Total Functionals and Well-founded Strategies
(Extended abstract)

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Abstract. In existing game models, total functionals have no simple characterization neither in term of game strategies, nor in term of the total set-theoretical functionals they define. We show that the situation changes if we extend the usual notion of game by allowing infinite plays. Total functionals are, now, exactly those having a tree-strategy in which all branches end in a last move, winning for the strategy. Total functionals now define (via an extensional collapse) all set-theoretical functionals. Our model is concrete: we used infinite computations only to have a nice characterization of totality. A computation may be infinite only when the input is a discontinous functional; in practice, never.

1 Introduction

Games and strategies have emerged as useful tools to model interaction, with applications both to logic and to the theory of higher type functionals.

We address the problem of characterizing total functionals in game theoretic models. A natural conjecture is that a functional is total if and only if it is the extensional counterpart of some winning well-founded strategy. This would mean that a total functional can always be described via strategies whose plays eventually end, after finitely many steps, in some move by the Player, which Opponent cannot reply to.

We prove, however, that this is the case only (and exactly) for Tait-definable functionals, and that some interesting computable total functionals have infinite branches in any strategy defining them. This calls for a generalization of the notion of play to ordinal sequences of moves (possibly of transfinite length), and for a proper notion of winning strategy. Later, we will remark that infinite plays arise only in the application of a functional to some discontinous functional. Hence transfinite plays are relevant to have a nice characterization of total maps, but they cannot arise in practice.

In the literature game theoretic concepts have been proposed to construct models of lambda calculi, by extensionally collapsing certain sets of strategies. There have been two proposals: the first one is based on the idea of history-free strategies [3]; according to the second one players move depending on "views" of the play: these are called dialog games and innocent strategies, as defined in [10, 11].

In [7] an apparently different notion of game, originally introduced by Novikoff, is used to give an intuitionistic explanation of the classical notion of truth. As it will be explained in sections 2 and 3 of the present paper, dialog games and Novikoff-Coquand games are closely related: the former can be obtained from the latter by distinguishing between question and answer moves, and by imposing Gandy's "no dangling question" condition (no computation may end before all its sub-computations ended).

In all cases quoted above strategies produce either finite plays, or non-terminated plays of length ω. This is not necessary, at least in the case of strategies depending on views (called "innocent strategies" in [10]), since a generalization of dialog games to plays of transfinite length has been achieved in [5]. As we pointed out in the abstract, in this way all total set-theoretical functionals become naturally definable via strategies in which all branches end (maybe after infinitely many steps) in a last move, winning for the strategy.

We do not loose concreteness of the game interpretation: transfinite plays may arise only as the effect of the application to discontinuous arguments. Yet, transfinite branches are necessary even to represent some computable functionals.

To substantiate this claim, we provide two type 3 examples of functionals, taken from Kreisel Realization model of the Analysis. They require strategies with transfinite branches; but, if their arguments are hereditarily continuous functionals, the resulting play is always finite, and it is recursive if the arguments are.

The plan of the paper is as follows. In section 2 we introduce the basic definitions of transfinite dialog games. Then, in section 3, we specialize games to functional games. In section 4 we characterize total functionals, as promised. In the same section, we characterize total functionals definable via well-founded strategies as the Tait-continuous functionals. Finally, in section 5, we prove that this class does not contain even all "computable" total functionals: in particular certain type 3 realizers for Classical Second Order Arithmetic cannot be described via well-founded strategies.

Because of lack of space, almost all proofs have been omitted.

2 Games with transfinite plays

In this section we introduce Coquand's notion of game, as generalized in [5].

We want games able to model computation consisting of questions/answers (or dialogues) between two process. The first question is the input value, its answer is the output value, and it ends the dialogue. During the dialogue, processes alternate: each process answers to some previous question of the other process. The answer may be another question (concerning the value of a subcomputation); or it may be the final value of a (sub)computation.

We fix a trivial example we will use through the paper. Let $F : (N \to N) \to N$, and $f : N \to N$. Assume $f(0) = a, f(1) = b, f(2) = c$. We will describe the computation of $F(f) = f(0) + f(1) + f(2)$ as a dialogue between a process F and a process f. First, f asks "$F(f) =?$" (asks F for the value of $F(f)$). F answers by asking "$f(x) =?$" (by asking f for the value of $f(x)$ in $x = 0$; f, in turn, answers "$x =?$" (asks F for the input value x). F answers by "$x = 0!$" (by sending an input value 0 to f); now f answers F's original question, by "$f(x) = a!$" (by returning the output value a of $f(x)$ in $x = 0$).

The same questions and answers are used to compute $f(1)$ and $f(2)$. Eventually, F may answers f's first question: "$F(f) =?$", by returning $(a + b + c)$. This ends the dialogue.

We will model processes by players, whose goal is always to provide an answer to other player's questions. The first player unable to answer looses. Game rules fix a possible set of answers to each question. Computations are represented by plays which follow the rules of the game. A winning strategy will model a total functional, while a strategy which may loose will model a partial functional. We will define strategies at the end of this next section. Before we will formally define Coquand's games and plays.

Definition 1. A *game* is a 5-ple $G = \langle A, B, M, R, m_0 \rangle$ such that:

1. A, B are the names of the first and the second player;
2. M is a set, whose elements are the moves of G;
3. $R \subseteq M \times M$ is the set of rules of G: $\langle m, m' \rangle \in R$, also written mRm', reads as "m' is a legal reply to move m";
4. $m_0 \in M$ is the starting move.

We assume the relation R having finite depth: there exists $k < \omega$ such that, if $m_0 R m_1 \cdots m_{n-1} R m_n$, then $n \leq k$.

In our example, A and B are the processes f and F. M is the set of possible questions and answers between any two $F : (N \to N) \to N$, and $f : N \to N$, that is: $F(f) =?, F(f) = i!, f(x) =?, f(x) = j!, f(x) = k!$. We list now a coding for the elements of M.

