U_m$ with $\vec{U}$; similarly consider a context $D[\,] \equiv (\lambda y_1 \dots y_h.[\,])Y_1 \dots Y_h V_1 \dots V_k$, abbreviated $(\lambda \vec{y}.[\,])\vec{Y}\vec{V}$, then define

$$D \bullet C[\,] \equiv (\lambda \vec{x}\vec{z}.[\,])\vec{X}^\circ \vec{Z}\vec{U}^\circ \vec{V}$$

where $\circ = [\vec{Y}/\vec{y}]$, and $\vec{z} \equiv z_1 \dots z_r \equiv y_{i_1} \dots y_{i_r}$, if $\{y_{i_1} \dots y_{i_r}\} = \{y_1, \dots, y_h\} - \{x_1, \dots, x_n\}$ and $\vec{Z} \equiv Y_{i_1} \dots Y_{i_r}$.

Lemma 7 *Let $C[\,], D[\,]$ be head contexts, then, for any $M \in \Lambda_\oplus$, if $\omega^k(M) = \{M_1, \dots, M_l\}$, for $k \geq 1$:*

i)

$$\omega^k(C[M]) = \begin{cases} \omega^k(C[M_1]) \cup \dots \cup \omega^k(C[M_l]) & \text{if } C[M]\downarrow \\ \{\Omega\} & \text{otherwise} \end{cases}$$

ii) $\omega^k(D \bullet C[M]) = \omega^k(D[C[M]])$.

Proof. Let $C[\,] \equiv (\lambda x_1 \dots x_n.[\,])X_1 \dots X_n U_1 \dots U_m$, then

$$\begin{array}{rcl} C[M] & \xrightarrow{*}_h & M^* U_1 \dots U_m \\ & \xrightarrow{*}_h & M_i^* U_1 \dots U_m \\ & {}_h\xleftarrow{*} & C[M_i]; \end{array}$$

for $i = 1, \dots, l$, where $* = [X_1/x_1, \dots, X_n/x_n]$. It follows that L is a principal hnf of $C[M]$ iff for some $i \leq l$, L is a principal hnf of $C[M_i]$, establishing (i).

To prove (ii) let $D[\,] \equiv (\lambda\vec{y}.[\,])\vec{Y}\vec{V}$, and $C[\,]$ as above; then, reasoning as for (i), we have

$$\begin{array}{rcl} D[C[M]] & \equiv & (\lambda\vec{y}.(\lambda\vec{x}.M)\vec{X}\vec{U})\vec{Y}\vec{V} \\ & \xrightarrow{*}_h & (\lambda\vec{x}.M)^\circ \vec{X}^\circ \vec{U}^\circ \vec{V} \\ & \equiv & (\lambda\vec{x}.M^\diamond)\vec{X}^\circ \vec{U}^\circ \vec{V} \\ & \xrightarrow{*}_h & M^{\diamond\star}\vec{U}^\circ\vec{V} \\ & {}_h\xleftarrow{*} & (\lambda\vec{x}\vec{z}.M)\vec{X}^\circ \vec{Z}\vec{U}^\circ\vec{V} \\ & \equiv & D \bullet C[M], \end{array}$$

where $\circ = [\vec{Y}/\vec{y}]$, $\diamond = [\vec{Z}/\vec{z}]$, $\star = [\vec{X}^\circ/\vec{x}]$, and $\vec{Z}$, $\vec{z}$ are as in the definition 44.

□

Lemma 8 *Let $\mathcal{M} = \omega^k(M)$ for $k \geq 1$ and $M\!\downarrow$; suppose that $\mathcal{M}_{/\sim} = \{[M_1], \dots, [M_h]\}$, then*

i) for each i, with $1 \leq i \leq h$ there exists a head context $C_i[\,]$ and an integer r_i s. t. for each $L \in \mathcal{M}$

$$\omega^k(C_i[L]) = \begin{cases} \{x_i \mathcal{L}_1 \dots \mathcal{L}_{r_i}\} & \text{if } L \in [M_i] \\ \{y\} & \text{otherwise} \end{cases}$$

where y is any fixed variable;

ii) there exists a head context $C[\,]$ s.t.

$$\omega^k(C[M]) = \{z_1, \dots, z_h\},$$

where $\{z_1, \dots, z_h\}$ are new, pairwise distinct variables, and for each $i \leq h$

$$\omega^k(C[L]) = \{z_i\} \;\Leftrightarrow\; L \in [M_i].$$

Proof. By standard separability techniques (see [CDR]). □

Example 10 Let $M \equiv \lambda x_1x_2x_3.x_1x_3(x_2x_3) \oplus \lambda x_1x_2.x_1x_2x_2 \oplus \lambda x_1x_2x_3.x_1x_3x_2$ and $N \equiv \lambda x_1x_2.x_1x_1 \oplus \lambda x_1x_2x_3.x_1x_2$.

We have:

$$\begin{array}{rcl} \omega^2(M) & = & \{\lambda x_1x_2x_3.x_1\{x_3\}\{x_2\{\Omega\}\} \;,\; \lambda x_1x_2.x_1\{x_2\}\{x_2\} \;, \\ & & \lambda x_1x_2x_3.x_1\{x_3\}\{x_2\}\}; \\ \omega^2(N) & = & \{\lambda x_1x_2.x_1\{x_1\} \;,\; \lambda x_1x_2x_3.x_1\{x_2\}\}; \\ M^{[2]} & = & \lambda x_1x_2x_3.x_1x_3(x_2\Omega) \oplus \lambda x_1x_2.x_1x_2x_2 \oplus \lambda x_1x_2x_3.x_1x_3x_2; \\ N^{[2]} & = & N. \end{array}$$

It comes out that there exists a $\sim$-equivalence class of $\omega^2(M) \cup \omega^2(N)$ which does not contain any element of $\omega^2(M)$. In this case, we can immediately find a context such that $C[M]$ converges while $C[N]$ does not. Indeed, take $C[\,] \equiv [\,](\lambda a_1a_2a_3.a_1)x_2x_3x_4\Omega$, then

$$\begin{array}{rcl} \omega^2(C[M^{[2]}]) & = & \{x_4\{x_5\}, x_3\{x_4\}\{x_5\}\}, \\ \omega^2(C[N^{[2]}]) & = & \{\Omega\}. \end{array}$$

We know that λ-calculus encodes all recursive functions; this can be done in many different ways, choosing a suitable *numeral system*, called in [Bar] *adequate* iff all recursive functions are representable using that system. We do not study here problems connected with the representability of recursive functions; however we need, for technical reasons, Church's numeral system, which is as follows:

$$\mathbf{c}_0, \mathbf{c}_1, \ldots \quad \text{where} \quad \mathbf{c}_n \equiv \lambda f x.\underbrace{f(\ldots f}_{n}(x)\ldots);$$

defining $\mathbf{Succ} \equiv \lambda xyz.y(xyz)$ one easily sees that $\mathbf{Succ}\,\mathbf{c}_n =_\beta \mathbf{c}_{n+1}$ for each n; the system of Church numerals is adeguate. In the sequel we shall write $\mathbf{n}$ for $\mathbf{c}_n$.

It is well known that the test for equality for Church numerals is λ-definible; however we need to represent such a test with a combinator of a special shape, for reasons which will be clear in the sequel.

Lemma 9 *There exists a combinator* $\mathbf{H} \in \Lambda$ *of the shape*

$$\mathbf{H} \equiv \lambda xy.xH_1 \ldots H_l,$$

with $x \notin FV(H_1) \cup \ldots \cup FV(H_l)$*, such that, for all non-negative integers* n, m*:*

$$\mathbf{Hnm} =_{\beta\eta} \begin{cases} \mathbf{1} & \textit{if } n = m \\ \mathbf{0} & \textit{otherwise} \end{cases}$$

Proof. To build $\mathbf{H}$ we solve the following system of equations in the theory $\lambda\beta\eta$:

$$\begin{cases} \mathbf{H00} & = & \mathbf{1} \\ \mathbf{H0}(\mathbf{Succ}\, y) & = & \mathbf{0} \\ \mathbf{H}(\mathbf{Succ}\, x)\mathbf{0} & = & \mathbf{0} \\ \mathbf{H}(\mathbf{Succ}\, x)(\mathbf{Succ}\, y) & = & \mathbf{H}xy \end{cases}$$

We guess that $\mathbf{H} \equiv \lambda uv.uPQ(vRT)$, where the combinators P, Q, R, T will be specified later. We compute

$$\begin{array}{rcl} \mathbf{H00} & = & \mathbf{0}PQ(\mathbf{0}RT) \\ & = & QT; \\ \mathbf{H0}(\mathbf{Succ} y) & = & \mathbf{0}PQ(\mathbf{Succ}\, yRT) \\ & = & Q(R(yRT)); \\ \mathbf{H}(\mathbf{Succ} x)\mathbf{0} & = & \mathbf{Succ}\, xPQ(\mathbf{0}RT) \\ & = & P(xPQ)T; \\ \mathbf{H}(\mathbf{Succ} x)(\mathbf{Succ} y) & = & \mathbf{Succ}\, xPQ(\mathbf{Succ}\, yRT) \\ & = & P(xPQ)(R(yRT)). \end{array}$$

Now we choose

$$\begin{array}{rcl} P & \equiv & \lambda ab.b\mathbf{O}a \\ Q & \equiv & \lambda a.a\mathbf{K} \\ R & \equiv & \lambda ab.b(\mathbf{K0})\mathbf{C}_* a \\ T & \equiv & \lambda a.a\mathbf{1}(\mathbf{K0}) \end{array}$$

where $\mathbf{C}_* \equiv \lambda ab.ba$; it is straightforward to see that these choices give the desired result: in particular for the fourth equation we have

$$\begin{array}{rcl} P(xPQ)(R(yRT)) & = & R(yRT)\mathbf{O}(xPQ) \\ & = & \mathbf{O}(\mathbf{K0})\mathbf{C}_*(yRT)(xPQ) \\ & = & \mathbf{C}_*(yRT)(xPQ) \\ & = & xPQ(yRT) \\ & = & \mathbf{H}xy. \end{array}$$

□

Corollary 3 *If* $N \equiv \mathbf{n}_1 \oplus \ldots \oplus \mathbf{n}_r$, *with* $r \geq 1$, *then, for all* m,

$$\omega^1(\mathbf{H}N\mathbf{m}) = \begin{cases} \{\mathbf{1}\} & \text{if } r = 1 \text{ and } n_1 = m \\ \{\mathbf{0}\} & \text{if } \forall i \leq r.\, n_i \neq m \\ \{\mathbf{1}, \mathbf{0}\} & \text{if } r > 1 \wedge \exists i \leq r.\, n_i = m \ . \end{cases}$$

Proof. From the shape of $\mathbf{H}$ one sees that, when applied to closed terms, it behaves like the head context $\lambda y.[\,]H_1 \ldots H_l$; on the other hand the $=_{\beta\eta}$ in the lemma is actually $\longrightarrow_{\beta\eta}$, because the numerals $\mathbf{0}$ and $\mathbf{1}$ are normal forms. Since $\longrightarrow_{\beta\eta} \subseteq \longrightarrow_r$, and using the standardization theorem we have that each exaustive head reduction of $\mathbf{H}N\mathbf{m}$ has to start with

$$\begin{aligned} \mathbf{H}N\mathbf{m} &\longrightarrow_r NH_1[\mathbf{m}/y] \ldots H_l[\mathbf{m}/y] \\ &\longrightarrow_r \mathbf{n}_i H_1[\mathbf{m}/y] \ldots H_l[\mathbf{m}/y]; \end{aligned}$$

for some $1 \leq i \leq r$; now the corollary follows from lemma 9 and lemma 7.

□

Lemma 10 *For* $M, N \in \Lambda_\oplus$,

$$M \not\lesssim_2 N \ \Rightarrow\ \exists C[\,].\, C[M]\!\downarrow \ \wedge\ C[N]\!\uparrow .$$

Proof. By cases.

Case 1: when $M \not\lesssim_1 N$: then either $\omega^1(N) = \mathcal{N} = \{\Omega\}$ and $\omega^1(M) = \mathcal{M} \neq \{\Omega\}$, so that there is nothing to prove, or there exists $N' \in \mathcal{N}$ such that, for all $M' \in \mathcal{M}$, $M' \not\sim N'$. If $(\mathcal{M} \cup \mathcal{N})_{/\sim} = \{[L_1], \ldots, [L_h]\}$, then there is an i such that $N' \in [L_i]$ and $[L_i] \cap \mathcal{M} = \emptyset$; by lemma 8 (ii) we know that there is a context $C[\,]$ such that $\omega^1(C[N']) = \{z_i\}$ for some variable z_i, and, for all $M' \in \mathcal{M}$ there is a $j \neq i$ s.t. $\omega^1(C[M']) = \{z_j\}$, where $z_i \not\equiv z_j$; now let

$$C'[\,] \equiv (\lambda z_1 \ldots z_h.[\,])z_1 \ldots z_{i-1}\Omega z_{i+1} \ldots z_h;$$

it is a head context, hence by lemma 7 and by the fact that head contexts are closed under composition we conclude that $\omega^1(C' \bullet C[M])$ is a subset of $\{z_1, \ldots, z_h\} - \{z_i\}$, and that $\omega^1(C' \bullet C[N]) = \omega^1(C' \bullet C[N']) = \{\Omega\}$.

Case 2: when $M \leq_1 N$ but $M \not\leq_2 N$: then by definition we have

$$M \leq_1 N \wedge M \not\leq_2 N \;\Rightarrow\; \exists \langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_1(\mathcal{M}, \mathcal{N}) = \mathsf{Pair}(\mathcal{M}, \mathcal{N}).\, \mathcal{U} \not\sqsubseteq^\sharp \mathcal{V};$$

this means that, for some $[P] \in (\mathcal{M} \cup \mathcal{N})_{/\sim}$, letting $\mathcal{M}_{[P]} = [P] \cap \mathcal{M}$ and similarly $\mathcal{N}_{[P]} = [P] \cap \mathcal{N}$, we have

$$\langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}(\mathcal{M}_{[P]}, \mathcal{N}_{[P]}).$$

By lemma 8 (i) there exists a context $C[\,]$ and an integer r such that

$$\begin{array}{rcl} \omega^2(C[M]) & = & \omega^2(C[\mathcal{M} - \mathcal{M}_{[P]}]) \cup \omega^2(C[\mathcal{M}_{[P]}]) \\ & = & \{y\} \cup \{x\mathcal{M}_1^1 \ldots \mathcal{M}_r^1, \ldots, x\mathcal{M}_1^n \ldots \mathcal{M}_r^n\} \end{array}$$

and

$$\begin{array}{rcl} \omega^2(C[N]) & = & \omega^2(C[\mathcal{N} - \mathcal{N}_{[P]}]) \cup \omega^2(C[\mathcal{N}_{[P]}]) \\ & = & \{y\} \cup \{x\mathcal{N}_1^1 \ldots \mathcal{N}_r^1, \ldots, x\mathcal{N}_1^m \ldots \mathcal{N}_r^m\} \end{array},$$

so that, for some $1 \leq i \leq r$, it must be the case that

$$\mathcal{U} = \{\mathcal{M}_i^1, \ldots, \mathcal{M}_i^n\} \quad \text{and} \quad \mathcal{V} = \{\mathcal{N}_i^1, \ldots, \mathcal{N}_i^m\}.$$

In the sequel we assume that

$$\forall j \leq n.\; x \notin FV(\mathcal{M}_i^j) \wedge x \notin FV(\mathcal{N}_i^k):$$

there is no theoretical loss, since this is similar as in the classical λ-calculus, where one uses the permutators technique by Böhm to make these occurrences harmless (see [Bar]).

Since $\mathcal{U} \not\sqsubseteq^\sharp \mathcal{V}$, there exists $1 \leq k \leq m$ such that, for all $1 \leq j \leq n$, we have $\mathcal{M}_i^j \not\leq_1 \mathcal{N}_i^k$: then $\mathcal{M}_i^j \neq \{\Omega\}$ for all j. Now we have two subcases.

Subcase 2.1: $\mathcal{N}_i^k = \{\Omega\}$: it follows that, taking $C'[\,] \equiv (\lambda x.[\,])\mathbf{U}_i^r$ we have

$$\omega^2(C'[\{x\mathcal{M}_1^1 \ldots \mathcal{M}_r^1, \ldots, x\mathcal{M}_1^n \ldots \mathcal{M}_r^n\}]) = \{\mathcal{M}_i^1, \ldots, \mathcal{M}_i^n\},$$

while

$$\omega^2(C'[\{x\mathcal{N}_1^1 \ldots \mathcal{N}_r^1, \ldots, x\mathcal{N}_1^m \ldots \mathcal{N}_r^m\}]) = \{\Omega\};$$

it follows that $C' \bullet C[M]{\downarrow}$ and $C' \bullet C[N]{\uparrow}$.

Subcase 2.2: $\mathcal{N}_i^k \neq \{\Omega\}$: then for each $1 \leq j \leq n$ there is a $[Q_j] \in (\bigcup \mathcal{U} \cup \mathcal{N}_i^k)_{/\sim}$ s.t.

$$[Q_j] \subseteq \mathcal{N}_i^k - \mathcal{M}_i^j. \tag{3.3}$$

Using lemma 8 (ii) we know that there exists a head context $\tilde{C}[\,]$ trasforming $\bigcup \mathcal{U} \cup \mathcal{N}$ into a set of variables of the same cardinality as $(\bigcup \mathcal{U} \cup \mathcal{N}_i^k)_{/\sim}$: say $\{z_1, \ldots, z_h\}$. 3.3 now implies that

$$\forall j\, \exists l \leq h. z_l \in \omega^1(\tilde{C}[\mathcal{N}_i^k]) - \omega^1(\tilde{C}[\mathcal{M}_i^j]). \tag{3.4}$$

We define $\bar{C}[\,]$ as the composition $((\lambda z_1 \ldots z_h.[\,])\mathbf{1} \ldots \mathbf{h}) \bullet \tilde{C}[\,]$. Say that $|(\mathcal{N}_i^k)_{/\sim}|= l$: then we take

$$C'[\,] \equiv (\lambda x.[\,])(\lambda y_1 \ldots y_l.v \underbrace{\bar{C}[y_i] \ldots \bar{C}[y_i]}_{l}).$$

Set $\mathcal{X}^j = \omega^h(\bar{C}[\mathcal{M}_i^j])$ and $\mathcal{Y} = \omega^h(\bar{C}[\mathcal{N}_i^k])$; they are sets of Church numerals, and, by 3.4, no $\mathcal{X}^j$ contains all the numerals in $\mathcal{Y}$. Now

$$\begin{aligned} \omega^h(C' \bullet C[M]) &= \{y, v \underbrace{\mathcal{X}^1 \ldots \mathcal{X}^1}_{l}, \ldots v \underbrace{\mathcal{X}^n \ldots \mathcal{X}^n}_{l}\}, \\ \omega^h(C' \bullet C[N]) &= \{y, v \underbrace{\mathcal{Y} \ldots \mathcal{Y}}_{l}\}. \end{aligned}$$

Using the combinator $\mathbf{H}$ of lemma 9, we finally define

$$C''[\,] \equiv (\lambda v.[\,])(\lambda v_1 \ldots v_l.\mathbf{P}_l(\mathbf{H}v_1\mathbf{1}) \ldots (\mathbf{H}v_l\mathbf{l}))(\mathbf{K}\Omega)\mathbf{I},$$

where $\mathbf{P}_l$ λ-defines the numeric function $\prod_{i=1}^{l} n_l$. It follows that, by corollary 3,

$$\omega^1(C'' \bullet C' \bullet C[M]) = \{\mathbf{0}\}(\mathbf{K}\Omega)\mathbf{I} = \{\mathbf{I}\},$$

while

$$\begin{aligned} \omega^1(C'' \bullet C' \bullet C[N]) &= \begin{cases} \{\mathbf{0}, \mathbf{1}\}(\mathbf{K}\Omega)\mathbf{I} \\ \{\mathbf{1}\}(\mathbf{K}\Omega)\mathbf{I} \end{cases} \\ &= \{\Omega\} \end{aligned}$$

according to the numerals in $\mathcal{Y}$.

$\square$

Example 11 We exhibit an example of the case where $M \leq_1 N$ but $M \not\leq_2 N$. Take $M \equiv xy \oplus xz$ and $N \equiv x(y \oplus z)$; now $\omega^2(M) = \{x\{y\}, x\{z\}\}$, while $\omega^2(N) = \{x\{y,z\}\}$; computing $\mathsf{Pair}_1(\omega^2(M), \omega^2(N))$, we get

$$\{\langle\{\{y\},\{z\}\},\{\{y,z\}\}\rangle\}$$

and it can be verified that $\{\{y\},\{z\}\} \not\sqsubseteq^\sharp \{\{y,z\}\}$. We take

$$C_0[\,] \equiv (\lambda xyz.[\,])(\lambda w.aww)\mathbf{1}\,\mathbf{2}.$$

Simple calculations give us

$$\omega^2(C_0[M]) = \{a\{\mathbf{1}\}\{\mathbf{1}\}, a\{\mathbf{2}\}\{\mathbf{2}\}\}$$

and

$$\omega^2(C_0[N]) = \{a\{\mathbf{1},\,\mathbf{2}\}\{\mathbf{1},\,\mathbf{2}\}\}.$$

Taking

$$C_1[\,] \equiv (\lambda a.[\,])(\lambda uv.\mathbf{P}(\mathbf{H}u\mathbf{1})(\mathbf{H}v\mathbf{2})),$$

where $\mathbf{P}$ λ-defines multiplication, we have

$$\omega^2(C_1[C_0[M]]) = \{\mathbf{0}\} \quad \text{and} \quad \omega^2(C_1[C_0[N]]) = \{\mathbf{0},\,\mathbf{1}\}.$$

Now, taking $C_2[\,] \equiv [\,](\mathbf{K}\Omega)\mathbf{I}$, we have

$$\omega^2(C_2[C_1[C_0[M]]]) = \{\mathbf{I}\}$$

while

$$\omega^2(C_2[C_1[C_0[N]]]) = \{\Omega\}.$$

The following lemma extends to the present calculus the Böhm out lemma of the classical λ-calculus (see [Bar]).

Lemma 11 *For $M, N \in \Lambda_\oplus$ and $k \geq 2$,*

$$M \not\leq_k N \;\Rightarrow\; \exists C[\,].\, C[M] \not\leq_2 C[N].$$

Proof. By induction on k. The case $k = 2$ is trivial; suppose $k > 2$: then, letting $\mathcal{M} = \omega^k(M)$ and $\mathcal{N} = \omega^k(N)$, there are two subacses: either $\mathcal{M} \not\leq_{k-1} \mathcal{N}$, in which case the thesis follows directly from the inductive hypothesis, or $\mathcal{M} \leq_{k-1} \mathcal{N}$ and $\mathcal{M} \not\leq_k \mathcal{N}$. In the last case by definition we have that

$$\exists \langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_{k-1}(\mathcal{M}, \mathcal{N}).\, \mathcal{U} \not\sqsubseteq^\sharp \mathcal{V};$$

this implies the existence of $\langle \mathcal{U}', \mathcal{V}' \rangle \in \mathsf{Pair}_1(\mathcal{M}, \mathcal{N})$ such that $\langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_{k-2}(\bigcup \mathcal{U}', \bigcup \mathcal{V}')$; it follows that, for some $[P] \in (\mathcal{M} \cup \mathcal{N})_{/\sim}$ the sets $\mathcal{M}_{[P]}$ and $\mathcal{N}_{[P]}$, defined as in the proof of the previous lemma, are non empty and

$$\langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_{k-2}(\mathcal{M}_{[P]}, \mathcal{N}_{[P]}).$$

Using lemma 8, we can find a context $C[\,]$ "selecting" P, thus we get

$$\begin{array}{rcl} \omega^k(C[M]) & = & \{y, x\mathcal{M}_1^1 \ldots \mathcal{M}_r^1, \ldots, x\mathcal{M}_1^n \ldots \mathcal{M}_r^n\} \\ \omega^k(C[N]) & = & \{y, x\mathcal{N}_1^1 \ldots \mathcal{N}_r^1, \ldots, x\mathcal{N}_1^m \ldots \mathcal{N}_r^m\} \end{array}$$

and, for some $1 \leq i \leq r$,

$$\mathcal{U}' = \{\mathcal{M}_i^1, \ldots, \mathcal{M}_i^n\}, \qquad \mathcal{V}' = \{\mathcal{N}_i^1, \ldots, \mathcal{N}_i^m\}.$$

Again, w.l.o.g. we soppose that x doesn't occurr free in any term in $\mathcal{U}'$ or in $\mathcal{V}'$, and we choose $C'[\,] \equiv (\lambda x.[\,])\mathbf{U}_i^r$; so that by lemma 7,

$$\begin{array}{rcl} \omega^k(C' \bullet C[M]) & = & \{y\} \cup \mathcal{M}_i^1 \cup \ldots \cup \mathcal{M}_i^n \quad =_{def} \quad \bar{\mathcal{M}} \\ \omega^k(C' \bullet C[M]) & = & \{y\} \cup \mathcal{N}_i^1 \cup \ldots \cup \mathcal{N}_i^m \quad =_{def} \quad \bar{\mathcal{N}}; \end{array}$$

hence $\langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}_{k-2}(\bar{\mathcal{M}}, \bar{\mathcal{N}})$, from which we conclude that $C' \bullet C[M] \not\leq_{k-1} C' \bullet C[M]$. The inductive hypothesis now applies.

□

Theorem 8 *For any M, $N \in \Lambda_\oplus$,*

$$M \sqsubseteq_{must} N \;\Rightarrow\; M \leq N.$$

Proof. By contraposition, we prove (see definition 32)

$$\exists k.\, M \not\leq_k N \;\Rightarrow\; \exists C[\,].\, C[M]\!\downarrow \;\wedge\, C[N]\!\uparrow .$$

Indeed,

$$\begin{array}{rcll} M \not\leq N & \Rightarrow & \exists k.\, M \not\leq_k N & \\ & \Rightarrow & \exists C[\,].\, C[M] \not\leq_2 C[N] & \text{by lemma 10} \\ & \Rightarrow & \exists C[\,], C'[\,].\, C'[C[M]]\!\downarrow \;\wedge\; C'[C[N]]\!\uparrow & \text{by lemma 11.} \end{array}$$

□

Chapter 4

Denotational semantics

4.1 Non deterministic algebras

Non deteterministic algebras are **CPO** with a suitable continuous "union" oparation. This induces an "inclusion" ordering which differs form the ordering in the given structure (for a general study see e.g. [Ada-Rei-Nel]). The powerdomain operators are those functors constructing the free algebras generated by their arguments.

Definition 45 *A* non-deterministic algebra *is a structure* $\langle E, + \rangle$ *where* E *is a* **CPO** *and* $+$ *is a binary continuous function on* E *satisfying:*

i) $x + x = x$*;*

ii) $x + y = y + x$*;*

iii) $(x + y) + z = x + (y + z)$*;*

A non-deterministic algebra is a Smyth algebra *iff it satisfies (i)-(iii) and*

iv) $x + y \sqsubseteq x$*;*

A non-deterministic algebra is a Hoare algebra *iff it satisfies (i)-(iii) and*

v) $x \sqsubseteq x + y$*.*

Proposition 12 *Let $x \in D$, where D is any algebraic* **CPO***; then, for $* \in \{\flat, \sharp, \natural\}$,*

$$\{\!|x|\!\} = \{u \in \mathrm{M}(D) \mid u \sqsubseteq^* \{x\}\};$$

defines a continuous function $\{\!| \cdot |\!\} : D \to D^*$. Similarly, for $I, J \in D^*$

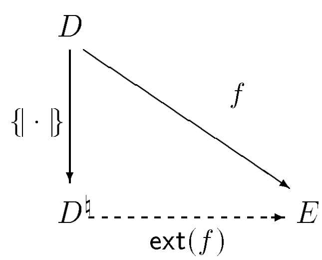
$$I \uplus J = \{u \cup v \mid u \in I \wedge v \in J\}$$

defines a countinuous function $\uplus : D^* \times D^* \to D^*$.

Proposition 13 *The non-deterministic algebras form a category, in the sequel referred to by* **NDA***, whose morphisms are continuous functions preserving + (called linear or additive).*

If $f \in \mathrm{Hom}_{\mathbf{NDA}}(D, E)$ then we will write $f : D \to_{lin} E$.

Theorem 9 *Let D be any ω-algebraic* **CPO** *and $\langle E, + \rangle$ any non-deterministic algebra; if $f : D \to E$ is a continuous function, then there exists a unique morphism $\mathsf{ext}(f) : D^\natural \to_{lin} E$ s.t. the following diagram commutes:*



Proof. To prove existence let $u = \{x_1, \ldots, x_n\}$ and define $\bar{f}(u) = f(x_1) + \cdots + f(x_n)$; then $\bar{f}$ is from $\mathrm{M}(D)$ to E; now, if $I \in D$ then the set $\{\bar{f}(u) \mid u \in I\}$ is directed being I a directed set and f and + monotonic functions; it follows that $\mathsf{ext}(f)(I) = \bigsqcup\{\bar{f}(u) \mid u \in I\}$ is the unique continuous extension of $\bar{f}$. Observing that

$$\begin{aligned} u = \{x_1, \ldots, x_n\} \sqsubseteq^\natural \{x\} &\Rightarrow \forall i \leq n.\ x_i \sqsubseteq x \\ &\Rightarrow \forall i \leq n.\ f(x_i) \sqsubseteq f(x) \\ &\Rightarrow f(\bar{u}) \sqsubseteq \underbrace{f(x) + \cdots + f(x)}_{n} = f(x) \end{aligned}$$

one gets

$$\begin{aligned}\mathsf{ext}(f)(\{\!|x|\!\}) &= \bigsqcup\{\bar{f}(u) \mid u \in \{\!|x|\!\}\} \\ &= \bigsqcup\{\bar{f}(u) \mid u \sqsubseteq^\natural \{x\}\} \\ &\sqsubseteq f(x).\end{aligned}$$

On the other hand $f(x) = \bigsqcup\{f(y) \mid y \in \mathcal{K}(D) \cap \downarrow\{x\}\}$ by the algebraicity of D and continuity of f; since $\{f(y) \mid y \in \mathcal{K}(D) \cap \downarrow\{x\}\} \subseteq \{\bar{f}(u) \mid u \sqsubseteq^\natural \{x\}\}$, we conclude

$$f(x) = \bigsqcup\{\bar{f}(u) \mid u \sqsubseteq^\natural \{x\}\} = \mathsf{ext}(f)(\{\!|x|\!\}).$$

To see uniqueness let $g : D^\natural \to_{lin} E$ be s.t. $g(\{\!|x|\!\}) = f(x)$. If $u = \{x_1, \ldots, x_n\} \in \mathrm{M}(D)$ then, since $\downarrow u = \{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\}$, we have

$$\begin{aligned}g(\downarrow u) &= g(\{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\}) \\ &= g(\{\!|x_1|\!\}) \uplus \cdots \uplus g(\{\!|x_n|\!\}) \\ &= \bar{f}(u).\end{aligned}$$

It follows that, for any $I \in D^\natural$

$$g(I) = \bigsqcup\{g(\downarrow u) \mid u \in I\} = \bigsqcup\{\bar{f}(u) \mid u \in I\} = \mathsf{ext}(f)(I),$$

by continuitiy of g and algebraicity of $D^\natural$.

□

Corollary 4 *For any algebraic* **CPO** D *and* **NDA** E

$$D \to_{cont} E \simeq D^\natural \to_{lin} E$$

where $\to_{cont}$ *refers to continuous functions and* $\simeq$ *is an isomorphism in the category of* **CPO**.

Proof. The isomorphism is given by ext and $\lambda g.g \circ \{\!|\cdot|\!\}$: from the theorem we have that $\mathsf{ext}(f) \circ \{\!|\cdot|\!\} = f$. On the other hand, from the unicity part of the prove above, if $h = g \circ \{\!|\cdot|\!\}$ for $g : D^\natural \to_{lin} E$, then, for $u = \{x_1, \ldots, x_n\} \in \mathrm{M}(D)$,

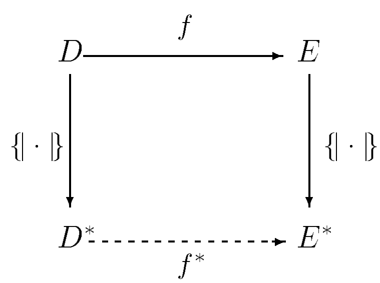
$$\begin{aligned}\mathsf{ext}(h)(\downarrow u) &= \bar{h}(u) \\ &= h(x_1) + \cdots + h(x_n) \\ &= g(\downarrow u)\end{aligned}$$

It follows that $\mathsf{ext}(g \circ \{\!|\cdot|\!\}) = g$. That these functions are order preserving and reversing is proved in a standard way.