1. $m_0 = ?\varepsilon$ is $F(f) =?$, the first question of the game, of f about the value of $F(f)$. *2.* The possible answers of F to $?\varepsilon$ are: the answer $!i$, or $F(f) = i!$" consisting of the output value $i \in N$ for F, and another question, $?1$, or $f(x) =?$, of F to f, about the value of $f(x)$. *3.* The possible answers of f to $?1$ are: the answer $!1.j$, or $f(x) = j!$, consisting of the output value $j \in N$ of $f(x)$, and the question $?1.1$, or $x =?$, of f to F, about the value of its input x. *4.* The only possible answers of F to $?1.1$ are $?1.1.k$, or $x = k!$, consisting of a value $k \in N$ for x. (In the next section, we will describe more in general a coding for the elements of M).

The relation $R(m, m')$ on M, or "game rule", describes the set of all m' which are a correct answer to m: in our case, according to what said, we have $R(?\varepsilon, !i), R(?\varepsilon, ?1), R(?1, !1.j), R(?1, ?1.1), R(?1.1, !1.1.k)$. The height of R is finite (equal to 3).

The next step will be to introduce first "generic" plays, and then specialize them to the particular notion of play we will use: "Novikoff plays".

Definition 2. A *generic play* of the game U above is a triple $p = \langle I, r, m_{(.)} \rangle$ such that:

1. I, called the carrier set, is a non-empty well-order (total and well-founded), with minimum 0_I. Its elements are the indexes of the moves of the play p.
2. $r : I - \{0_I\} \to I$ is a map, such that $r(i) < i$ for all $i \in I$. r is called the *replay map*; $r(i)$ denotes (the index of) the move to which the move with index i answers to. Thus, $r(0)$ is undefined.
3. $m_{(.)} : I \to M$ is a map, associating to each index $i \in I$ a move $m_i \in M$ of the play, having such index. We ask moreover that $R(m_i, m_{r(i)})$ (that whenever a move answers to another one, then it is a correct answer to it)

In our example, the whole play has 14 moves, and index set $I = \{0 \ldots 13\}$. The moves are: $m_0 = ?\varepsilon$ (or $F(f) = ?$), $m_1 = ?1$ (or $f(x) = ?$), $m_2 = ?1.1$ (or $x = ?$), $m_3 = !1.1.0$ (or $x = 0!$), $m_4 = !1.a$ (or $f(x) = a!$), $\ldots$ The last move is $m_13 = !(a + b + c)$, or $F(f) = (a + b + c)!$. The reply map r keeps track to which move answers each move: we may check that its values are: $r(1) = 0, r(2) = 1, r(3) = 2, r(4) = 1, \ldots$ (the move 4 provides the value of $f(x)$ in $x = 0$, hence it answers to the move 1). Remark that $r(13) = 0$ (the last move provides the value of the whole computation, hence it answers to the move 0).

We will now define a map $\mathsf{turn} : I \to \{A, B\}$, telling which player is on turn at a given step. Since $r(i) < i$ for $i > 0$, we have $r^n(i) = 0$ for a unique $n \in N$. The player on turn on 0 is A by the rules of the game, and the player on turn on $r(i)$ is the opponent of the player on turn on i. Thus, we may define turn as follows: $\mathsf{turn}(i) = A$ if the first n such that $r^n(i) = 0$ is even, and $\mathsf{turn}(i) = B$ if such an n is odd.

The last step is to restrict the set of plays we allow by introducing the notion of visibility. Visibility models the *memory* of the computation (which past moves may be used by a player to decide the next move, or which moves may be answered). We follows Novikoff and Coquand, and we decide to assume that each move between a question in $j = r(i)$ and its answer in i are invisible for the player who got the answer. The reason is that we think of the moves in $]j, i[$ as a subcomputation, with input the question in j, and output the answer in i. And we want to model any computation by a "black box", with only visible points the input and the output, as real computations are. Thus, the player who sent the input in j and received the output in i should see nothing else in between.

Let $U = \mathsf{turn}(k)$. We may express Novikoff-Coquand by requiring: *1.* each segment $[0, i[$ of the play is split into a partition made of segments $[r(k), k]$ ($r(k)$ = question of U, k = answer of his opponent); *2.* the only visible moves, by U from i, are the endpoints $\{r(k), k\}$ of such segments; *3.* $r(i) = k$ for the last point k of one of such segments. This latter requirement means that U, in i, replies to some visible answer of his opponent. We will now formalize the idea above into definition of Novikoff play.

Definition 3. – We associate to any $i \in I$ a segment by $S(i) = [0, 0]$ if $i = 0$,

$S(i) = [r(i), i]$ if $i > 0$. We call $S(i)$ an R-segment: it is the segment of moves between the move i answers to (if any), and i itself.

– We say that $\{S(k)|k \in V\}$ is a "black box structure" over I if it is a *partition* of I. We call the set V above, consisting of the last points of the segments $S(k)$, a *visibility set* over I.
– We say that $p = \langle I, r, m_{(.)}\rangle$ is a *Novikoff play* if there is a map $V(.) : I \to \wp(I)$ such that, for all $i \in I$, $V(i)$ is a visibility set over $[0, i[$ and $r(i) \in V(i)$.

Starting from the sets $V(i)$, we may formalize the visibility predicate $\mathsf{Vis}(U, \xi, \zeta)$ (to be read "ζ is visible by player U at ξ"), by $\mathsf{Vis}(\mathsf{turn}(\xi), \xi, \zeta) \Leftrightarrow \zeta \in V(\xi) \vee \zeta r(V(\xi) - \{0\})$ and, if $U = \mathsf{turn}(r(\xi)) \neq \mathsf{turn}(\xi)$, $\mathsf{Vis}(\mathsf{turn}(\xi), \xi, \zeta) \Leftrightarrow \zeta = r(\xi) \vee \mathsf{Vis}(U, r(\xi), \zeta)$. The first definition expresses that $V(\xi) \cup r(V(\xi) - \{0\})$ is the set of endpoints of the "black box structure" associated to ξ and to the player on turn on ξ. The second definition expresses the fact that no move in $]r(\xi), \xi[$ is visible by the the player U on turn on $r(\xi)$. This is because the segment $[r(\xi), \xi]$ starts by a question by U, and ends by the answer of the other player. Thus, according to our assumptions, its interior is invisible by U.