$\square$

The theorem and its corollary extends to the cases of Smyth and Hoare powerdomains. A consequence of the theorem is the functoriality of the powerdomain operators.

Definition 46 *Let $f : D \to_{cont} E$ be any continuous function, then, for $* \in \{\flat, \sharp, \natural\}$, define $f^* : D^* \to_{lin} E^*$ by $f^* = \mathsf{ext}(\{\!|\cdot|\!\} \circ f)$, which is illustrated by the diagram*



The following proposition says that $(\cdot)^*$ is an $\mathcal{O}$-functor in the sense of [Henn-Plo].

Proposition 14 *Let $(\cdot)^*$ be an operator among $(\cdot)^\flat$, $(\cdot)^\sharp$ and $(\cdot)^\natural$; then*

i) $(Id_D)^* = Id_{D^*}$;

ii) $f \circ g : D \to_{cont} E \Rightarrow (f \circ g)^* = f^* \circ g^*$;

iii) $f \sqsubseteq g \Rightarrow f^* \sqsubseteq g^*$;

where $\sqsubseteq$ is the pointwise ordering.

Actually $(\cdot)^*$ is a monad over the category of **CPO**, with $\{\!|\cdot|\!\}$ as its unit and $\biguplus = \mathsf{ext}(Id_{D^*})$ as multiplication (see below the subsection about monads).

To make use of the category of **NDA** for modeling λ-calculus we need to show that it is cartesian closed.

Lemma 12 *Let* $\langle D, +_D \rangle$, $\langle E, +_E \rangle$ *be two* **NDA***; then* $\langle D \times E, + \rangle$ *where*

$$\langle x, y \rangle + \langle x', y' \rangle = \langle x +_D x', y +_E y' \rangle$$

is the cartesian product of D *and* E *in* **NDA**, *with as projections the set theoretic ones.*

Lemma 13 *Let* $\langle D, +_D \rangle$, $\langle E, +_E \rangle$ *be two* **NDA***; then* $\langle D \rightarrow_{lin} E, + \rangle$ *where* $D \rightarrow_{lin} E = Hom_{\mathbf{NDA}}(D, E)$ *pointwise ordered, and*

$$(f + g)(x) = f(x) +_E g(x)$$

for $x, y \in D$, *is an* **NDA**.

Theorem 10 **NDA** *is a cartesian closed category.*

Proof. In lemma 12 and 13 products and exponents have been exhibited. The terminal object is the trivial algebra. It remains to show that the arrow is actually the left adjoint of the product. Define abstraction and application:

- if $f \in [D_1 \times D_2 \rightarrow_{lin} D_3]$ then define $\hat{f} : D_1 \rightarrow [D_2 \rightarrow D_3]$ by

$$\hat{f}(x) = \lambda y . f(x, y)$$

- define $App \in [D_1 \rightarrow_{lin} D_2] \times D_1 \rightarrow D_2$ by

$$App(f, x) = f(x).$$

The proof that these functions are continuous runs as in the case of **CPO**. Concerning linearity:

(a) f is linear $\Rightarrow \hat{f}$ is linear:

$$\begin{aligned} \hat{f}(x + y)(z) &= f(x + y, z) \\ &= f(x, z) + f(y, z) \\ &= \hat{f}(x)(z) + \hat{f}(y)(z) \\ &= (\hat{f}(x) + \hat{f}(y))(z). \end{aligned}$$

(b) $\widehat{(\cdot)}$ is linear: given f, g of the right type

$$\begin{aligned}
(\widehat{f+g})(x) &= \lambda y.(f+g)(x,y) \\
&= \lambda y.f(x,y) + g(x,y) \\
&= \lambda y.f(x,y) + \lambda y.g(x,y) \\
&= \hat{f}(x) + \hat{g}(x) \\
&= (\hat{f} + \hat{g})(x)
\end{aligned}$$

(c) *App* is linear:

$$\begin{aligned}
App(f+g, x+y) &= (f+g)(x+y) \\
&= f(x+y) + g(x+y) \\
&= f(x) + f(y) + g(x) + g(y) \\
&= App(f,x) + App(f,y) + App(g,x) + App(g,y).
\end{aligned}$$

□

Corollary 5 *The full subcategory* **SNDA** *of* **NDA** *is cartesian closed.*

Proof. It is routine to check that the product and exponentiation in **NDA** of two Smyth algebras is a Smyth algebra: now use the theorem above.

□

4.2 Semilinear and linear applicative structures

λ-calculus studies functions under their applicative behaviour. Consequently models of this calculus and of its derivatives are applicative structures, that is sets equipped with a binary operation whose intended meaning is functional application. In the present case the structure we are looking for is an applicative structure with an extra operator modeling $\oplus$. It is useful to introduce the following concept.

Definition 47 *A* semilinear applicative structure *is a triple* $\langle X, \cdot, + \rangle$ *such that*

i) $\langle X, \cdot \rangle$ *is an applicative structure,* *ii)* $+ : X^2 \to X$ *is an idempotent, commutative and associative operation,*

iii) $\forall x, y, z \in X.\ (x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

A linear applicative structure *is a semilinear applicative structure satisfying:*

iv) $\forall x, y, z \in X.\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Both semilinear and linear applicative structures are extensional *if they are such as applicative strucures, i.e.*

v) $\forall x, y \in X.\ (\forall z \in X.\ x \cdot z = y \cdot z) \Rightarrow x = y$.

It is used the word *semilinear* since in general the application is not right distributive with respect to the sum: i.e. it is not linear. This is due to the fact that the application will be used to model continuous functions whose arguments are "sets", that is sums, and it is not true in general that the value of these functions is the set of their values on the "elements" of the argument.

To build both linear and semilinear applicative structures the inverse limit construction can be used. In view of the theorem 10 and of the fact that powerdomain functors are known to be locally continuous, the general theory of solution of domain equations in [Plo-Smy] allows us to conclude for the existence of a universal object in the category **SNDA**. However the point is that solving the equation

$$D \simeq [D \to_{lin} D]$$

gives rise to a linear applicative structure.

To get the solution of our problem we reason as follows: we know from corollary 4 that

$$D^\sharp \to_{lin} E \simeq D \to_{cont} E$$

and clearly that the set of linear functions from D to E is a proper subset of the continuous functions among them; on the other hand the extension of any continuous function is linear by definition: this will assure the semi-linearity we are looking for.

Definition 48 *Take* D_0 *as any non trivial Smyth algebra (e.g.* $(\mathbf{2})^\sharp$*), and* $D_{n+1} = [(D_n)^\sharp \to_{lin} D_n]$*; then inductively define* $\varphi_n : D_n \to_{lin} D_{n+1}$ *and* $\psi_n : D_{n+1} \to_{lin} D_n$ *as follows:*

i) $\varphi_0(x) = \lambda y.x$, $\psi_0(y) = y(\bot)$,

ii) $\varphi_{n+1}(x) = \varphi_n \circ x \circ (\psi_n)^\sharp$, $\psi_{n+1}(y) = \psi_n \circ y \circ (\varphi_n)^\sharp$.

Proposition 15 *The mappings φ_n, ψ_n are well defined, that is they are linear; furthemore for each natural number n $\langle\varphi_n, \psi_n\rangle$ is an embedding-projection pair, that is*

i) $\psi_n \circ \varphi_n = Id_n$;

ii) $\varphi_n \circ \psi_n \sqsubseteq Id_n$.

Proof. Linearity is proved by induction on n: for $n = 0$

$$\varphi_0(x + y)(z) = z = z + z = \varphi_0(x)(z) + \varphi_0(y)(z);$$

and

$$\psi_0(f + g) = (f + g)(\bot) = f(\bot) + g(\bot) = \psi_0(f) + \psi_0(g).$$

If $n > 0$ then the thesis follows from the inductive hypothesis, since for any (continuous) f, $f^\sharp$ is always linear and composition of linear functions is linear. To prove that $\langle\varphi_n, \psi_n\rangle$ is an embedding-projection pair we again make induction on n. If $n = 0$:

$$\begin{array}{rcl} \psi_0 \circ \varphi_0(x) & = & \psi_0(\lambda y.x) \\ & = & (\lambda y.x)(\bot) \\ & = & x \end{array}$$

and

$$\begin{array}{rcl} \varphi_0 \circ \psi_0(f) & = & \varphi_0(f(\bot)) \\ & = & \lambda z.f(\bot) \\ & \sqsubseteq & f. \end{array}$$

For the inductive step:

$$\begin{array}{rcll} \psi_{n+1} \circ \varphi_{n+1} & = & \psi_{n+1}(\varphi_n \circ x \circ (\psi_n)^\sharp) & \\ & = & \psi_n \circ \varphi_n \circ x \circ (\psi_n)^\sharp) \circ (\varphi_n)^\sharp & \\ & = & (\psi_n \circ \varphi_n) \circ x \circ (\psi_n \circ \varphi_n)^\sharp & \text{by prop. 14} \\ & = & Id_n \circ x \circ (Id_n)^\sharp & \text{by ind. hyp.} \\ & = & Id_n \circ x \circ Id_{D_n^\sharp} & \text{by prop. 14} \\ & = & x & \end{array}$$

and similarly

$$\begin{array}{rcll} \varphi_{n+1} \circ \psi_{n+1}(y) & = & \varphi_{n+1}(\psi_n \circ y \circ (\varphi_n)^\sharp) & \\ & = & \varphi_n \circ \psi_n \circ y \circ (\varphi_n)^\sharp \circ (\psi_n)^\sharp & \\ & = & \varphi_n \circ \psi_n \circ y \circ (\varphi_n \circ \psi_n)^\sharp & \text{by prop. 14} \\ & \sqsubseteq & Id_n \circ y \circ (Id_n)^\sharp & \text{by ind. hyp. and prop. 14} \\ & = & y. & \end{array}$$

□

Definition 49

i) $D_* = \lim_{\leftarrow}(D_n, \psi_n) = \{x \in \prod_n D_n \mid \forall n \in \omega.\ \psi_n(x_{n+1}) = x_n\}$ *with coordinate-wise ordering;*

ii) $\Phi_{m,n} : D_m \to_{lin} D_n$ *is defined:*

$$\Phi_{m,n} = \begin{cases} \varphi_{n-1} \circ \ldots \circ \varphi_m & \text{if } m < n \\ Id & \text{if } m = n \\ \psi_n \circ \ldots \circ \psi_{m-1} & \text{if } n < m \end{cases}$$

iii) $\Phi_{*,n} : D_* \to_{lin} D_n$ *and* $\Phi_{n,*} : D_n \to_{lin} D_*$ *are defined:*

$$\begin{array}{rcl} \Phi_{*,n}(x) & = & x_n \\ \Phi_{n,*}(y) & = & \langle \Phi_{n,m}(y) \rangle_{m \in \omega} \end{array}$$

As usual with inverse limit constructions, each D_n *embeds into* D_**, by the* $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ *embedding-projection pair; we will write* $x_n = \Phi_{n,*} \circ \Phi_{*,n}(x)$ *for* $x \in D_*$ *and* $a_n = (\Phi_{n,*} \circ \Phi_{*,n})^\sharp(a) = \Phi^\sharp_{n,*} \circ \Phi^\sharp_{*,n}(a)$ *for* $a \in D^\sharp_*$

Definition 50

i) The map $F : D_* \to [D^\sharp_* \to_{lin} D_*]$ *is defined by:*

$$F(x) = \lambda a \in D^\sharp_*. \bigsqcup_n x_{n+1}(a_n)$$

and $\tilde{F} : D_* \to [D_* \to_{cont} D_*]$ *by:*

$$\tilde{F} = (\lambda g. g \circ \{\!|\cdot|\!\}) \circ F;$$

ii) the map $G : [D_*^\sharp \to_{lin} D_*] \to D_*$ *is defined by:*

$$G(f) = \bigsqcup_n (\lambda a \in D_n^\sharp . (f(\Phi_{n,*}^\sharp(a)))_n)$$

and $\tilde{G} : [D_* \to_{cont} D_*] \to D_*$ *by:*

$$\tilde{G} = G \circ \mathsf{ext};$$

iii) the operation $\cdot : D \times D \to D$ *is defined by:*

$$x \cdot y = \tilde{F}(x)(y) = F(x)\{|y|\}.$$

We list in the following lemma some relevant properties of the domain D_*

Lemma 14 *For any* $x, y, z \in D_*$ *and* $a \in D_*^\sharp$,

i) $(x_m)_n = x_{min(n,m)}$;

ii) $x = \bigsqcup_n x_n, \ a = \bigsqcup_n a_n$;

iii) $x_{n+1} \cdot y_n = x_{n+1}(\{|y_n|\})$;

iv) $x_{n+1} \cdot y = x_{n+1} \cdot y_n = (x \cdot y_n)_n$;

v) $x_0 \cdot y = x_0 = (x \cdot \bot)_0$;

vi) $(x + y)_n = x_n + y_n = (x_n + y_n)_n$.

Proof.

(i) Consequence of the fact that $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ is an embedding-projection pair.

(ii) $x = \bigsqcup_n x_n$ is standard in inverse limit constructions; to see $a = \bigsqcup_n a_n$:

a) Let $\langle \varphi, \psi \rangle$ be an injection-projection pair from some D to some E: then

$$\varphi(\mathcal{K}(D)) \subseteq \mathcal{K}(E).$$

Indeed, let $x \in \mathcal{K}(D)$ then for any directed $Y \subseteq E$

$$\begin{array}{rcl} \varphi(x) \sqsubseteq \bigsqcup Y & \Rightarrow & x \sqsubseteq \psi(\bigsqcup Y) = \bigsqcup \psi(Y) \\ & \Rightarrow & \exists y \in Y.\ x \sqsubseteq \psi(y) \\ & \Rightarrow & \exists y \in Y.\ \varphi(x) \sqsubseteq \varphi \circ \psi(y) \sqsubseteq y. \end{array}$$

b) $\mathcal{K}(D_*) = \bigcup_n \mathcal{K}(D_n)$: indeed if $d \in \mathcal{K}(D_n)$ then $d \in \mathcal{K}(D_*)$ follows from (a) and the fact that $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ is an embedding-projection pair; on the other hand, from $d \in \mathcal{K}(D_*)$ it follows

$$d = \bigsqcup_n d_n \Rightarrow \exists m.\, d = d_m.$$

Now, given any directed $S \subseteq D_m$

$$\begin{array}{rcl} \Phi_{*,m}(d) \sqsubseteq \bigsqcup S & \Rightarrow & d = \Phi_{m,*} \circ \Phi_{*,m}(d) \sqsubseteq \Phi_{m,*}(\bigsqcup S) \\ & \Rightarrow & d \sqsubseteq \bigsqcup \Phi_{m,*}(S) \\ & \Rightarrow & \exists s \in S.\, d \sqsubseteq \Phi_{m,*}(s) \\ & \Rightarrow & \exists s \in S.\, \Phi_{*,m}(d) \sqsubseteq s. \end{array}$$

That $a_n \sqsubseteq a$ for all n is immediate. Vice versa, let $u \in a$, then $u = \{d^1, \ldots, d^r\} \in \mathrm{M}(D_*)$; using (b) we know that each d_i is compact in some D_{m_i}, then we choose $m = max\{m_i \mid 1 \leq i \leq r\}$: by (a) $u \in \mathrm{M}(D_m)$. On the other hand

$$\Phi^\sharp_{*,m}(a) = \bigcup\{\bar{\Phi}_{*,m}(v) \mid v \in a\}$$

but

$$\begin{array}{rcl} \bar{\Phi}_{*,m}(u) & = & \{\!|\Phi_{*,m}(d^1)|\!\} \uplus \ldots \uplus \{\!|\Phi_{*,m}(d^r)|\!\} \\ & = & \{\!|d^1|\!\} \uplus \ldots \uplus \{\!|d^r|\!\} \\ & = & \downarrow u \end{array}$$

so that $u \in \downarrow u \subseteq a_m$, from which we conclude that $a \sqsubseteq \bigsqcup_n a_n$.