The *view* of U on p at ξ is the set

$$\mathsf{view}(U, p, \xi) = \{\zeta \mid \mathsf{Vis}(U, \xi, \zeta)\}.$$

The main result about Novikoff plays is the following (proved in [5]):

Theorem 4.
Let p be any Novikoff play. Then all one-step extensions of p have, in their last move, the same visibility set and the same player on turn.

Because of 4, if a play p of length α can be extended, it makes sense to speak of the player on turn at α-th step: abusing notation we simply write $\mathsf{turn}_p(\alpha)$.

The theorem 4 is easy to prove when I has a successor length, but difficult when I has a limit length. Herbelin [8] remarked that the case length$(I) = \omega$ is elementary equivalent to Tait's normalization result for ω-logic. As an easy corollary, the visibility assignment $V(.) : I \to \wp(I)$ such that $r(i) \in V(i)$ for all $i > 0$, if it exists, it is unique; and $V(i), \mathsf{turn}(i)$ are uniquely determined by r restricted to $[0, i[$. Thus, in principle, we could just say that a play is Novikoff, without quoting the map $V(.) : I \to \wp(I)$, since this map is unique.

Our example of play is a Novikoff play. We will now write down, for each move, a row with all visibility informations for the player on turn. Moves visible by the player on turn will be marked "v", or "**v** " for the moves of his opponent, forming the visibility set. Invisible moves will be marked "*i*". We call the process F "P" (for "Player"), and process f "O" (for "Opponent").

	turn	Move	Coding of the move	r	0	1	2	3	4	...	...	13
0	O	$F(f) =?$	$?\varepsilon$	-	i	i	i	i	i	..	..	i
1	P	$f(x) =?$	$?1$	0	$\mathbf{v}$	i	i	i	i	..	..	i
2	O	$x =?$	$?1.1$	1	v	$\mathbf{v}$	i	i	i	..	..	i
3	P	$x = 0!$	$!1.1.0$	2	$\mathbf{v}$	v	$\mathbf{v}$	i	i	..	..	i
4	O	$f(x) = a!$	$!1.a$	1	v	$\mathbf{v}$	v	$\mathbf{v}$	i	..	..	i
...	...	...	...	..	..	..	..	..	..	..	..	i
...	...	...	...	..	..	..	..	..	..	..	..	i
13	P	$F(f) = a + b + c!$	$!(a + b + c)$	0	$\mathbf{v}$	v	i	i	$\mathbf{v}$	..	..	i

Remark that move 13 cannot see, for instance, the moves $2, 3$. The reason is that such moves are in the interior of the R-segment $[1, 4]$, that is, of the subcomputation with question $f(x) =?$ and answer $f(x) = a!$. Thus, moves $2, 3$ are, for the player on turn on move 13, inside a "black box", hence invisible.

In the case of finite pre-plays, we may prove that the set view is the visibility set of view-strategies (called "innocent" in [10]), having a simple inductive definition:

$$\mathsf{view}(U, p, i) = \begin{cases} \{i - 1\} \cup \mathsf{view}(U, p, i - 1) & \text{if } \mathsf{turn}(i) = U \\ \{r(i)\} \cup \mathsf{view}(U, p, r(i)) & \text{if } \mathsf{turn}(i) \neq U. \end{cases}$$

This is the standard notion of visibility in dialog games: it is defined in this way both in [10, 11] and in [7]. the case of plays of possibly transfinite length has been considered for the first time in [5], from which we borrow the axiomatic definition of Vis. Definition above does not tell, explicitly, who is the player on turn at a limit point $\lambda \in I$, nor his views. The main theorem 4, however, states that r restricted to $[0, \lambda[$ uniquely determine the turn and the view at point λ.

This ends the introduction of Novikoff plays. In the remaining of this section, we will introduce strategies. In the next section, we will use them to model functionals.

To define strategies, concepts and terminology about certain parts of plays are in order. First, if $\xi \in I$ then $p\lceil\xi$ (a prefix play of p) is the (pre) play whose carrier set is $[0, \xi[$, whose $r, m_{(.)}$ are the restrictions to $[0, \xi[$ of those of p. More in general if $J \subseteq I$ then $p\lceil J$ is the structure $\langle J, r', m'_{(.)}\rangle$ where $r', m'_{(.)}$ come from $r, m_{(.)}$, by restricting them to J.

Given a play p we can choose J such that $p\lceil J$ is closed under the reply function and has the structure of a play, but it is not such for trivial reasons: e.g. because its first move is not m_0, or it is played by P. To define the notion of subplay without being too restrictive we introduce the notion of play morphism (see also [10]).

Definition 5. If p and q are (pre) plays, with carrier sets I, J, then $\varphi : p \to q$ is a *play morphism* if it consists of a pair of maps $\langle\varphi_0, \varphi_1\rangle$ such that $\varphi_0 : I \to J$ is strictly increasing and $\varphi_1 : \{O, P\} \to \{O, P\}$ is identity or exchange, and for all $\xi < \alpha$:

$$\mathsf{turn}_q(\varphi_0(\xi)) = \varphi_1(\mathsf{turn}_p(\xi)), \qquad r_q(\varphi_0(\xi)) = \varphi_0(r_p(\xi)).$$

The image $\varphi[p]$ in q is a *subplay* of q.

The subplay $\varphi[p]$ of q has the same structure of p, and its reply and turn functions are $r_q\lceil\varphi_0[\alpha]$ and $\mathsf{turn}_q\lceil\varphi_0[I]$ (where $\varphi_0[I]$ is the image of I in J via φ_0).

Proposition 6. *If $\varphi[p]$ is a subplay of q, then $I = \varphi_0[\mathrm{length}(p)]$ is such that:*

1. *if $\xi, \zeta \in I$ are such that $\xi < \zeta$ and there exists no $\eta \in I$ such that $\xi < \eta < \zeta$, then $\mathsf{turn}(\xi) \neq \mathsf{turn}(\zeta)$;*
2. *$I \neq \emptyset$ and $r[I \setminus \{\min(I)\}] \subseteq I$;*
3. *for any $\xi \in I$, if $I' = \{\zeta \in I \mid \zeta < \xi\}$ then $I' \cap \mathsf{view}(\mathsf{turn}(\xi), q, \xi)$ is cofinal in I'.*

Vice versa, if $I \subseteq \mathrm{length}(q)$ satisfies the above conditions, then $q\lceil I$ is a subplay of q.

A pre-play is *U-cut free*, for $U \in \{O, P\}$ if

$$\xi > 0 \wedge \mathsf{turn}(\xi) \neq U \Rightarrow \xi = r(\xi) + 1,$$

namely if the opponent of U is forced to reply to the last move of U.

U-cut free (pre) plays is the terminology of [7]. If a pre-play has finite length then the previous definition is a generalization of [11], definition 3.1.3. Observe that in a U-cut free pre-play, U is the unique player allowed to play at limit points.

Any view determines a subplay (but not vice versa), i.e. any non empty $I = \mathsf{view}(U, q, \xi)$ satisfies the conditions of 6. Such a $q\lceil I$ is a U-cut free play which, with overloaded terminology, we call the *U-view of q at ξ*. Also $I \cup \{\xi\}$ determines a subplay $q\lceil(I \cup \{\xi\})$, which we call "large U-view".

We say that player U is *deterministic* on a play p if for all $\xi, \zeta < \mathrm{length}(p)$, if $\mathsf{turn}(\xi) = \mathsf{turn}(\zeta) = U$ and $p\lceil\mathsf{view}(U, p, \xi)$ isomorphic to $p\lceil\mathsf{view}(U, p, \zeta)$ (i.e., that they are the same up to renaming of the elements of the carrier sets) then the lare U-views of ξ, ζ are isomorphic, too. A play p is a *deterministic play* if both players are deterministic on p.

Definition 7. A *strategy* s for player U over a game U (shortly an U-strategy) is a tree (i.e. a prefix closed set) of U-cut free plays of U such that, for all $p \in s$ with $\alpha = \mathrm{length}(p)$:

1. if $\mathsf{turn}(\alpha) = U$ then there is at most one $q \in s$ of length $\alpha + 1$ such that p is a prefix of q;
2. if $\mathsf{turn}(\alpha) \neq U$ (hence α is a successor) then for any $m \in M$ which is a legal reply to $p_{\alpha-1}$, i.e. such that $p_{\alpha-1} R m$, there exists $q \in s$ of length $\alpha + 1$ such that p is a prefix of q, $q_\alpha = m$ and $r_q(\alpha) = \alpha - 1$.

Player U follows the strategy s in the play q if for all $\xi < \mathrm{length}(q)$ the large U-view p of q at ξ belongs to s, up to renaming of the carrier set. Clearly U follows some strategy in q if and only if U is deterministic on q.

The main consequence of Theorem 4 w.r.t. strategies is the cut-elimination theorem:

Theorem 8 Cut-elimination [5]. *Let s be a P-strategy and t an O-strategy such that the heights of s and t are bounded above by some infinite regular ordinal κ. Then there exists a unique play p of maximal length such that P and O follow the strategies s and s' respectively, and* $\mathrm{length}(p) = \alpha + 1 < \kappa$.

This play has successor length, hence it has a last move; the player who did the last move won. Therefore any two strategies s and t, for Player and Opponent respectively, determine a winning player.

3 Sequential functionals of finite type

The present section specializes dialog games to games and strategies representing functionals. In this case the role of Player is to show that a functional F_s, associated to the strategy s, is defined against the arguments $F_{t_1}, \ldots, F_{t_k}$: if s wins against $t_1, \ldots, t_k$ then either some t_i misses a move or the resulting play has a last move $!v$ such that $F_s(F_{t_1}, \ldots, F_{t_k}) = v$. Therefore winning strategies (i.e. strategies such that the player who follows them is always able to play a move, when on turn) naturally induce total functionals.

We base our treatment on [11]. Admittedly formalizations based on the categorical semantics of linear logic, as it is the case of [6, 2, 3, 1, 9], have the advantage of being compositional with respect to the type structure, which is not the case of the present one. However the actual description of strategies seems more direct in a formulation which does not make use of the decomposition of the function space bifunctor into linear implication and the comonad "!". Perhaps the best thing would be a compromise between the two, which is still on demand.

Let $\Gamma = \{\gamma_0, \gamma_1, \ldots\}$ be a set of ground types, and $\mathsf{T}(\Gamma)$ be the set of simple types over Γ. We fix an interpretation of types in Γ as a set of values $V = \bigcup\{V_\gamma \mid \gamma \in \Gamma\}$.

Any type has the form $\tau = \tau_1 \to (\cdots \to (\tau_k \to \gamma)) \in \mathsf{T}(\Gamma)$, and is abbreviated by $(\tau_1, \ldots, \tau_k \to \gamma)$. The set of *occurrences* of τ, $Occ(\tau)$ is defined inductively: $\varepsilon \in Occ(\tau)$ and $\tau_\varepsilon = \gamma$; if $1 \le i \le k$ and $a \in Occ(\tau_i)$ then $i.a \in Occ(\tau)$ and $\tau_{i.a} = (\tau_i)_a$.

To each type τ it is associated a game G_τ as follows.