(iii) Let us note preliminarily that, by the very definition of $(\cdot)^\sharp$:

$$\{\!|y|\!\}_n = \Phi^\sharp_{*,n}\{\!|y|\!\} = \{\!|\Phi_{*,n}(y)|\!\} = \{\!|y_n|\!\}.$$

Now

$$\begin{array}{rcl} x_{n+1} \cdot y_n & = & F(x_{n+1})\{\!|y_n|\!\} \\ & = & \bigsqcup_m (x_{n+1})_{m+1}\{\!|y_n|\!\}_m \\ & = & \bigsqcup_m (x_{n+1})_{m+1}(\{\!|y|\!\}_n)_m \\ & = & x_{n+1}\{\!|y|\!\}_n \\ & = & x_{n+1}\{\!|y_n|\!\}. \end{array}$$

(iv)-(v) Similar to the proof of the corresponding properties for Scott D_∞ models.

(vi) By linearity of $\Phi^\sharp_{*,n}$ we immediately have $(x+y)_n = x_n + y_n$. On the other hand

$$\begin{aligned}(x_n + y_n)_n &= (x_n)_n + (y_n)_n \\ &= x_n + y_n.\end{aligned}$$

□

Lemma 15 *The mappings F and G are continuous and linear, that is they are* **NDA** *morphisms. Furthermore the structure*

$$\langle D_*, \cdot, + \rangle$$

is a semilinear applicative strucure.

Proof. Let $x, y \in D_*$ and $a \in D^\sharp_*$, then

$$\begin{aligned}\bigsqcup_n (x+y)_{n+1}(a_n) &= \bigsqcup_n (x_{n+1} + y_{n+1})(a_n) && \text{by lemma 14 (vi)} \\ &= \bigsqcup_n (x_{n+1}(a_n) + y_{n+1}(a_n)) && \text{by lemma 13} \\ &= \bigsqcup_n x_{n+1}(a_n) + \bigsqcup_n y_{n+1}(a_n) && \text{by continuity of } +\end{aligned}$$

hence

$$\begin{aligned}F(x+y) &= \lambda a \in D^\sharp_*. \bigsqcup (x+y)_{n+1}(a_n) \\ &= \lambda a \in D^\sharp_*. \bigsqcup_n x_{n+1}(a_n) + \bigsqcup_n y_{n+1}(a_n) \\ &= (\lambda a \in D^\sharp_*. \bigsqcup_n x_{n+1}(a_n)) + (\lambda a \in D^\sharp_*. \bigsqcup_n y_{n+1}(a_n)) \\ &= F(x) + F(y).\end{aligned}$$

Let $f, g \in [D^\sharp_* \to_{lin} D_*]$ and $a \in D^\sharp_*$, then

$$((f+g)(a))_n = (f(a) + g(a))_n = (f(a))_n + (g(a))_n$$

by lemma 14 (vi); it follows

$$\begin{aligned}G(f+g) &= \bigsqcup_n (\lambda b \in D^\sharp_n.((f+g)(\Phi^\sharp_{n,*}(b)))_n) \\ &= \bigsqcup_n (\lambda b \in D^\sharp_n.(f(\Phi^\sharp_{n,*}(b)))_n + (g(\Phi^\sharp_{n,*}(b)))_n) \\ &= \bigsqcup_n ((\lambda b \in D^\sharp_n.(f(\Phi^\sharp_{n,*}(b)))_n) + (\lambda b \in D^\sharp_n.(g(\Phi^\sharp_{n,*}(b)))_n)) \\ &= \bigsqcup_n (\lambda b \in D^\sharp_n.(f(\Phi^\sharp_{n,*}(b)))_n) + \bigsqcup_n (\lambda b \in D^\sharp_n.(g(\Phi^\sharp_{n,*}(b)))_n) \\ &= G(f) + G(g).\end{aligned}$$

This establishes the linearity property; the continuity property is proved in the same way as in the cathegory of **CPO**.

Finally, let $x, y, z \in D_*$:

$$\begin{array}{rcll} (x+y)\cdot z & = & \tilde{F}(x+y)(z) & \\ & = & F(x+y)(\{\!|z|\!\}) & \\ & = & (F(x)+F(y))(\{\!|z|\!\}) & \text{by linearity of } F \\ & = & F(x)(\{\!|z|\!\}) + F(y)(\{\!|z|\!\}) & \\ & = & (x\cdot z)+(y\cdot z). & \end{array}$$

□

Theorem 11 *The domain D_* satisfies the equation*

$$D \simeq [D^\sharp \rightarrow_{lin} D]$$

in the category of **NDA** *and consequently in that of* **SNDA***; it satisfies also the equation*

$$D \simeq [D \rightarrow_{cont} D]$$

in the category of **CPO** *as pictured in the diagram*

$$D_* \underset{F}{\overset{G}{\rightleftarrows}} [D_*^\sharp \rightarrow_{lin} D_*] \underset{\lambda g. g \circ \{\!|\cdot|\!\}}{\overset{\mathsf{ext}}{\rightleftarrows}} [D_* \rightarrow_{cont} D_*]$$

We conclude that the structure $\langle D_*, \cdot, + \rangle$ is an extensional *semilinear applicative structure.*

Proof. To prove the theorem it remains to show that F and G are mutually inverse: actually the second isomorphism will follow from this one and from corollary 4, which applies to the $(\cdot)^\sharp$ functor as well.

a) $G \circ F = Id$: by definition $(G \circ F)(x) = G(f)$ where

$$f = \lambda a \in D_*^\sharp . \bigsqcup_n x_{n+1}(a_n);$$

now we observe that if y is in (the image of) D_n the $y = y_n$, and similarly if a is in the image if $D_n^\sharp$; now given such an a

$$\begin{array}{rcl} (f(a))_n & = & (\bigsqcup_m x_{m+1}(a_m))_n \\ & = & (x_{n+1}(a))_n \\ & = & x_{n+1}(a). \end{array}$$

It follows that

$$\begin{aligned} G(f) &= \bigsqcup_n(\lambda a \in D^\sharp_n . x_{n+1}(a)) \\ &= \bigsqcup_n x_{n+1} \\ &= x. \end{aligned}$$

b) $F \circ G = Id$: we note that

$$\begin{aligned} G(f)_{n+1}(a_n) &= (f(\Phi^\sharp_{n,*}(a_n)))_n \\ &= (f(a))_n \end{aligned}$$

so that

$$\begin{aligned} (F \circ G)(f)(a) &= \bigsqcup_n (f(a))_n \\ &= f(a), \end{aligned}$$

that is $(F \circ G)(f) = (f)$.

To prove extensionality:

$$\begin{aligned} \forall z.\, x \cdot z = y \cdot z &\Rightarrow \tilde{F}(x)(z)\tilde{F}(y)(z) \\ &\Rightarrow \tilde{F}(x) = \tilde{F}(y) \\ &\Rightarrow x = \tilde{G} \circ \tilde{F}(x) = \tilde{G} \circ \tilde{F}(y) = y. \end{aligned}$$

$\square$

Remark 5 The structure $\langle D_*, \cdot, + \rangle$ is actually a semilinear applicative structure which is not linear: this will follow from the full abstraction theorem and from the fact that

$$\Delta(H_0 \oplus H_1) \not\simeq_{must} \Delta H_0 \oplus \Delta H_1$$

for $H_0 \equiv \lambda x.x\mathbf{U}^3_3\Delta$ and $H_1 \equiv \lambda xy.yy$.

4.3 Syntactical and canonical model

We present a notion of *model*, which actually does not directly interpret the relation $\longrightarrow_r$, but the equivalence relation induced by $\sqsubseteq_{must}$.

Definition 51 *A* syntactical model *is a semilinear applicative structure* $\mathcal{M} = \langle X, \cdot, + \rangle$*, equipped with a map* $[\![\cdot]\!] : \Lambda_\oplus \to (Env \to X)$*, such that the triple* $\langle X, \cdot, [\![\cdot]\!] \rangle$*, for any* $\rho \in Env = Var \to X$*, satisfies:*

i) $[\![x]\!]_\rho = \rho(x)$;

ii) $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$;

iii) $[\![\lambda x.M]\!]_\rho \cdot d = [\![M]\!]_{\rho[d/x]}$ *for all* $d \in X$;

iv) $\rho\lceil \mathrm{FV}(M) = \rho'\lceil \mathrm{FV}(M) \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$;

v) $[\![\lambda x.M]\!]_\rho = [\![\lambda y.M[y/x]]\!]_\rho$ *if* $y \notin \mathrm{FV}(M)$;

vi) $(\forall d \in X.\ [\![M]\!]_{\rho[d/x]} = [\![N]\!]_{\rho[d/x]}) \Rightarrow [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$;

which are the clauses of the classical definition of syntactical λ*-model of* [Hin-Lon], *and furthermore*

vii) $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho + [\![N]\!]_\rho$.

Finally we call extensional *any syntactical model whose underlining semilinear applicative structure is extensional.*

Lemma 16 *If* $\mathcal{M} = \langle X, \cdot, + \rangle$ *is an extensional syntactical model, then for any* $M, N \in \Lambda_\oplus$ *and for all* $\rho \in Env$:

$$[\![\lambda x.M \oplus N]\!]_\rho = [\![(\lambda x.M) \oplus (\lambda x.N)]\!]_\rho.$$

Proof. Let $d \in X$ be an arbitrary element; then, for any $\rho \in Env$,

$$\begin{array}{rcll} [\![\lambda x.M \oplus N]\!]_\rho \cdot d & = & [\![M \oplus N]\!]_{\rho[d/x]} & \text{by def. 51 (iii)} \\ & = & [\![M]\!]_{\rho[d/x]} + [\![N]\!]_{\rho[d/x]} & \text{by def. 51 (vii)} \\ & = & [\![\lambda x.M]\!]_\rho \cdot d + [\![\lambda x.N]\!]_\rho \cdot d & \text{by def. 51 (iii)} \\ & = & ([\![\lambda x.M]\!]_\rho + [\![\lambda x.N]\!]_\rho) \cdot d & \text{by semilinearity.} \end{array}$$

Since d is arbitrary, it follows that

$$\begin{array}{rcll} [\![\lambda x.M \oplus N]\!]_\rho & = & [\![\lambda x.M]\!]_\rho + [\![\lambda x.N]\!]_\rho & \text{by def. 51 (vi)} \\ & = & [\![(\lambda x.M) \oplus (\lambda x.N)]\!]_\rho & \text{by def. 51 (vii).} \end{array}$$

□

Definition 52 *Given the structure* $\langle D_*, \cdot, + \rangle$ *and* $\rho \in Env = Var \to D_*$, *we define the map* $[\![\cdot]\!] : \Lambda_\oplus \to (Env \to D_*)$ *as follows:*

i) $[\![x]\!]_\rho = \rho(x)$,

ii) $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$,

iii) $[\![\lambda x.M]\!]_\rho = \tilde{G}(\lambda d \in D_*.[\![M]\!]_{\rho[d/x]})$,

iv) $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho + [\![N]\!]_\rho$.

This is a good definition, since in (iii) the continuity and linearity of application, abstraction and $+$ ensure that the function $\lambda d \in D_*.[\![M]\!]_{\rho[d/x]}$ is continuous and linear as well.

Proposition 16 *The quadruple* $\langle D_*, \cdot, +, [\![\cdot]\!]\rangle$ *is a syntactical model, furthermore it is extensional.*

Proof. By theorem 11 the structure $\langle D_*, \cdot, +\rangle$ is an extensional semilinear applicative structure. The rest is routine; e.g.

$$\begin{array}{rcl} [\![\lambda x.M]\!]_\rho \cdot d & = & \tilde{F}(\tilde{G}(\lambda d'.[\![M]\!]_{\rho[d'/x]}))(d) \\ & = & (\lambda d'.[\![M]\!]_{\rho[d'/x]})(d) \\ & = & [\![M]\!]_{\rho[d/x]}, \end{array}$$

hence definition 51 (iii) is verified.

□

4.4 Full abstraction theorem

The main result of this chapter is a theorem stating that the operational and denotational semantics constructed so far coincide.

Theorem 12 (**Full Abstraction Theorem**)
For all $M,\ N \in \Lambda_\oplus$

$$M \sqsubseteq_{must} N \iff \forall \rho.\ [\![M]\!]_\rho \sqsubseteq [\![N]\!]_\rho.$$

To prove the theorem we shall use ideas from classical λ-calculus. It was after work by Wadsworth and Hyland that the deep connection between the algebraic semantics (see [Levy]), presented in [Bar] in terms of Böhm trees, and the notion of approximation in Scott's models of the λ-calculus has been understood. In the present setting we show how this construction carries over to our nondeterministic extension of the λ-calculus, using NBT in place of Böhm trees.

In the sequel the intended interpretation is D_*.

Definition 53 *Let $\mathcal{I}$ be an* indexing function *that is a map $\mathcal{I} : \Lambda_\oplus(\Omega) \to \omega$; then, writing $M^\mathcal{I}$ to mean the (fully) indexed term associated to M by $\mathcal{I}$:*

i) $[\![\Omega^\mathcal{I}]\!]_\rho = \perp$;

ii) $[\![x^\mathcal{I}]\!]_\rho = (\rho(x))_{\mathcal{I}(x)}$;

iii) $[\![(MN)^\mathcal{I}]\!]_\rho = ([\![M^\mathcal{I}]\!]_\rho \cdot [\![N^\mathcal{I}]\!]_\rho)_{\mathcal{I}(MN)}$;

iv) $[\![(\lambda x.M)^\mathcal{I}]\!]_\rho = (\tilde{G}(\lambda d.[\![M^\mathcal{I}]\!]_{\rho[d/x]}))_{\mathcal{I}(\lambda x.M)}$;

v) $[\![(M \oplus N)^\mathcal{I}]\!]_\rho = ([\![M^\mathcal{I}]\!]_\rho + [\![N^\mathcal{I}]\!]_\rho)_{\mathcal{I}(M\oplus N)}$.

Lemma 17 *For any $M \in \Lambda_\oplus$ and all $\rho \in Env$:*

$$[\![M]\!]_\rho = \bigsqcup_\mathcal{I} [\![M^\mathcal{I}]\!]_\rho.$$

Proof. By induction on M using the equation $x = \bigsqcup_n x_n$ of lemma 14. □

In the sequel we call terms together with their indexes modulo some indexing function *indexed terms* (see [Bar]).

Definition 54 *First extend the definition of substitution to indexed terms inductively from the base clause $x^m[N^n/x] \equiv (N^n)^m$. Now define the following binary relation $\rhd$ over indexed terms:*

i) $(\lambda x.M)^{n+1}N \rhd (M[N^n/x])^n$;

ii) $(\lambda x.M)^0 N \rhd (M[\Omega^0/x])^0$;

iii) $\Omega^n \rhd \Omega^0$;

iv) $\lambda x.\Omega^n \rhd \Omega^0$;

v) $\Omega^n M \rhd \Omega^0$;

vi) $\Omega^n \oplus M \rhd \Omega^0$;

vii) $M \oplus \Omega^n \rhd \Omega^0$;

viii) $(M \oplus N)^{n+1} L \rhd (ML^n \oplus NL^n)^n$;

ix) $(M \oplus N)^0 L \rhd (M\Omega^0 \oplus N\Omega^0)^0$;

x) $(M^m)^n \rhd M^{min(m,n)}$;

xi) $M^m \rhd N^n \Rightarrow C[M^m] \rhd C[N^n]$.

Lemma 18

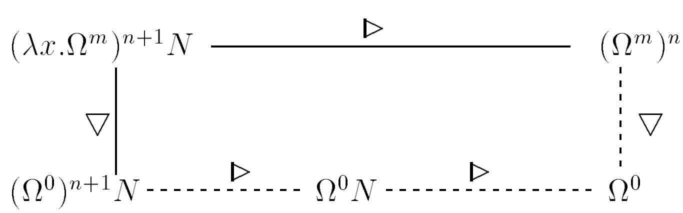
i) $\rhd \models$ **WCR**;

ii) $\rhd \models$ **SN**;

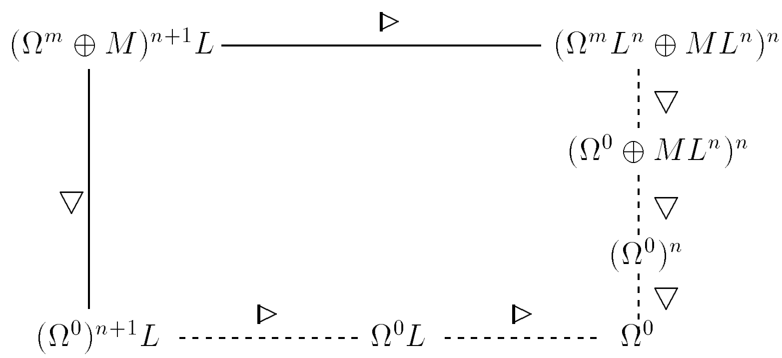
iii) $\rhd \models$ **CR**.