Definition 9. For $\tau \in \mathsf{T}(\Gamma)$, G_τ is the game $\langle M_\tau, R_\tau, ?\varepsilon\rangle$ where:

1. $M_\tau = \{?a, !a.v \mid a \in Occ(\tau, v \in V_\gamma,$ for γ last atom in $\tau_a\}$;
2. R_τ is the least binary relation over M_τ such that:
   $(R1)$ $a.i \in Occ(\tau) \Rightarrow ?aR_\tau?a.i,$
   $(R2)$ $a \in Occ(\tau) \wedge \tau_a = \gamma \wedge v \in V_\gamma \Rightarrow ?aR_\tau!a.v.$

In M_τ moves of the form $?a$ are queries for the output value of a functional of type τ_a, applied to all its arguments; moves of the form $!a.v$ are the corresponding answers.

Definition 10. A *functional play* (henceforth simply a play) over the game G_τ is a deterministic play p over it such that

$$(F)\ \ p_\xi = !v \wedge r(\xi) < \zeta < \xi \wedge p_\zeta = ?a \Rightarrow \exists \zeta', v'.\ \zeta' < \xi \wedge p_{\zeta'} = !v' \wedge r(\zeta') = \zeta.$$

(F) imposes that an answer replies to the last unanswered question (the "no dangling condition" of [10]). By $(R1)$-$(R2)$ only queries can be replied to.

Let $\langle p, r_p \rangle$ be a play and $\langle p', r_{p'} \rangle$ a subplay (of any other play). By $p * p'$ we indicate the partially defined operation of concatenating p with p': $p * p'$ is defined and equal to $\langle q, r_q \rangle$ if $\mathrm{length}(q) = \mathrm{length}(p) + \mathrm{length}(p')$, $q_\xi = p_\xi$ if $\xi < \mathrm{length}(p)$, p'_ζ if $\xi = \mathrm{length}(p) + \zeta$, and finally

$$r_q(\xi) = \begin{cases} r_p(\xi) & \text{if } \xi < \mathrm{length}(p) \\ \mathrm{length}(p) + r_{p'}(\zeta) & \text{if } \xi = \mathrm{length}(p) + \zeta > \mathrm{length}(p) \\ \mathrm{length}(p) - 1 & \text{if } \xi = \mathrm{length}(p) \text{ is a successor, and} \\ & p_{\mathrm{length}(p)-1} R q_\xi \end{cases}$$

If some of the above conditions cannot be satisfied, $p * p'$ is undefined. If it is defined we set turn_q as the function determined by r_q.

Let p be a play of type τ_i, for $1 \le i \le k$, and $\tau = (\tau_1, \ldots, \tau_k \to \gamma)$. Then we may construct a play $p^{(i)} = \langle ?\varepsilon \rangle * p'$ of type τ by adding a first move $?\varepsilon$ and by transforming each move over τ_i into the corresponding move over τ: so p' is the (sub) play obtained from p by changing any question of the form $?a$ into a question of the form $?i.a$. Because of the definition of concatenation, the first move of p' replies to $?\varepsilon$, which implies that players on p' are interchanged with respect to p (indeed, for all $\xi < \mathrm{length}(p)$, p_ξ corresponds to $p^{(i)}_{1+\xi}$, so that $r^n_p(\xi) = 0$ if and only if $r^{n+1}_{p^{(i)}}(1+\xi) = 0$: in particular, if ξ is limit, then $1+\xi = \xi$ so that players are exchanged also at limit points); therefore, if p is a P-view of a play of type τ_i, then $p^{(i)}$ is an O-view of a play of type τ. Finally, if s is a strategy of type τ_i then we set $s^{(i)} = \{p^{(i)} \mid p \in s\}$.

Proposition 11. *Let* $\tau = (\tau_1, \ldots, \tau_k \to \gamma)$ *and* $s_1, \ldots, s_k$ *be* P*-strategies of type* $\tau_1, \ldots, \tau_k$*. Then*

$$(s_1, \ldots, s_k)^O = \bigcup_{i=1}^{k} s_i^{(i)}$$

is an O*-strategy of type* τ*, and any such a strategy arises in this way.*

Because of this proposition there is no theoretical loss in concentrating on P-strategies, henceforth called simply strategies. An immediate consequence of this and of 8 is that given some P-strategy s of type $(\tau_1, \ldots, \tau_k \to \gamma)$ and the P-strategies $s_1, \ldots, s_k$ of type $\tau_1, \ldots, \tau_k$ it is uniquely determined the play $p = s \bullet (s_1, \ldots, s_k)^O$ of maximal length in which P and O follow s and $(s_1, \ldots, s_k)^O$ respectively.

A functional play is *terminated* if it has a move answering to the first move $?\varepsilon$. This move is necessarily the last one, by $(F1)$. If such $s \bullet (s_1, \ldots, s_k)^O$ is terminated by the move $!v$ then write

$$s[s_1, \ldots, s_k] = v.$$

$s[s_1, \ldots, s_k]$ is undefined otherwise. By $s[s_1, \ldots, s_k] \simeq t[t_1, \ldots, t_h]$ we mean they are either both defined and equal, or both undefined.

The functional interpretation of strategies depends on the following fact. For each type τ define the binary relation $\sim_\tau$ among strategies of type τ inductively as follows:

- $s \sim_\gamma s' \Leftrightarrow s = s'$;
- $s \sim_{(\tau_1,\ldots,\tau_k \to \gamma)} s' \Leftrightarrow$

$$\forall s_1, s'_1, \ldots, s_k, s'_k. \bigwedge_{i=1}^{k} s_i \sim_{\tau_i} s'_i \Rightarrow s[s_1, \ldots, s_k] \simeq s'[s'_1, \ldots, s'_k].$$

Then, if $s_i \sim_{\tau_i} s'_i$ for $1 \leq i \leq k$ and s is a strategy of type $(\tau_1, \ldots, \tau_k \to \gamma)$, $s[s_1, \ldots, s_k] \simeq s[s'_1, \ldots, s'_k]$.