Proof. Part (i) is proved by case inspection of overlapping right hand sides; we treat the following two cases (the others are similar):

Case 1: (i)-(iv)



Case 2: (iii)-(viii)



(ii) Extension of the classical proof of strong normalization of the labelled λ-calculus (see [Bar]). Just note the decreasing index in clauses (i) and (viii) of the definition 54, and that the length of the term decreases in the other cases.
(iii) From (i) and (ii) by Newman lemma (see any text about term rewriting, including [Bar]).

□

Corollary 6

i) $\mathrm{N}_{\oplus}^{\Omega} = \{|M| \mid M \textit{ in } \triangleright - nf\}$, *where* $|\cdot|$ *is the index erasing map;*

ii) $\forall M \in \Lambda_{\oplus}\ \forall \mathcal{I}\ \exists N \in \mathrm{N}_{\oplus}^{\Omega}\ \exists \mathcal{J}.\ M^{\mathcal{I}} \triangleright N^{\mathcal{J}}$.

Proof. Recall that $\mathrm{N}_{\oplus}^{\Omega} = \{M^{[k]} = \vartheta_1 \circ \omega^k(M) \mid M \in \Lambda_{\oplus}, k \in \omega\}$. Let us observe that, after the very definition of ϑ_1, this set could be inductively defined by:

i) $\Omega \in \mathrm{N}_{\oplus}^{\Omega}$;

ii) $M_1, \ldots, M_m \in \mathrm{N}_{\oplus}^{\Omega} \wedge x_1, \ldots, x_n, x \in Var \Rightarrow \lambda x_1 \ldots x_n.xM_1 \ldots M_m \in \mathrm{N}_{\oplus}^{\Omega}$;

iii) $M, N \in \mathrm{N}_{\oplus}^{\Omega} - \{\Omega\} \Rightarrow M \oplus N \in \mathrm{N}_{\oplus}^{\Omega}$;

now to prove (i) is routine. (ii) follows from (i) and the lemma above.

□

Lemma 19

$$M \in \mathrm{N}_{\oplus}^{\Omega} \Rightarrow \exists k \in \omega \; \forall h \geq k.\; \omega^h(M) = \omega^k(M);$$

hence, defining $\mathsf{height}(M)$ *as the minimal* k *satisfying the above statement,*

$$M \leq N \;\Leftrightarrow\; M \leq_{\mathsf{height}(M)} N \;\Leftrightarrow\; M \leq N^{[\mathsf{height}(M)]}.$$

Proof. Note that, since $M \in \mathrm{N}_{\oplus}^{\Omega}$, the NBT$(M)$ differs from the synctatical tree only in that the operator $\oplus$ is treated as a set constructor, and some abstractions are pushed into sums: e.g.

$$\omega^k(\lambda x.P \oplus Q) = \omega^k(\lambda x.P) \cup \omega^k(\lambda x.Q) = \omega^k(\lambda x.P \oplus \lambda x.Q).$$

Now take as k the depth of the synctatical tree of M.

□

Lemma 20 *For any* $M, N \in \Lambda_{\oplus}$ *and any indexing functions* $\mathcal{I}$ *and* $\mathcal{J}$*:*

i) $M^{\mathcal{I}} \rhd^* N^{\mathcal{J}} \in \mathrm{N}_{\oplus}^{\Omega} \Rightarrow N \leq M$*;*

ii) $M^{\mathcal{I}} \rhd N^{\mathcal{J}} \Rightarrow \forall \rho.\; [\![M^{\mathcal{I}}]\!]_{\rho} = [\![N^{\mathcal{J}}]\!]_{\rho}$.

Proof. The NBT of M is the same of that of N with the possible exception of some nodes labelled with Ω; since $N \in \mathrm{N}_{\oplus}^{\Omega}$ the thesis follows by induction on the height of N.
To prove (ii) one checks the clauses in definition 54 along the equations of lemma 14; e.g.

$$\begin{array}{rcl} [\![(M \oplus N)^{n+1} L]\!] & = & [\![(M \oplus N)^{n+1}]\!] \cdot [\![L]\!] \\ & = & [\![M \oplus N]\!]_{n+1} \cdot [\![L]\!] \\ & = & ([\![M]\!] + [\![N]\!])_{n+1} \cdot [\![L]\!]. \end{array}$$

Now call $x = [\![M]\!]$, $y = [\![N]\!]$, $z = [\![L]\!]$:

$$\begin{array}{rcl} (x+y)_{n+1} \cdot z & = & (x_{n+1} + y_{n+1}) \cdot z \\ & = & x_{n+1} \cdot z + y_{n+1} \cdot z \\ & = & (x \cdot z_n)_n + (y \cdot z_n)_n \\ & = & (x \cdot z_n + y \cdot z_n)_n \\ & = & [\![(ML^n \oplus NL^n)^n]\!]. \end{array}$$

□

Lemma 21 *For any $M \in \Lambda_\oplus$, if $L \in \mathrm{N}^\Omega_\oplus$ and $L \leq M$, then $[\![L]\!] \sqsubseteq [\![M]\!]$.*

Proof. Using the inductive definition of $\mathrm{N}^\Omega_\oplus$.

Case 1: $L \equiv \Omega$, then
$$[\![L]\!] = \bot \sqsubseteq [\![M]\!].$$

In the sequel, since $L \not\equiv \Omega$, $L \leq M$ implies that $M\downarrow$: let $\{M_1, \ldots, M_r\}$ be the principal hnfs of M.

Case 2: $L \equiv x$; now
$$x \leq_1 M \Rightarrow x \leq_1 M_1 \wedge \ldots \wedge x \leq_1 M_r,$$

hence, for $i = 1, \ldots, r$,
$$M_i \equiv \lambda y_1 \ldots y_{n_i}.xM^i_1 \ldots M^i_{n_i}.$$

By lemma 17 we know that $[\![x]\!] = \bigsqcup_{\mathcal{I}}[\![x^{\mathcal{I}}]\!]$; hence we proceed by induction on $q = \mathcal{I}(x)$.

Subcase 2.1: $q = 0$, then, by lemma 14 (v), for $i = 1, \ldots, r$:
$$\begin{array}{rcl} [\![x^0]\!] & = & [\![x]\!]_0 \\ & = & [\![\lambda y_1 \ldots y_{n_i}.x^0 \underbrace{\Omega \ldots \Omega}_{n_i}]\!] \\ & \sqsubseteq & [\![M_i]\!]. \end{array}$$

We conclude $[\![x^0]\!] \sqsubseteq [\![M_1]\!] + \ldots + [\![M_r]\!] = [\![M]\!]$.

Subcase 2.2: $q > 0$, then, by lemma 14 (iv), for $i = 1, \ldots, r$:
$$[\![x^q]\!] = [\![\lambda y_1 \ldots y_{n_i}.x^q y_1^{q-1} \ldots y_{n_i}^{q-n_i}]\!].$$

Now each pair in $\mathsf{Pair}_1(\omega^2(L), \omega^2(M_i))$ will have the shape:
$$\langle\{\{y_j^{q-j}\}\}, \{\mathcal{M}_1, \ldots, \mathcal{M}_s\}\rangle$$

and $\{y_j^{q-j}\} \leq \mathcal{M}_h$ for $h = 1, \ldots, s$. By ind. hyp.
$$[\![y_j^{q-j}]\!] \sqsubseteq [\![N_1]\!] + \ldots + [\![N_t]\!] = [\![\mathcal{M}_h]\!]$$

given that $\mathcal{M}_h = \{N_1, \ldots, N_t\}$. This means that again $[\![x^q]\!] \sqsubseteq [\![M_i]\!]$ for each i, so that $[\![x^q]\!] \sqsubseteq [\![M]\!]$

Case 3: $L \equiv \lambda x_1 \dots x_m . x L_1 \dots L_n$, then the pairs in $\mathsf{Pair}_1(\omega^k(L), \omega^k(M))$, where $k = \mathsf{height}(L)$, are of the form

$$\langle \{\mathcal{L}_j\}, \{\mathcal{M}_j^1, \dots, \mathcal{M}_j^r\} \rangle,$$

where $\mathcal{L}_j = \omega^{k-1}(L_j)$. By ind. hyp. $[\![\mathcal{L}_j]\!] \sqsubseteq [\![\mathcal{M}_j^h]\!]$ for $h = 1, \dots, r$; we conclude, as in subcase 2.2, that $[\![L]\!] \sqsubseteq [\![M]\!]$.

Case 4: $L \equiv L_1 \oplus \dots \oplus L_n$ where we can suppose that the L_i are not sums. Let again $k = \mathsf{height}(L) > 0$. From the definition of $\leq$ we know that

a) $\forall [P] \in (\mathcal{L}, \mathcal{M}).\ [P] \cap \mathcal{L} \neq \emptyset$;

b) $\forall \langle \mathcal{U}, \mathcal{V} \rangle \in \mathsf{Pair}(\mathcal{L}, \mathcal{M}).\ \mathcal{U} \sqsubseteq^\sharp \mathcal{V}$.

where $\mathcal{L} = \omega^k(L)$ and $\mathcal{M} = \omega^k(M)$. To the sake of simplicity suppose that

$$\mathcal{L} = \{x\mathcal{L}_1, x\mathcal{L}_2\} \text{ and } \mathcal{M} = \{x\mathcal{M}_1, x\mathcal{M}_2, x\mathcal{M}_3\};$$

then $\mathsf{Pair}_1(\mathcal{L}, \mathcal{M})$ contains only the pair

$$\langle \{\mathcal{L}_1, \mathcal{L}_2\}, \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\} \rangle.$$

We know that

$$\forall j \leq 3\ \exists i \leq 2.\ \mathcal{L}_i \leq_{k-1} \mathcal{M}_j,$$

so that the inductive hypothesis applies giving that

$$\forall j \leq 3\ \exists i \leq 2\ .[\![\mathcal{L}_i]\!] \sqsubseteq [\![\mathcal{M}_j]\!],$$

that is

$$\forall j \leq 3\ \exists i \leq 2\ .[\![x\mathcal{L}_i]\!] \sqsubseteq [\![x\mathcal{M}_j]\!].$$

Since in any Smyth algebra D, for any $a_1, \dots, a_n, b_1, \dots, b_m \in D$

$$\forall j \leq m\ \exists i \leq n.\ a_i \sqsubseteq b_j \Rightarrow a_1 + \dots + a_n \sqsubseteq^\sharp b_1 + \dots + b_m,$$

we get the thesis observing that

$$[\![L]\!] = [\![x\mathcal{L}_1]\!] + [\![x\mathcal{L}_2]\!] \text{ and } [\![M]\!] = [\![x\mathcal{M}_1]\!] + [\![x\mathcal{M}_2]\!] + [\![x\mathcal{M}_3]\!].$$

□

Lemma 22 *For any $M \in \Lambda_\oplus$ and natural number k,*

$$\forall \rho \in Env.\ [\![M]\!]_\rho = \bigsqcup_k [\![M^{[k]}]\!]_\rho.$$

Proof. For any $k \in \omega$, $M^{[k]} \in \mathrm{N}_\oplus^\Omega$ and $M^{[k]} \leq M$, hence by lemma 21 (omitting the environment)

$$[\![M^{[k]}]\!] \sqsubseteq [\![M]\!]$$

that is

$$\bigsqcup_k [\![M^{[k]}]\!] \sqsubseteq [\![M]\!].$$

Let $\mathcal{I}$ be any indexing map, then by corollary 6 there exist $\mathcal{J}$ and $L \in \mathrm{N}_\oplus^\Omega$ such that $M^\mathcal{I} \rhd^* L^\mathcal{J}$. By lemma 20 $[\![M^\mathcal{I}]\!] = [\![L^\mathcal{J}]\!]$ and $L \leq M$; now $[\![L^\mathcal{J}]\!] \sqsubseteq [\![L]\!]$ and, being $L \in \mathrm{N}_\oplus^\Omega$, $L \leq M^{[k]}$, where $k = \mathsf{height}(L)$, by lemma 19; again by lemma 21 it follows that

$$[\![L]\!] \sqsubseteq [\![M^{[k]}]\!],$$

hence

$$[\![M^\mathcal{I}]\!] = [\![L^\mathcal{J}]\!] \sqsubseteq [\![L]\!] \sqsubseteq [\![M^{[k]}]\!].$$

From this we conclude, by lemma 17,

$$[\![M]\!] = \bigsqcup_\mathcal{I} [\![M^\mathcal{I}]\!] = \bigsqcup_k [\![M^{[k]}]\!].$$

□

Theorem 13 *For all $M, N \in \Lambda_\oplus$,*

$$\forall \rho \in Env.\ [\![M]\!]_\rho \sqsubseteq [\![N]\!]_\rho \Rightarrow M \sqsubseteq_{must} N.$$

Proof.

$$\begin{array}{rcll} M^{[1]} = \Omega & \Leftrightarrow & \forall k.\ M^{[k]} = \Omega & \\ & \Leftrightarrow & \forall k.\ [\![M^{[k]}]\!] = \bot & \\ & \Leftrightarrow & [\![M]\!] = \bigsqcup_k [\![M^{[k]}]\!] = \bot & \text{by lemma 22} \end{array}$$

since $[\![\Omega]\!] = \bot$; hence

$$\begin{array}{rcl} M \not\sqsubseteq_{must} N & \Rightarrow & \exists C[\,].\, C[M]\!\downarrow \wedge C[N]\!\uparrow \\ & \Rightarrow & \exists C[\,].\, \omega^1(C[M]) \neq \{\Omega\} = \omega^1(C[N]) \\ & \Rightarrow & \exists C[\,].\, [\![C[M]]\!] \neq \bot = [\![C[N]]\!] \\ & \Rightarrow & [\![M]\!] \not\sqsubseteq [\![N]\!], \end{array}$$

being the context operation the composition of abstraction, application and +, that is a monotonic function.

□

Corollary 7 *For all* $M, N \in \Lambda_{\oplus}$,

i) $M \leq N \Rightarrow [\![M]\!] \sqsubseteq [\![N]\!]$,

ii) $M \sqsubseteq_{must} N \Leftrightarrow M \leq N \Leftrightarrow [\![M]\!] \sqsubseteq [\![N]\!]$.

Proof. To establish (i):

$$\begin{array}{rcl} M \leq N & \Rightarrow & \forall k.\, M^{[k]} \leq M \leq N \\ & \Rightarrow & \forall k.\, [\![M^{[k]}]\!] \sqsubseteq [\![N]\!] \\ & \Rightarrow & [\![M]\!] \sqsubseteq [\![N]\!]. \end{array} \quad .$$

Now (ii) follows from (i) and theorems 8 and 13. □

4.5 A non-extensional model based on the notion of Monad

In this section we construct a different semilinear applicative structure, which is not extensional. In [Moggi] a notion of model for the computational λ-calculus has been introduced; this notion uses the categorical concept of monad to give a uniform treatment of various aspects of computing, such as call-by-value and continuations.

Now it has been observed in [Henn-Plo] that the powerdomain functors actually form monads over the category of **CPO**; hence Moggi's construction applies.

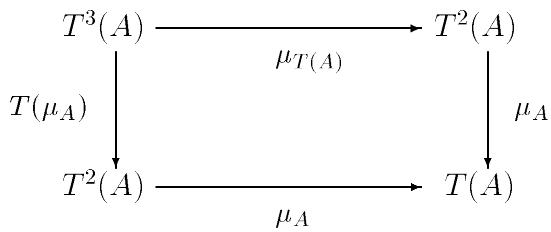
4.5.1 Strong monads

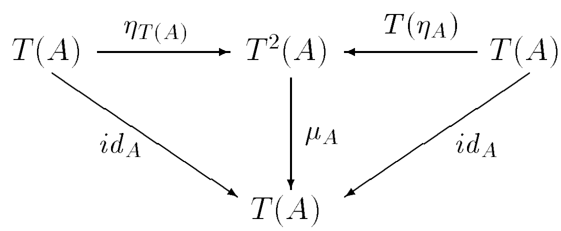
Definition 55 *Given a category $\mathcal{C}$, a* monad *is a triple $\langle T, \eta, \mu \rangle$ where*

i) $T : \mathcal{C} \to \mathcal{C}$ is a functor;

ii) $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{C}} \to T$ are natural transformations;

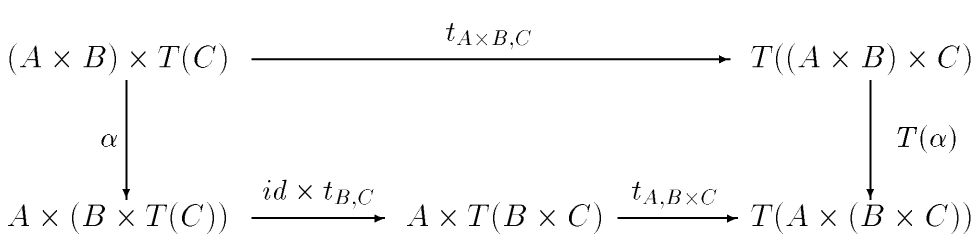
iii) the following diagrams commute:

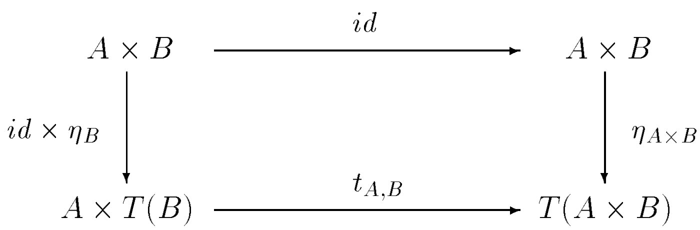


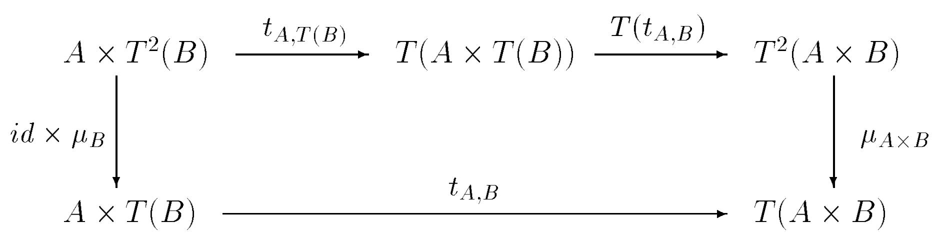


The easiest example of a monad is given by the powerset functor over the category **SET**. In this case the "multiplication" μ is given by infinitary union, while the "unit" η is the singleton function.

Definition 56 *Let $\mathcal{C}$ be a cartesian category and $\alpha : (A \times B) \times T(C) \to A \times (B \times T(C))$ the obvious isomorphisms; then a morphism $t_{A,B} : A \times T(B) \to T(A \times B)$ is a* tensorial strength *iff the following diagrams commute:*







A strong monad *is quadruple* $\langle T, \eta, \mu, t\rangle$ *such that* $\langle T, \eta, \mu\rangle$ *is a monad, and* $t_{A,B}$ *is a tensorial strength, for all objects* A *and* B.

Continuing the previous example, the tensorial strength in the case of the powerset functor is

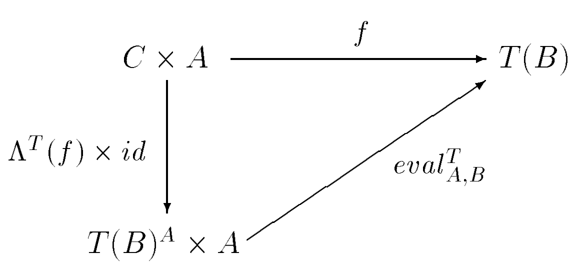
$$t(x, Y) = \{(x, y) \mid y \in Y\}.$$

In his work Moggi generalizes the notion of **CCC** to that of a cartesian category with all exponentials of a special shape; one gets a **CCC** in the particular case that the functor T is the identity.

Definition 57 *Given any strong monad* $\langle T, \eta, \mu, t\rangle$ *over a cartesian category* $\mathcal{C}$*, a* T-exponantial *is a pair* $\langle T(B)^A, eval^T_{A,B} : T(B)^A \times A \to T(B)\rangle$*, with the universal property:*

$$\forall f : C \times A \to T(B)\, \exists! \Lambda^T(f) : C \to T(B)^A .\, f = eval^T_{A,B} \circ \Lambda^T(f) \times id;$$

that is the following diagram commutes:



When modeling programs in a category $\mathcal{C}$, usually one interprets terms into morphisms, and types into objects. The idea in [Moggi] is to ditinguish between the types of values, say A, from the type $T(A)$ of computations of type A.

Now, in the categorical semantics of pure λ-calculus (see [Sco, Koy]), one works within a cartesian closed category $\mathcal{C}$; since the calculus is type free, types are just one (for the opinion that the pure calculus is a special case of the typed one see [Sco]), and $\mathcal{C}$ is supposed to have an object U, called the "universal object", such that U^U is a retract of U via (ψ, φ) (this can always be achieved in the subcategory of projections since there the exponentiation functor is covariant and locally continuous: see [Plo-Smy]) . In this way terms are interpreted into "points" of U, that is morphisms from O to U, where O is the terminal object of $\mathcal{C}$ (more precisely we have to require that U has "enough points": see [Koy]). Given two points $x, y : O \to U$, their application is defined:

$$x \cdot y = app \circ \langle x, y \rangle = eval \circ \langle \varphi \circ x, y \rangle,$$

where $app : U \times U \to U$ is $eval \circ \varphi \times id$.

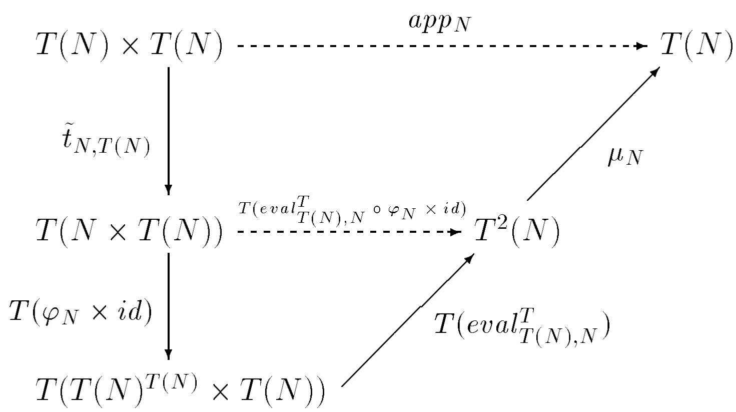
On the other hand, using the idea of $T(A)$ as the type of computations with values of type A, Moggi introduces two possible extensions of this construction, to the case of a cartesian category $\mathcal{C}$, with a strong monad $\langle T, \eta, \mu, t \rangle$ over it and all T-exponentials.

In any case programs will be interpreted as points of an object $T(A)$; if the idea is to model call-by-name value passing mechanism, then programs will take as input a computation instead of a value: the object we need in place of U above is now an N such that there is a retraction $T(N)^{T(N)} \lhd N$, via(ψ_N, φ_N).

Let $\kappa_{A,B} : A \times B \to B \times A$ the obvious isomorphism, and define $\tilde{t}_{A,B}$: $T(A) \times B \to T(A \times B)$ by

$$\tilde{t}_{A,B} = T(\kappa) \circ t_{B,A} \circ \kappa;$$

then the following diagram illustrates the construction of the morphism $app_N : T(N) \times T(N) \to T(N)$:

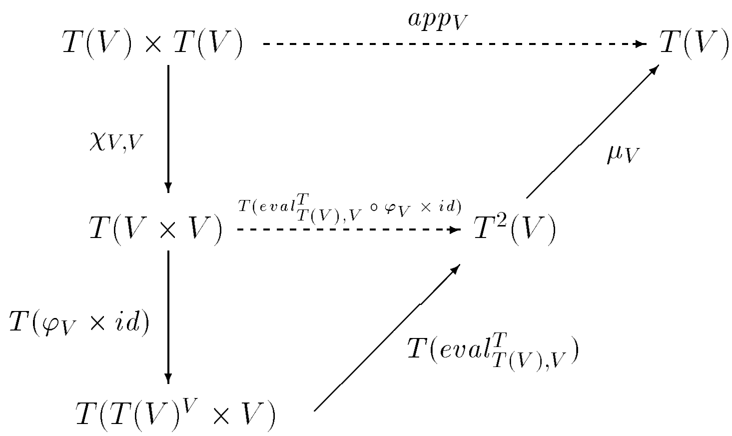


If one wishes to model call-by-value value passing mechanism, then programs have to be interpreted in points of an object $T(V)$ such that there is a retraction $T(V)^V \lhd V$: actually in this case the input of programs will be values instead of computations.

Let $\chi_{A,B} : T(A) \times T(B) \to T(A \times B)$ be defined as

$$\chi_{A,B} = \mu_{A,B} \circ T(\tilde{t}_{A,B}) \circ t_{A,B};$$

then the following diagram illustrates the construction of the morphism $app_V : T(V) \times T(V) \to T(V)$:



4.5.2 The model N_* in the category of CPO

Run-time-choice calculus is a typical case of call-by-name calculus, while call-time-choice may be seen as a kind of call-by-value calculus, supposing that our "values" are deterministic terms. We illustrate in this section how Moggi's construction can be used in the former case; the latter is similar and easier.

Our aim is to solve in the category of **CPO** the domain equation

$$N = N^\sharp \to N^\sharp,$$

with usual inverse limit techniques. We recall that $\mathbf{2}$, the two point CPO, is such that $\mathbf{2} \cong \mathbf{2}^\sharp$; it follows that $[\mathbf{2} \to \mathbf{2}] \cong [\mathbf{2}^\sharp \to \mathbf{2}^\sharp]$: by the way we can identify them. Consequently the following definition is well given:

Definition 58 *Take $N_0 = \mathbf{2}$ and $N_{n+1} = [N_n^\sharp \to N_n^\sharp]$; then inductively define $\varphi_n : N_n \to N_{n+1}$ and $\psi_n : N_{n+1} \to N_n$ by*

1. $\varphi_0(x) = \lambda y.x$, $\psi_0(y) = y(\perp)$,
2. $\varphi_{n+1}(x) = \varphi_n^\sharp \circ x \circ \psi_n^\sharp$, $\psi_{n+1}(y) = \psi_n^\sharp \circ y \circ \varphi_n^\sharp$.

The $\mathcal{O}$-functoriality of $(\cdot)^\sharp$ ensures that $\langle \varphi_n, \psi_n \rangle$ is an embedding-projection pair, for each n, so that we can take

$$N_* = \lim_{\leftarrow} (N_n, \psi_n).$$

From [Henn-Plo] we know that, for any D, $\langle D^\sharp, \{\!|\cdot|\!\}, \uplus \rangle$ is a *monad* on the category of domains (or of **SFP** objects depending on the powerdomain functor we consider). To obtain an applicative structure over $N_*^\sharp$ we need a continuous function $t : N_*^\sharp \times N_*^\sharp \to (N_*^\sharp \times N_*)^\sharp$ which is a *tensorial strength* of our monad in the sense of [Moggi]: this is achieved defining $t' : N_*^\sharp \times \mathrm{M}(N_*) \to (N_*^\sharp \times N_*)^\sharp$ by

$$t'(a, \{d^1, \ldots, d^n\}) = \{\!|\langle a, d^1\rangle|\!\} \uplus \cdots \uplus \{\!|\langle a, d^n\rangle|\!\}$$

and taking its unique continuous extension.

Proposition 17 *The quadruple* $\langle N^\sharp, \{\!|\cdot|\!\}, \uplus, t\rangle$ *is a strong monad in the sense of* [Moggi]*; in particular, for any* $a \in N_*^\sharp$ *and* $d \in N_*$ *we have* $t(a, \{\!|d|\!\}) = \{\!|\langle a, d\rangle|\!\}$.

Proof. Routine

□

We are now ready to define the application on $N_*^\sharp$.

Definition 59

$$\forall a, b \in N_*^\sharp. \quad a \cdot b =_{def} (\biguplus \circ (eval \circ H \times id)^\sharp \circ \hat{t}\,)(a, b)$$

where $\hat{t} : N_*^\sharp \times N_*^\sharp \to (N_* \times N_*^\sharp)^\sharp$ *is defined simmetrically from* t.

As for D_*, each N_n embeds into N_*, say by the $\langle \Phi_{n,*}, \Phi_{*,n}\rangle$ embedding-projection pair; setting $x_n = \Phi_{*,n}(x)$ for $x \in N_*$ and $a_n = \Phi_{*,n}^\sharp(a)$ for $a \in N_*^\sharp$ we can state:

Lemma 23 *For any* $x \in N$ *and* $a, b \in N_*^\sharp$,

i) $a = \bigsqcup_n a_n$,

ii) $\{\!|x_{n+1}|\!\} \cdot a_n = x_{n+1}(a_n)$, *iii)* $a_{n+1} \cdot b = a_{n+1} \cdot b_n = (a \cdot b_n)_n$,

iv) $a_0 \cdot b = a_0 = (a \cdot \bot)_0$.

Proof.

$$\begin{aligned} \{\!|x_{n+1}|\!\} \cdot a_n = x_{n+1}(a_n) &= (\biguplus \circ (eval \circ H \times id)^\sharp \circ \hat{t}\,)(\{\!|x_{n+1}|\!\}, a_n) \\ &= (\biguplus \circ (eval \circ H \times id)^\sharp)(\{\!|\langle x_{n+1}, a_n\rangle|\!\}) \\ &= \biguplus(\{\!|H(x_{n+1})(a_n)|\!\}) \\ &= H(x_{n+1})(a_n) \\ &= \textstyle\bigsqcup_m (x_{n+1})_{m+1}(a_n)_m \\ &= x_{n+1}(a_n) \end{aligned}$$

This establishes (ii). The rest is similar as in the proof of lemma 14.

□

Proposition 18 *The map $H : N_* \to [N_*^\sharp \to N_*^\sharp]$ defined by*

$$H(x) = \lambda a \in N_n^\sharp . \bigsqcup_n x_{n+1}(a_n)$$

and the map $K : [N_*^\sharp \to N_*^\sharp] \to N_*$ defined by

$$K(f) = \bigsqcup_n (\lambda a \in N_n^\sharp . (f(\Phi_{n,*}(a)))_n)$$

are continuous and mutually inverse.

Proof. Similar to the proof of therem 11 using lemma 23

□

Proposition 19 *The triple $\langle N_*^\sharp, \cdot, \uplus\rangle$ is a semilinear applicative structure; it is however not linear.*

*Proof.*This is a consequence of the fact that $\tilde{t}$ is linear in its first argument by construction, while $\biguplus$ and $(eval \circ H \times id)^\sharp$ are linear being obtained by using ext: the composition of linear functions is linear. However the function $\{\!| \cdot |\!\}$ is not linear: hence the application is not linear in its second argument.

□

Definition 60 *Given the structure* $\langle N^\sharp_*, \cdot, \uplus\rangle$ *and* $\rho \in Env = Var \to N^\sharp_*$, *we define the map* $[\![\cdot]\!] : \Lambda_\oplus \to (Env \to N^\sharp_*)$ *as follows:*

i) $[\![x]\!]_\rho = \rho(x)$,

ii) $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$,

iii) $[\![\lambda x.M]\!]_\rho = \{\!|K(\lambda a \in N^\sharp_*.[\![M]\!]_{\rho[a/x]})|\!\}$,

iv) $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho \uplus [\![N]\!]_\rho$.

This is a good definition, since in (iii) the continuity of application, abstraction and $\uplus$ ensures that the function $\lambda a \in N^\sharp_*.[\![M]\!]_{\rho[a/x]}$ is continuous as well.

Theorem 14 *The quadruple* $\langle N^\sharp_*, \cdot, \uplus, [\![\cdot]\!]\rangle$ *is a syntactical model.*

*Proof.*By proposition 19 we know that $\langle N^\sharp_*, \cdot, \uplus\rangle$ is a semilinear applicative structure; excluding in the definition 51 those clause which are immediately satisfied, we are left with clauses (iii) and (vi).