The type structure of the *Hereditarily Sequential Functionals*[1], HSF, is defined as follows. To each type τ it is associated a set HSF^τ of functionals, and to each strategy s of type τ a functional $F_s \in \mathsf{HSF}^\tau$. Set $F_{\langle ?\varepsilon \rangle} = \bot$ and $F_{\tilde{v}} = v$, where $\tilde{v} = \langle ?\varepsilon, !v \rangle$. If s is a strategy of type $\tau = (\tau_1, \ldots, \tau_k \to \gamma)$ then $F_s : \mathsf{HSF}^{\tau_1} v \cdots \mathsf{HSF}^{\tau_k} \to \mathsf{HSF}^\gamma$ is the functional

$$F_s(F_{s_1}, \ldots, F_{s_k}) = s[s_1, \ldots, s_k] \text{ if defined}, Finally$$

$\mathsf{HSF}^\tau = \{F_s \mid s \text{ is a strategy of type } \tau\}$, in particular $\mathsf{HSF}^\gamma = (V_\gamma)_\bot$.

The structure HSF is a type frame. To see this we need a definition of application between strategies of higher type, namely an operation $App(s,t) = s[t]$ where, if s is some strategy of type $\sigma \to \tau$ and t of type σ, $s[t]$ is a strategy of type τ.

Let p be a play of type $(\tau_1, \ldots, \tau_k \to \gamma)$: q is the subplay of p on the *i-th component* if it is the maximal subplay of p such that any question of q but the first one has the shape $?i.a$.

If p is a play of type $\tau = (\tau_1, \ldots, \tau_k \to \gamma)$, then we may construct a play $p|\sigma$ of type $\sigma = (\tau_2, \ldots, \tau_k \to \gamma)$, by restricting p to the moves not in the first component. Take $I = \{\zeta < \text{length}(p) \mid \forall a, b.\ p_\zeta \neq ?1.a \wedge p_{r(\zeta)} \neq ?1.b\}$: then $p\lceil I$ is a subplay of p and there exists a play q of type σ and a play morphism φ such that $\varphi[q] = p\lceil I$, $q_\zeta = ?(j-1).a$ whenever $p_{\varphi_0(\zeta)} = ?j.a$, $q_\zeta = p_{\varphi_0(\zeta)}$ else, and φ_1 is the identity. r_q is fully determined by φ and r_p.

Proposition 12. *Let s be a strategy of type $\tau = (\tau_1, \ldots, \tau_k \to \gamma)$. Consider $\sigma = (\tau_2, \ldots, \tau_k \to \gamma)$ and some strategy t of type τ_1. Define $s[t]$ as the set of all P-cut free plays p' such that for some play p of type τ:*

1. *p' is a P-view of $p|\sigma$;*
2. *P follows s on p;*
3. *if q is the subplay of p on the first component then O follows $t^{(1)}$ on q.*

[1] We give to this structure the same name as in [11], but they are different since our HSF properly includes the structure considered by Nickau.

Then $s[t]$ *is a strategy of type* $\sigma = (\tau_2, \ldots, \tau_k \to \gamma)$, *such that, for all strategies* $t_2, \ldots, t_k$ *of type* $\tau_2, \ldots, \tau_k$

$$s[t][t_2, \ldots, t_k] \simeq s[t, t_2, \ldots, t_k].$$

By this the functional application is simply defined by: $F_s(F_t) = F_{s[t]}$.

4 Well-founded total functionals

In this section and in the next one we restrict our attention to type structures over $\mathsf{T}(N) = \mathsf{T}(\{N\})$, namely to simple types with ground type N. We also fix $V_N = \omega$.

A P-strategy s is *winning* if P always wins against any O-strategy, by following s. It is *strongly winning* if any $p \in s$ has some extension $q \in s$ won by P. A strongly winning strategy is winning, but not vice versa: indeed a winning strategy may include plays lost by P which simply cannot be a P- view of any play against some O-strategy. Strongly winning strategies are *complete*: by Theorem 4 any play of limit length can be extended; on the other hand in a P-cut free play just P may play at limit points; therefore if s is a winning strategy and p is a P-cut free play of limit length λ, then $p \in s$ if and only if $p\lceil\xi \in s$ for all $\xi < \lambda$.

Winning strategies are related to total functionals: $F_s \in \mathsf{HSF}^{(\tau_1,\ldots,\tau_k\to\gamma)}$ is *total* if for all total $F_{s_1}, \ldots, F_{s_k}$ there exists $n \in V_N$ such that

$$F_s(F_{s_1}, \ldots, F_{s_k}) \simeq n.$$

Theorem 13. *F_s is total if and only if s is strongly winning.*

The proof of the last theorem depends on the fact that any strategy s is included in some strongly winning strategy (possibly of transfinite height). This implies that any partial object in HSF has a total extension within HSF: this should be contrasted with the Scott continuous functionals, where e.g. Plotkin continuous existential quantifier is maximal (w.r.t. the pointwise ordering) but not total (see [12]). The same remark applies to the PCF definable functionals: indeed (our) HSF is a larger model than the extensional collapse of innocent strategies.

Because of the existence of transfinite plays and of strategies of transfinite height, any functional in the type frame HTF of the *Hereditarily Total Functionals* (the full type hierarchy over $V_N = \omega$) is an object of HSF[2].

Theorem 14. *For all type τ and $F \in$ HTF there exists a winning strategy s of the same type such that $F = F_s$.*

[2] Strictly speaking any object of HTF turns out to be the restriction to total functionals of some object of HSF, as the latter may have partial functionals in its domain. In the sequel we shall not enter into such details, and we will consider HTF as a subframe of HSF

If κ is an infinite regular ordinal and s is a strategy of height $\leq \kappa$ (recall that the height of a tree T is the first ordinal α such that for all sequence $x \in T$, $\text{length}(x) < \alpha$), we say that it is a *$\kappa$-strategy*: an ω-strategy is then a well-founded tree. A functional $F_s \in \mathsf{HSF}^\tau$ is *well-founded* if there exists an ω-strategy s such that $F = F_s$. The following Corollary is an immediate consequence of the Cut-Elimination Theorem 8 and of the definition of totality.