Clause (iii): Let $f \in [N^\sharp_* \to N^\sharp_*]$ and $a \in N^\sharp_*$, then

$$\begin{array}{rcll} \{\!|K(f)|\!\} \cdot a & = & (\biguplus \circ (eval \circ H \times id)^\sharp \circ \tilde{t})(\{\!|K(f)|\!\}, a) & \\ & = & (\biguplus \circ (eval \circ H \times id)^\sharp)\{\!|\langle K(f), a\rangle|\!\} & \text{prop. 17} \\ & = & \biguplus \{\!|(H \circ K)(f)(a)|\!\} & \text{by def. 55} \\ & = & \biguplus \{\!|f(a)|\!\} & \text{since } H \circ K = id \\ & = & f(a) & \text{by def. 55} \end{array}$$

and (iii) follows since

$$\begin{array}{rcl} [\![\lambda x.M]\!]_\rho \cdot a & = & \{\!|H(\lambda b.[\![M]\!]_{\rho[b/x]})|\!\} \cdot a \\ & = & (\lambda b.[\![M]\!]_{\rho[b/x]})(a) \\ & = & [\![M]\!]_{\rho[a/x]}. \end{array}$$

Clause (vi):

$$\begin{array}{rcl} \forall a \in N^\sharp_*.\ [\![M]\!]_{\rho[a/x]} = [\![N]\!]_{\rho[a/x]} & \Rightarrow & \lambda a.[\![M]\!]_{\rho[a/x]} = \lambda a.[\![N]\!]_{\rho[a/x]} \\ & \Rightarrow & \{\!|H(\lambda a.[\![M]\!]_{\rho[a/x]})|\!\} = \{\!|H(\lambda a.[\![N]\!]_{\rho[a/x]})|\!\} \\ & \Rightarrow & [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho \end{array}$$

by definition 60 (iii).

$\square$

Remark 6 $N_* \not\models \lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$: this follows from the fact that the interpretation of an abstraction is a "singleton", while the interpretation of a sum is the "union" of the interpretations of the summands. Now consider two functions $f, g : N_*^\sharp \rightarrow N_*^\sharp$, such that there is no upper bound to them: then $K(f)$ and $K(g)$ have no upper bound in N_*; it follows that $\{\!|f|\!\} \oplus \{\!|g|\!\}$ cannot be in the homomorphic image of N_* into $N_*^\sharp$ via $\{\!| \cdot |\!\}$.

Chapter 5

Theories

5.1 The theory λ_c

The reduction relation $\longrightarrow_c$ is connected to the equational theory λ_c introduced in [Sharma]; there Sharma proved that a subtheory of λ_c was consistent, namely the theory obtained deleting axiom (γ). The result was established by defining a notion of reduction (different from $\longrightarrow_c$) essentially by orienting from left to right the axioms of λ_c, and then proving a Church-Rosser theorem.

The proof of the consistency theorem was however very long, and the difficulty with the axiom (γ) couldn't be overcome. On the contrary we give here a very short proof of the consistency of the whole theory, in a way that, in our opinion, enlights the fact that the λ_c-calculus is nothing more than a calculus of finite sets of classical terms.

Definition 61 *The theory λ_c is the equational theory over $\Lambda_\oplus$ whose axioms and rules are as follows*

(β_c) $(\lambda x.M)N = M[N/x]$ *if* $N \in \Lambda$;

(ρ) $M = M$;

(σ) $M = N \Rightarrow N = M$;

(τ) $M = N, N = L \Rightarrow M = L$;

(μ) $M = N \Rightarrow LM = LN$;

(ν) $M = N \Rightarrow ML = NL$;

(ξ) $M = N \Rightarrow \lambda x.M = \lambda x.N$;

(ζ_1) $M \oplus M = M$;

(ζ_2) $M \oplus N = N \oplus M$;

(ζ_3) $(M \oplus N) \oplus L = M \oplus (N \oplus L)$;

(ε) $M = N \Rightarrow M \oplus L = N \oplus L$;

(δ) $(M \oplus N)L = ML \oplus NL$;

(θ) $L(M \oplus N) = LM \oplus LN$;

(γ) $\lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$;

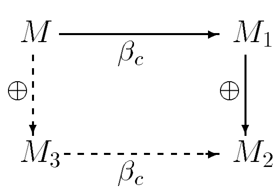
As a first step we prove a simple property of the reduction relation $\longrightarrow_c$, that fails in case of $\longrightarrow_r$.

Notation: We will write $\longrightarrow_{\beta_c}$ when a one step β_c-contraction occurs; similarly we write $\longrightarrow_\oplus$ when only one $\oplus$-contraction occurs. $\stackrel{*}{\longrightarrow}_{\beta_c}$ and $\stackrel{*}{\longrightarrow}_\oplus$ are their reflexive and transitive closures respectively.

Lemma 24

$$\forall M, M_1, M_2 \in \Lambda_\oplus.\ M \longrightarrow_{\beta_c} M_1 \longrightarrow_\oplus \Rightarrow \exists M_3 \in \Lambda_\oplus.\ M \longrightarrow_\oplus M_3 \longrightarrow_{\beta_c} M_2$$

that is



Proof. By induction on M, and then by cases. The only interesting case is when $M \equiv (\lambda x.M')M''$ and $M_1 \equiv M'[M''/x]$; in $M_1 \longrightarrow_\oplus M_2$ the only possibility is that a (residual of a) $\oplus$ redex in M' is contracted, since it must be the case that $M'' \in \Lambda$. It follows that $M' \equiv C[P_1 \oplus P_2]$ for some P_1 and

P_2 and $M_1 \equiv C'[P_1[M''/x] \oplus P_2[M''/x]]$ if x is not bounded above in $C[\,]$ and $C'[\,]$ results from $C'[\,]$ substituting M'' for all free occurences of x; in this case $M_2 \equiv C'[P_i[M''/x]]$ for $i = 1$ or 2. Then

$$(\lambda x.M')M'' \longrightarrow_{\oplus} (\lambda x.C[P_i])M'' \longrightarrow_{\beta_c} (C[P_i])[M''/x] \equiv C'[P_i[M''/x]]$$

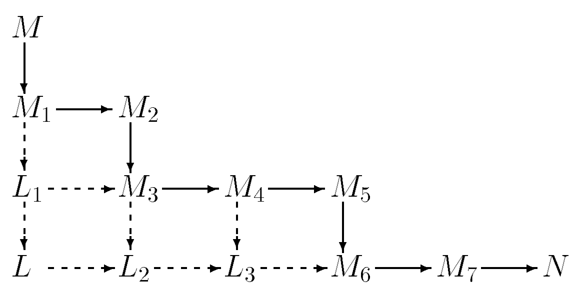
so that we take $M_3 \equiv (\lambda x.C[P_i])M''$. If x is bounded above the hole $[\,]$ in the context $C[\,]$ the proof is similar and easier.

□

Corollary 8

$$\forall M, N \in \Lambda_{\oplus}.\ M \xrightarrow{*}_c N \Rightarrow \exists L \in \Lambda_{\oplus}.\ M \xrightarrow{*}_{\oplus} L \xrightarrow{*}_{\beta_c} N.$$

Proof. The proof is illustrated in the following picture, where vertical arrows represent one-step $\oplus$-reductions, horizontal arrows represent one-step β_c-reductions, and each square is an application of lemma 24:



□

Remark 7 This corollary is not true in the case of $\longrightarrow_r$: for a similar result, but under stronger restrictions, see [Sharma].

Definition 62 *Let $\mathcal{A} \subseteq \Lambda$ and $M, N \in \Lambda_{\oplus}$, then*

i) $\mathcal{A}^+ = \{M \mid \exists N \in \mathcal{A}.\ M =_\beta N\}$;

ii) $\mathsf{det}(M) = \{L \in \Lambda \mid M \xrightarrow{*}_c L\}^+$;

iii) $M \subseteq_c N \;\Leftrightarrow\; \mathsf{det}(M) \subseteq \mathsf{det}(N)$;

iv) $M =_c N \;\Leftrightarrow\; M \subseteq_c N \subseteq_c M$.

The operation $(\cdot)^+$ is the usual closure under β-conversion; the intuitive meaning of $\mathsf{det}(M)$ is “the set of deterministic values of M”.

Definition 63 *Let $\mathcal{A} \subseteq \Lambda$; $\mathcal{A}$ is β-closed iff $\mathcal{A} = \mathcal{A}^+$. Furthermore if $\mathcal{A}, \mathcal{B}$ are β-closed then*

i) $\mathcal{A}\mathcal{B} = \{MN \mid M \in \mathcal{A}, N \in \mathcal{B}\}^+$;

ii) $\lambda x.\mathcal{A} = \{\lambda x.M \mid M \in \mathcal{A}\}^+$;

iii) $\mathcal{A}[\mathcal{B}/x] = \{M[N/x] \mid M \in \mathcal{A}, N \in \mathcal{B}\}^+$.

Lemma 25 *For any $M, N \in \Lambda_\oplus$*

i) $\mathsf{det}(M \oplus N) = \mathsf{det}(M) \cup \mathsf{det}(N)$;

ii) $\mathsf{det}(\lambda x.M) = \lambda x.\mathsf{det}(M)$;

iii) $\mathsf{det}(MN) = \mathsf{det}(M)\mathsf{det}(N)$.

Proof. Parts (i) and (ii) are clear. To see (iii):

$$L \in \mathsf{det}(MN) \Rightarrow \exists L' \in \Lambda.\ MN \xrightarrow{*}_c L' =_\beta L$$

by definition; by corollary 8 there is a $P \in \Lambda_\oplus$ s.t.

$$MN \xrightarrow{*}_\oplus P \xrightarrow{*}_{\beta_c} L';$$

now $MN \xrightarrow{*}_\oplus P$ implies that $P \equiv M'N'$ where $M \xrightarrow{*}_\oplus M'$ and $N \xrightarrow{*}_\oplus N'$. On the other hand we note that no β_c contraction can delete an occurrence of a $\oplus$ e.g.:

$$\mathbf{K}L(M \oplus N) \not\xrightarrow{*}_{\beta_c} L$$

since $M \oplus N \notin \Lambda$. It follows that $P \xrightarrow{*}_{\beta_c} L'$ implies $P \in \Lambda$ being $L' \in \Lambda$. We conclude that $L \in \mathsf{det}(M)\mathsf{det}(N)$, that is $\mathsf{det}(MN) \subseteq \mathsf{det}(M)\mathsf{det}(N)$. The inverse inclusion is clear.

□

Lemma 26

$$M \in \Lambda_{\oplus}, N \in \Lambda \Rightarrow \mathsf{det}(M[N/x]) = \mathsf{det}(M)[\mathsf{det}(N)/x].$$

Proof. By induction on M.

Case 1: $M \equiv x$ then

$$\mathsf{det}(x[N/x]) = \mathsf{det}(N) = \mathsf{det}(x)[\mathsf{det}(N)/x].$$

Case 2: $M \equiv y \not\equiv x$ then

$$\mathsf{det}(y[N/x]) = \mathsf{det}(y) = \mathsf{det}(y)[\mathsf{det}(N)/x].$$

Case 3: $M \equiv M_1 M_2$ then

$$\begin{array}{rcll} \mathsf{det}((M_1M_2)[N/x]) & = & \mathsf{det}(M_1[N/x]M_2[N/x]) & \\ & = & \mathsf{det}(M_1[N/x])\mathsf{det}(M_2[N/x]) & \text{by lemma 25 (iii)} \\ & = & \mathsf{det}(M_1)[\mathsf{det}(N)/x]\mathsf{det}(M_2)[\mathsf{det}(N)/x] & \text{by ind. hyp.} \\ & = & \mathsf{det}(M_1M_2)[\mathsf{det}(N)/x] & \text{since } N \in \Lambda \end{array}$$

where in the last step above we observe that, since $N \in \Lambda$, $\mathsf{det}(N)$ is a set of β-convertible terms; now in the classical calculus we know that

$$Q_1 =_\beta Q_2 \Rightarrow P[Q_1/x] =_\beta P[Q_2/x]$$

from which it follows that

$$\begin{array}{rcl} (P_1[Q_1/x])(P_2[Q_2/x]) & =_\beta & (P_1P_2)[Q_1/x] \\ & =_\beta & (P_1P_2)[Q_2/x]. \end{array}$$

Case 4: $M \equiv \lambda x.M'$: trivial.

Case 5: $M \equiv \lambda y.M'$, where $y \not\equiv x$, and supposing $x \notin \mathrm{FV}(N)$ then

$$\begin{array}{rcll} \mathsf{det}((\lambda y.M')[N/x]) & = & \mathsf{det}(\lambda y.M'[N/x]) & \\ & = & \lambda y.\mathsf{det}(M'[N/x]) & \text{by lemma 25 (ii)} \\ & = & \lambda y.\mathsf{det}(M')[\mathsf{det}(N)/x] & \text{by ind. hyp.} \\ & = & \mathsf{det}(\lambda y.M')[\mathsf{det}(N)/x] & \text{by lemma 25 (ii).} \end{array}$$

Case 6: $M \equiv M_1 \oplus M_2$ then

$$\begin{aligned}
\mathsf{det}((M_1 \oplus M_2)[N/x] &= \mathsf{det}(M_1[N/x] \oplus M_2[N/x]) \\
&= \mathsf{det}(M_1[N/x]) \cup \mathsf{det}(M_2[N/x]) \\
&= \{P[Q/x] \mid P \in \mathsf{det}(M_1) \cup \mathsf{det}(M_2), Q \in \mathsf{det}(N)\} \\
&= \{P[Q/x] \mid P \in \mathsf{det}(M_1 \oplus M_2), Q \in \mathsf{det}(N)\} \\
&= \mathsf{det}(M_1 \oplus M_2)[\mathsf{det}(N)/x]
\end{aligned}$$

using lemma 25 (i).

□

Theorem 15 *For any $M, N \in \Lambda_{\oplus}$*

$$\lambda_c \vdash M = N \Rightarrow M =_c N.$$

Proof.

(β_c) $(\lambda x.M)N = M[N/x]$ if $N \in \Lambda$; let $M \in \Lambda_{\oplus}$ and $N \in \Lambda$, then using lemma 25

$$\begin{aligned}
\mathsf{det}((\lambda x.M)N) &= \mathsf{det}(\lambda x.M)\mathsf{det}(N) \\
&= (\lambda x.\mathsf{det}(M))\mathsf{det}(N) \\
&= \mathcal{A}
\end{aligned}$$

say, then

$$\begin{aligned}
P \in \mathcal{A} &\Leftrightarrow \exists Q \in \mathsf{det}(M).\ P =_\beta (\lambda x.Q)N =_\beta Q[N/x] \\
&\Leftrightarrow P \in \mathsf{det}(M)[\mathsf{det}(N)/x] = \mathsf{det}(M[N/x]),
\end{aligned}$$

by lemma 26.

(ρ), (σ), (τ) Obvious.

(μ)

$$\begin{aligned}
\mathsf{det}(ML) &= \mathsf{det}(M)\mathsf{det}(N) && \text{by lemma 25 (iii)} \\
&= \mathsf{det}(N)\mathsf{det}(L) && \text{by ind. hyp.} \\
&= \mathsf{det}(NL) && \text{by lemma 25 (iii).}
\end{aligned}$$

(ν) Similar to the previous one.

(ξ) Immediate consequence of lemma 25 (ii).

(ζ_1), (ζ_2), (ζ_3) From $\mathsf{det}(M \oplus N) = \mathsf{det}(M) \cup \mathsf{det}(N)$ (lemma 25 (i)) and idempotency, commutativity and associativity of $\cup$.

(ε) Immediate from lemma 25 (i).

(δ)

$$\begin{array}{rcll} \mathsf{det}((M \oplus N)L) & = & \mathsf{det}(M \oplus N)\mathsf{det}(L) & \text{by lemma 25 (iii)} \\ & = & (\mathsf{det}(M) \cup \mathsf{det}(N))\mathsf{det}(L) & \text{by lemma 25 (i)} \\ & = & \mathsf{det}(M)\mathsf{det}(L) \cup \mathsf{det}(N)\mathsf{det}(L) & \\ & = & \mathsf{det}(ML \oplus NL) & \text{by lemma 25 (i),(iii).} \end{array}$$

(θ) Similar to the previous one.

(γ) $\lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$;

$$\begin{array}{rcll} \mathsf{det}(\lambda x.M \oplus N) & = & \lambda x.\mathsf{det}(M \oplus N) & \text{by lemma 25 (ii)} \\ & = & \lambda x.(\mathsf{det}(M) \cup \mathsf{det}(N)) & \text{by lemma 25 (i)} \\ & = & \lambda x.\mathsf{det}(M) \cup \lambda x.\mathsf{det}(N) & \\ & = & \mathsf{det}(\lambda x.M \oplus \lambda x.N) & \text{by lemma 25 (i), (ii).} \end{array}$$

□

We conjecture that $M =_c N \Rightarrow \lambda_c \vdash M = N$.