Corollary 15. *Total well-founded functionals from* HSF *are closed under application.*

Let TWF be the type frame of *Total Well-founded Functionals.*

Theorem 16. TWF *is a model of simply typed λ-calculus.*

Well-founded functionals embody the idea of functionals determined by finite amounts of information about their arguments: the same idea at the basis of Kleene-Kreisel countable functionals and of Scott continuous functionals. In the final part of this section we characterize the well-founded total functionals using a generalization to all types, due to Tait, of Brouwer's notion of continuity for type 2 functionals.

Definition 17. The *Tait Continuous Functionals*, TCF, is the least type frame over $\mathsf{T}(N)$ such that:

1. TCF^N is the set of natural numbers;
2. TCF contains the combinators $\mathbf{S}, \mathbf{K}, \mathbf{I}$ at all (suitable) types;
3. if $\{F_n \mid n \in \omega\} \subseteq \mathsf{TCF}^\tau$ then the functional $F(n) = F_n$ (also denoted by $\lambda n.\, F_n$) is in $\mathsf{TCF}^{(N \to \tau)}$ (the ω-rule).

Recursive Tait-continuous functionals, which are obtained from Definition 17 by asking in the third clause that the set $\{F_n \mid n \in \omega\}$ is recursive, are total functionals (this is a consequence of Tait cut-elimination theorem for the ω-logic). That TCF is a subframe of HTF will be a consequence of the proof that TCF and TWF actually coincide.

It is not difficult to show that $\mathsf{TCF} \subseteq \mathsf{TWF}$, since by Theorem 16 it suffices to prove the closure of TWF under the ω-rule. Suppose that $F_n = F_{s_n}$ for all n and take s as the prefix closure of the set of all P-cut free plays p of type $(N \to \tau)$ such that $p = \langle ?\varepsilon, ?1, !n \rangle * q$, and q is obtained from some $q' \in s_n$ by substituting each move of the form $?i.a$ by $?(i+1).a$. Then s is a strategy of type $(N \to \tau)$, and $F_s = \lambda n.\, F_n$.

To prove that $\mathsf{TCF} \supseteq \mathsf{TWF}$ the following lemma is needed (compare with [11] Theorem 3.3.6). If T is a tree then $T_{\langle x \rangle} = \{y \mid \langle x \rangle * y \in T\}$ is an immediate subtree of T; a proper subtree of T is either an immediate subtree or a proper subtree of some immediate subtree of T. Recall that well-founded trees admit an inductive definition: T is well-founded if all immediate subtrees of T are such.

Lemma 18. *Let s be an ω-strategy of type $(\tau_1,\ldots,\tau_k \to N)$ such that $s \neq \tilde{n}$ for any n. Then there exist $1 \leq i \leq k$ and the ω-strategies $s_1,\ldots,s_{n_i}$ (where $\tau_i = (\sigma_1,\ldots,\sigma_{n_i} \to N)$) and a family of ω-strategies $\{s'_m\}_{m\in\omega}$ such that, for all strategies $t_1,\ldots,t_k$ of type $\tau_1,\ldots,\tau_k$, if $t_i[s_1[t_1,\ldots,t_k],\ldots,s_{n_i}[t_1,\ldots,t_k]] \simeq m$ then $s[t_1,\ldots,t_k] \simeq s'_m[t_1,\ldots,t_k]$. Moreover $s_1,\ldots,s_{n_i}$ and each s'_m are isomorphic to proper subtrees of s.*

Theorem 19. *The well-founded functionals are exactly the Tait-continuous functionals, namely* $\mathsf{TWF} = \mathsf{TCF}$.

Proof. Let $F = F_s$ be a well-founded functional of type $(\tau_1,\ldots,\tau_k \to N)$. If $s = \tilde{n}$ then $F_s = \lambda x_1 \cdots x_k.n$ and it is trivially Tait-continuous. Otherwise, by induction over the well founded tree s and by Lemma 18, there exist $G_1 = F_{s_1},\ldots,G_{n_i} = F_{s_{n_i}}$ and $G'_m = F_{s'_m}$ for each $m \in \omega$ which are Tait-continuous and such that, if $F_i(G_1(F_1,\ldots,F_k),\ldots,G_{n_i}(F_1,\ldots,F_k)) = m$ then $F(F_1,\ldots,F_k) = G'_m(F_1,\ldots,F_k)$. Therefore

$$F(F_1,\ldots,F_k) = \\ (\lambda m.G'_m(F_1,\ldots,F_k))(F_i(G_1(F_1,\ldots,F_k),\ldots,G_{n_i}(F_1,\ldots,F_k)))$$

is Tait-continuous as it is obtained applying the ω-rule to a combination of $F_1,\ldots,F_k$ and of constants for Tait-continuous functionals.

□

5 Computable non well-founded functionals

Given any $F \in \mathsf{HTF}^{((N\to N)\to N)}$, there exists $f,g \in \mathsf{HTF}^{(N\to N)}$ such that

$$f(F(f)) \neq g(F(g)) \quad (1)$$

$$F(f) = F(g) \quad (2)$$

Indeed for any ordinal ξ let h_ξ be the characteristic function of $X_\xi = \{F(h_\zeta) \mid \zeta < \xi\}$. By a cardinality reasoning there exists a minimal $\alpha < \omega_1$ such that $X_{\alpha+1} = X_\alpha$; therefore $h_\alpha(F(h_\alpha)) = h_{\alpha+1}(F(h_\alpha)) = 1$. Since $F(h_\alpha) \in X_{\alpha+1} = X_\alpha$ there exists a (unique) $\beta < \alpha$ such that $F(h_\alpha) = F(h_\beta)$. If $h_\beta(F(h_\beta)) = 1$ then $X_\beta = X_{\beta+1} = X_\alpha$ contradicting the minimality of α, so that $h_\beta(F(h_\beta)) \neq h_\alpha(F(h_\alpha))$: now set $f = h_\alpha$ and $g = h_\beta$.