Corollary 9 *The theory λ_c is consistent.*

Proof. For any $M, N \in \Lambda$

$$M =_c N \;\Leftrightarrow\; \mathsf{det}(M) = \mathsf{det}(N) \;\Leftrightarrow\; M =_\beta N,$$

that is $=_c$ restricted to Λ coincides with $=_\beta$. This implies that the theory induced by $=_c$ is a conservative extension of λ: hence it is consistent. Then, by the theorem, λ_c is consistent.

□

5.2 The theory λ_r

In previous chapters we studied the properties of the relation $\longrightarrow_r$ and gave both operational and denotational characterizations of the equivalence of programs it induces. In this section our aim is to present an axiomatization of this equivalence, allowing to compare this relation with the β-convertibility relation of the classical λ-calculus. Here too, as in the case of the theory λ_c we get inspiration from [Sharma] and [Boud91].

Definition 64 *The theory λ_r is the equational theory over $\Lambda_\oplus$ whose axioms and rules are as follows*

(β_r) $(\lambda x.M)N = M[N/x]$;

(ρ) $M = M$;

(σ) $M = N \Rightarrow N = M$;

(τ) $M = N, N = L \Rightarrow M = L$;

(μ) $M = N \Rightarrow LM = LN$;

(ν) $M = N \Rightarrow ML = NL$;

(ξ) $M = N \Rightarrow \lambda x.M = \lambda x.N$;

(ζ_1) $M \oplus M = M$;

(ζ_2) $M \oplus N = N \oplus M$;

(ζ_3) $(M \oplus N) \oplus L = M \oplus (N \oplus L)$;

(ϵ) $M = N \Rightarrow M \oplus L = N \oplus L$;

(δ) $(M \oplus N)L = ML \oplus NL$;

Remark 8 This theory has been presented in [Sharma]. It is very similar to the theory λ_c. It differs however because of the lack of axiom (θ) and of the unrestricted axiom (β_r); this is responsible for the fact that the theory λ_r is not a subtheory of λ_c.

This theory was proved consistent in [Sharma] with syntatical methods: for us it is actually a corollary of previous results.

Theorem 16 *The syntatical model D_* is a non trivial model of λ_r, hence λ_r is consistent.*

Proof. By definition 51, lemma 16 and proposition 16 we know that D_* is a synctactical model (actually an extensional one); now the proof that the equations and rules up to (ξ) are validates runs as in the classical way; the rest is an immediate consequence of the definition of the interpretation of $\oplus$ and of the semilinearity of D_*.

□

In his proof Sharma didn't prove consistency of the full theory, which in his formulation had among its axioms also

$$(\gamma) \qquad \lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N.$$

We do not include this axiom in the theory λ_r because of its special status, illustrated in the following proposition.

Another equation which appeared in the literature (see [Ash-Henn]) is

$$(\iota) \qquad M \oplus N = \lambda x.Mx \oplus Nx \quad \text{if } x \notin \mathrm{FV}(M \oplus N);$$

it is clearly connected (actually equivalent) with the axiom η.

Proposition 20

i) $\lambda_r \nvdash \gamma$;

ii) $\lambda_r + \eta \vdash \gamma$;

iii) $\lambda_r + \iota \vdash \eta$.

Proof.

(i) The non extensional model N_* is a model of λ_r but not of γ.

(ii) Let $y \notin \mathrm{FV}(M) \cup \mathrm{FV}(N)$:

$$\begin{aligned}(\lambda x.M \oplus N)y &= M[y/x] \oplus N[y/x] && \text{by } (\beta_r)\\ &= (\lambda x.M)y \oplus (\lambda x.N)y && \text{by } (\beta_r)\\ &= (\lambda x.M \oplus \lambda x.N)y && \text{by } (\delta)\end{aligned}$$

from which $\lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$ follows by (η).

(iii) Let $x \notin \mathrm{FV}(M)$:

$$\begin{array}{rcll} M & = & M \oplus M & \text{by } (\zeta_1) \\ & = & \lambda x.Mx \oplus Mx & \text{by } (\iota) \\ & = & \lambda x.Mx & \text{by } (\zeta_1). \end{array}$$

□

We leave as an open question whether the theory $\lambda_r + \gamma$ is extensional, but we conjecture it is not.

5.3 The theory $\mathcal{T}_{must}$

In this section we compare the theory induced by the equivalence studied throughout this thesis with the theories $\lambda + \eta$ and $\mathcal{H}^*$. The main theorem is a conservativity result; it is readily seen that this can be considered as an alternative (synctatical) proof of the consistency for the theory $\mathcal{T}_{must}$ defined below.

Definition 65 $\mathcal{T}_{must} = \{M = N \mid M, N \in \Lambda^0_\oplus,\ M \simeq_{must} N\}$.

We know from the semantic construction in the previous chapters that:

Proposition 21 *The theory $\mathcal{T}_{must}$ is the theory of the model $\mathcal{M}$, hence it is consistent.*

Proof. Immediate consequence of the full abstraction theorem.

□

This fact is not illuminating, however, with respect to the question of conservativity. To establish this result we are going to prove a "simulation lemma" which says that contexts containing the $\oplus$ operator do not discriminate more w.r.t. must-convergency than classical contexts do.

Definition 66 *Let $M \in \Lambda_\oplus$, $\mathcal{F} \subseteq M$ a subset of the set of redexes occurring in M, and σ a head reduction starting with M; then*

i) σ is finitely often *in $\mathcal{F}$ iff*

$$\exists m\, \forall n \geq m.\ \sigma_{0,n+1} : M \xrightarrow{*}_r M_n \xrightarrow{\Delta}_r M_{n+1} \Rightarrow \Delta \notin \mathcal{F}/\sigma_{0,n};$$

ii) let σ be any head reduction of M finitely often in $\mathcal{F}$ then

$$\deg(\mathcal{F},\sigma) = \max\{n \mid \exists m \in \omega\, \exists R \in \mathcal{F}/\sigma_{0,m}.\ M \xrightarrow{m}_h \lambda\vec{x}.RM_1 \ldots M_n\};$$

iii) let $\sigma_1, \ldots, \sigma_n$ be head reductions of M finitely often in $\mathcal{F}$ then

$$\deg(\mathcal{F},\sigma_1,\ldots,\sigma_n) = \max\{\deg(\mathcal{F},\sigma_i) \mid 1 \leq i \leq n\}.$$

Remark 9 σ is *finitely often* in $\mathcal{F}$ iff it contracts at most a finite number of (residuals of) redexes in $\mathcal{F}$.

The next lemma is based on the idea that, if a context containing the operator $\oplus$ converges on a term M, while it diverges on a term N, the choices caused by the $\oplus$'s inside the context which are essential for this convergecy-divergency property, are bounded above by those which are necessary to converge on M. On the other hand we know that all the reductions on M will converge, while there is a diverging reduction on N. The point is to simulate the choices of this last reduction, encoding them in a classical context.

Lemma 27 *Given $M, N \in \Lambda_\oplus$*

$$\exists D[\,] \in \Lambda_\oplus[\,].\ D[M]\downarrow \wedge D[N]\uparrow \Rightarrow \exists C[\,] \in \Lambda[\,].\ C[M]\downarrow \wedge C[N]\uparrow .$$

Proof. W.l.o.g. let us suppose that $M, N \in \Lambda^0_\oplus$; then exists $F \in \Lambda_\oplus$ s.t. $FM\downarrow$ and $FN\uparrow$. Let F' be the term in Λ obtained from F by substituting all occurrences of a subterm of the form $P \oplus Q$ with an occurrence of xPQ, where x is a fresh variable. For any $r \in \omega$ define

$$T_r \equiv \lambda xyz_1 \ldots z_r w.w(xz_1 \ldots z_r)(yz_1 \ldots z_r).$$

We show that there exists an $F'' \in \Lambda$ and a vector $\vec{L} \in (Var \cup \{\mathbf{K}, \mathbf{O}, \Omega\})^*$ s.t.

$$F''M\vec{L}\downarrow \quad \text{and} \quad F''N\vec{L}\uparrow .$$

Let $\tau_1, \ldots, \tau_m$ be the set of the head reductions of FM, and σ be any divergent head reduction of FN; let $\mathcal{F}$ be the set of $\oplus$ redexes of F.

Case 1: σ doesn't contract any redex in $\mathcal{F}$: then choose $F'' \equiv F'$ and $\vec{L}$ is empty.

Case 2: $\tau_1, \dots, \tau_m$ do not contract any redex in $\mathcal{F}$: then choose $F'' \equiv F'[\Omega/x]$.

Case 3: both σ and $\tau_1, \dots, \tau_m$ contract redexes in $\mathcal{F}$. Since $\tau_1, \dots, \tau_m$ are finite, they are finitely often in $\mathcal{F}$, hence $k = \deg(\mathcal{F}, \tau_1, \dots, \tau_m)$ for some k. We proceed as follows

- we choose an $r \geq k$ and take $F'' \equiv F'[T_r/x]$;
- we perform all possible head reductions of $F''M$ until either a head normal form is reached, or a term with T_r in head position;
- we reduce $F''N$ until a term with T_r in head position is reached: this must happen since σ reduces some redex in $\mathcal{F}$ and no head normal form can be reached, otherways we would have $FN\!\downarrow$.

Suppose that the term obtained in the reduction of $F''N$ is

$$\lambda\vec{x}.(T_r P Q) N_1 \dots N_m$$

and, supposing r choosen greater than m, the next steps in the head reduction of $F''N$ will give

$$U_0 \equiv \lambda\vec{x} z_{m+1} \dots z_r w.w(PN_1 \dots N_m z_{m+1} \dots z_r)(QN_1 \dots N_m z_{m+1} \dots z_r).$$

We note that $w \notin \mathrm{FV}(PN_1 \dots N_m) \cup \mathrm{FV}(QN_1 \dots N_m)$. Correspondingly from the reductions of $F''M$ we get

$$\begin{array}{lcl} U_1 & \equiv & \lambda\vec{x}_1.(T_r P_1 Q_1) M_{1,1} \dots M_{1,m_1} \\ & \dots & \\ U_q & \equiv & \lambda\vec{x}_q.(T_r P_q Q_q) M_{q,1} \dots M_{q,m_q} \\ U_{q+1} & \equiv & \lambda\vec{x}_{q+1}.\xi_{q+1} M_{q+1,1} \dots M_{q+1,m_{q+1}} \\ & \dots & \\ U_p & \equiv & \lambda\vec{x}_p.\xi_p M_{p,1} \dots M_{p,m_p} \end{array}$$

and from these, for $1 \leq i \leq q$, the head reductions proceed giving certain U'_i of the form

$$\begin{array}{ll} \lambda\vec{x}_i z_{m_i+1} \dots z_r w.w & (P_i M_{i,1} \dots M_{i,m_i} z_{m_i+1} \dots z_r) \\ & (Q_i M_{i,1} \dots M_{i,m_i} z_{m_i+1} \dots z_r), \end{array}$$

where we make a similar remark about the w as for U_0. Because of our assumptions all head variables appearing in the terms above are bounded variables, hence thy must occurr in the prefixed string of abstractions of the respective terms. For any closed term in head normal form define its "head distance" as follows:

$$\mathrm{hd}(\lambda v_1 \dots v_n.\xi R_1 \dots R_m) = i \quad \text{if } \xi \equiv v_i.$$

Now we can always assume that for all $q+1 \leq i \leq p$

$$\mathrm{hd}(U_0) \neq \mathrm{hd}(U_i),$$

because we simply suppose the r to be choosen suitably large. If this condition is satisfied also for $1 \leq i \leq q$ then we take $F'' \equiv F'[T_r/x]$ and $\vec{L} \equiv y_1 \dots y_{h-1}\Omega$, where $h = \mathrm{hd}(U_0)$, and we are done.

Hovever nothing prevents us from having some U'_j, where $1 \leq j \leq q$, s.t. $\mathrm{hd}(U'_j) = \mathrm{hd}(U_0)$, and of course this cannot be remedied with a choice of r, since both head distances will depend on it. In this case suppose that the original reduction σ has, after $\lambda\vec{x}.(P \oplus Q)N_1 \dots N_m$, a choice to the left, namely it continues with $\lambda\vec{x}.PN_1 \dots N_m$. In this case, if $l = \max\{|\vec{x}|, |\vec{x}_{q+1}|, \dots, |\vec{x}_p|\}$, then take

$$\vec{L} \equiv y_1 \dots y_{h-1}\mathbf{K}y_{h+1} \dots y_l\vec{L}',$$

where $\vec{L}'$ remains to be determined. (Clearly, if the choice is to the right, we take $\mathbf{O}$ instead of $\mathbf{K}$). By the way

$$U'_i\vec{L}\downarrow \qquad \text{if } i \neq 0 \text{ and } \mathrm{hd}(U'_i) \neq \mathrm{hd}(U_0),$$

since the head variable will be replaced by some y, and the rest can be ignored; otherwise

$$U_0\vec{L} \xrightarrow{*}_h PN_1 \dots N_m\vec{y},$$

and

$$U'_i\vec{L} \xrightarrow{*}_h P_iM_{i,1} \dots M_{i,m_i}\vec{y}\,'$$

where $\vec{y}, \vec{y}\,' \subseteq y_1 \dots y_{h-1}y_{h+1} \dots y_l$.

If either σ or $\tau_1, \dots, \tau_m$ do not contract any other redex in $\mathcal{F}$, we are in a case similar to case 1 or to case 2: consequently we shall choose the $\vec{L}'$ accordingly. Otherways the present case applies, and we repeat the same reasoning. This process, however, is bounded because the

$\tau_1, \ldots, \tau_m$ were finitely often in $\mathcal{F}$. This implies that we must reach a point in which either (the simulation of) σ definitely diverges, or all the reducts obtained from (the simulation of) $\tau_1, \ldots, \tau_m$ are similar to the U_i above, when $q+1 \leq i \leq p$: that is we can suppose that they all have a different head distance from that of the term coming from σ. In the former case we add nothing to the $\vec{L}$ constructed up to that point; in the last case we add

$$w_1 \ldots w_s \Omega,$$

supposing s to be the head distance of the term coming from σ.

□

We remind the reader that

$$\mathcal{H}^* = \{M = N \mid M, N \in \Lambda^0, \forall C[\,] \in \Lambda[\,].\ C[M] \in \mathbf{SOL} \ \Leftrightarrow\ C[N] \in \mathbf{SOL}\}.$$

It is known that $\mathcal{H}^*$ is the theory of the model D_∞. We conclude this section with the conservativity theorem.

Theorem 17

i) $\lambda_r + \gamma \subseteq \mathcal{T}_{must}$*,*

ii) $\mathcal{T}_{must}$ *is a conservative extension of* $\mathcal{H}^*$*,*

iii) $\mathcal{T}_{must}$ *is a conservative extension of* $\lambda + \eta$*.*

Proof. To prove the first part, simply note that $\mathcal{T}_{must}$ is the theory of a model of λ_r, and that this model validates γ since it is extensional (by lemma 16). As to (ii): let $M, N \in \Lambda^0$ be such that $\mathcal{T}_{must} \not\vdash M = N$; then there is a context $D[\,] \in \Lambda_\oplus[\,]$ such that, say, $D[M]\!\downarrow$ and $D[N]\!\uparrow$. By lemma 27, there is a context $C[\,] \in \Lambda[\,]$ such that $C[M]\!\downarrow$, that is $C[M] \in \mathbf{SOL}$ and $C[N] \notin \mathbf{SOL}$; hence $\mathcal{H}^* \not\vdash M = N$: it follows that $\mathcal{H}^* \subseteq \mathcal{T}_{must}$. On the other hand, and a fortiori, if $M, N \in \Lambda^0$, then

$$\begin{aligned} \mathcal{T}_{must} \vdash M = N &\Rightarrow \forall D[\,] \in \Lambda_\oplus[\,].\ D[M]\!\downarrow \Leftrightarrow D[N]\!\downarrow \\ &\Rightarrow \forall C[\,] \in \Lambda[\,].\ C[M] \in \mathbf{SOL} \ \Leftrightarrow\ C[N] \in \mathbf{SOL} \\ &\Rightarrow \mathcal{H}^* \vdash M = N, \end{aligned}$$

since $\downarrow \lceil \Lambda = \mathbf{SOL}$.
Finally (iii) follows from (ii) and the fact that $\mathcal{H}^*$ is an extensional λ-theory.

□

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