The construction of f,g is uniform in F, so that there exist two total functionals Φ,Ψ of type $(((N \to N) \to N), N \to N)$ such that $f = \Phi(F)$ and $g = \Psi(F)$ satisfy (1), (2). If F is continuous (w.r.t. the product topology over $\mathsf{HTF}^{(N\to N)} = \omega^\omega$) then $\alpha < \omega$. In this case it is easily proved that $\Phi(F)(n) = m$ and $\Psi(F)(n) = m$ are predicates recursive in F. In this sense Φ and Ψ are "computable" type 3 functionals.

By Theorem 14 Φ,Ψ are objects of HSF. More explicitly a strategy for Φ is the least prefix closed set of P-cut free plays of type $(((N \to N) \to N), N \to N)$

including plays of the following two forms (using the symbolic notation):

$$\langle \Phi(F,x) =?, F(f) =?, F(f) = n_0, \ldots, F(f) =?, F(f) = n_\eta, \text{ (for all } \eta < \xi) \\ F(f) =?, f(y) =?, y =?, y = m, f(y) = h_\xi(m)\rangle$$

which accounts for the computation of $F(h_\xi)$, and

$$\langle \Phi(F,x) =?, F(f) =?, F(f) = n_0, \ldots, F(f) =?, F(f) = n_\alpha, \\ x =?, x = n, \Phi(F,x) = h_\alpha(n)\rangle.$$

which yields the value of $\Phi(F,x)$. In the second line, as in the informal definition of Φ, α is the minimum ordinal such that $n_\alpha = n_\beta$ for a (unique) $\beta < \alpha$. The definition of a strategy for Ψ is similar, but the last move in the second case is $\Psi(F,x) = h_\beta(n)$.

These strategies are both ω_1-strategies, where ω_1 is the first uncountable ordinal. Next we prove that Φ, Ψ have no ω-strategy.

Theorem 20. *Φ and Ψ are not well-founded functionals.*

The proof uses two Lemmas. By $F \subseteq G$ it is meant graph inclusion.

Lemma 21. *Let $F \in \mathsf{HSF}^{((N\to N)\to N)}$ be partial injective, X be the range of F, $x \notin X$ and $f \in \mathsf{HTF}^{(N\to N)} \subseteq \mathsf{HSF}^{(N\to N)}$: then there exists $G \in \mathsf{HSF}^{((N\to N)\to N)}$ partial injective such that $\mathrm{Rng}(G) \subseteq X \cup \{x\}$, $F \subseteq G$ and $f \in \mathrm{Dom}(G)$.*

Lemma 22. *Let $\{s_n \mid n \in \omega\}$ be a family of winning ω-strategies of type $(((N \to N) \to N) \to N)$, and $X \subseteq \omega$ an infinite set. Then there exists $F \in \mathsf{HSF}^{((N\to N)\to N)}$ partial injective with range X s.t. $F_{s_n}(F)$ is defined for all n.*

Proof of Theorem 20. Toward a contradiction suppose that $\Phi = F_s$ and $\Psi = F_t$, for some (winning) ω-strategies s, t. Then there exist winning ω-strategies s_n and t_m associated to $\Phi_n = \lambda F.\ \Phi(F,n)$ and $\Psi_m = \lambda F.\ \Phi(F,m)$ respectively. Let us abbreviate by $\theta_{\langle n,m\rangle}$ a strategy for the functional $\Theta_{\langle n,m\rangle} \in \mathsf{HSF}^{(((N\to N)\to N)\to N}$ such that

$$\Theta_{\langle n,m\rangle}(G) = \langle \Phi(G)(n), \Psi(G)(m)\rangle,$$

where $\langle \_, \_\rangle$ is a surjective pairing function over the natural numbers. Of course $\theta_{\langle n,m\rangle}$ can be constructed from s_n and t_m in such a way that it is an ω-strategy. Being $\Theta_{\langle n,m\rangle}$ a total functional, $\theta_{\langle n,m\rangle}$ is winning by 13.

By Lemma 22, given any infinite $X \subseteq \omega$ and $\langle i,j\rangle \notin X$ we can find $F \in \mathsf{HSF}^{((N\to N)\to N)}$ partial injective with range $\subseteq X$ such that $\Theta_{\langle n,m\rangle}(F)$ is defined for all n, m, which implies that $f = \Phi(F)$ and $g = \Psi(F)$ are total functions, since $\langle f(n), g(m)\rangle \simeq \Theta_{\langle n,m\rangle}(F)$ for all n, m.

Applying Lemma 21 twice we find U partial injective such that $F \subseteq G$, $X \cup \{i,j\}$ is the range of U and $f, g \in \mathrm{Dom}(G)$. Let H be any total extension of U: then $\Phi(F) \subseteq \Phi(G) \subseteq \Phi(H)$, and, as $f = \Phi(F)$ is total, $\Phi(H) = f$. Similarly $\Psi(H) = g$.

By the absurd hypothesis $f(H(f)) \neq g(H(g))$ and $H(f) = H(g)$. From $H(f) = G(f)$ and $H(g) = G(g)$ it follows $G(f) = G(g)$, hence $f = g$ since U is injective: a contradiction.

□

6 Concluding remarks

Although well-founded functionals are a natural structure, they do not capture the idea of (relative) computable functionals at type 3 and higher. This may be of minor interest as soon as one is concerned with λ-calculus models, but becomes relevant when dealing with the constructive analysis of classical proofs, and with program extraction. Indeed the functionals Φ, Ψ can be shown to be natural realizers of the no-counterexample of the comprehension axiom scheme for classical second order arithmetic, and have been found following methods introduced in [4].

The fact that they are not well-founded may appear not surprising as they are set theoretic functionals, defined also on discontinuous type 2 arguments (i.e. non continuous w.r.t. the product topology on type 1 objects), as it is needed if they have to build "no-counterexamples" against any possible candidate as a counterexample. However they have the robust property, as argued in the previous section, to yield finite plays on continuous (namely well-founded) arguments, which are effectively computable if the arguments are recursive. Actually Φ, Ψ are examples of a large class of functionals enjoying this property, which, we think, deserves further investigation.

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