Logical Semantics for the First Order ς-Calculus

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ICTCS, Bertinoro, Oct 13-15, 2003

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Object calculi have been investigated by syntactical means, under the assumption that their models are λ-models of some kind. We think that λ-models deserve further investigation in the special concern of object-oriented concepts and calculi, answering the following:

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- Which structures are suitable as models of object calculi?
- Is there a simple description of types and values?
- Is there a logical description of such models, allowing for an assignment system to derive properties of terms?

The untyped ς-calculus

The grammar of terms of the untyped ς-calculus is:

$$a, b \ ::= \ x \mid [\ell_i = \varsigma(x_i)b_i{}^{i \in I}] \mid a.\ell \mid a.\ell \Leftarrow \varsigma(x)b$$

where $L = \{\ell_i \mid i \in N\}$ is a denumerable set of *labels*.

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The *reduction relation* is defined by:

$$[\ell_i = \varsigma(x_i)b_i{}^{i \in I}].\ell_j \ \rightarrow \ b_j\{x_j \leftarrow [\ell_i = \varsigma(x_i)b_i{}^{i \in I}]\}$$
$$[\ell_i = \varsigma(x_i)b_i{}^{i \in I}].\ell_j \Leftarrow \varsigma(x)b \ \rightarrow \ [\ell_i = \varsigma(x_i)b_i{}^{i \in I \setminus j}, \ell_j = \varsigma(x^A)b]$$

where $a\{x \leftarrow b\}$ is the replacement of x by b in a, avoiding variable clashes.

The first order typed system OB_1

The set of *types* is defined by the following grammar:

$$A, B \ ::= \ K \mid [\ell_i : B_i \ ^{i \in I}]$$

where I is a finite set of indexes.

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Predicates (intersection types)

The set $\mathcal{L}$ of *predicates* is inductively defined by:

$$\sigma, \tau \ ::= \ \kappa \mid \omega \mid (\sigma \wedge \tau) \mid (\sigma \rightarrow \tau) \mid \langle \ell_i : \sigma_i{}^{i \in I} \rangle$$

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- $\sigma \longrightarrow \tau$ is true of functions sending elements that satisfy σ into elements that satisfy τ;
- $\langle \ell_i : \sigma_i{}^{i \in I} \rangle$ holds of records whose item at ℓ_i satisfies σ_i (for all $i \in I$).

Predicates entailment

The pre-order $\leq$ on predicates is such that:

$$\sigma \leq \tau \;\Rightarrow\; \langle \ell : \sigma \rangle \leq \langle \ell : \tau \rangle$$

$$\langle \ell_i : \sigma_i{}^{i \in I} \rangle \;\leq\; \langle \ell_j : \tau_j{}^{j \in J} \rangle, \text{ if } J \subseteq I$$

$$\langle \ell_i : \sigma_i{}^{i \in I} \rangle \wedge \langle \ell_j : \tau_j{}^{j \in J} \rangle \;\leq\; \langle \ell_k : \rho_k{}^{(k \in I \cup J)} \rangle,$$

$$\text{where } \begin{cases} \rho_k = \sigma_k \wedge \tau_k, & \text{if } k \in I \cap J, \\ \rho_k = \sigma_k, & \text{if } k \in I \setminus J, \\ \rho_k = \tau_k, & \text{if } k \in J \setminus I \end{cases}$$

plus axioms for the arrow, ω and conjunction.

$\sigma \leq \tau$ reads as "σ implies τ".

Predicate Assignment

A *basis* Γ is a finite set of statements $x^B : \sigma$ with distinct subjects. Let $A \equiv [\ell_i : B_i{}^{i \in I}]$ and B, B_i any type, then:

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$$(\textit{ValSelect}) \quad \frac{\Gamma \vdash a^A : \langle \ell_j{:}\sigma_j {\rightarrow} \tau_j{}^{j \in J} \rangle \quad \Gamma \vdash a^A : \sigma_k}{\Gamma \vdash a.\ell_k^{B_k} : \tau_k} \; (k \in J)$$

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$$(\textit{ValUpdate})\quad \frac{\Gamma \vdash a^A : \langle \ell_j{:}\sigma_j{}^{j \in J} \rangle \quad \Gamma, y^A : \sigma \vdash b^{B_k} : \tau}{\Gamma \vdash (a.\ell_k \Leftarrow \varsigma(y^A) b)^A : \langle \ell_j{:}\sigma_j{}^{j \in J \backslash k}, \ell_k{:}\sigma {\to} \tau \rangle}\ (k \in J)$$

Example

Let $A \equiv [\ell_0 : \mathsf{I}, \ell_1 : \mathsf{I}$ and $a \equiv [\ell_0 = \varsigma(x^A)1$, $\ell_1 = \varsigma(x^A)x.\ell_0]$, such that $\vdash a^A$. Then (where I stands for the type Integer, and O and E for its sub-type Odd, and Even.)

$$\dfrac{x^A : \omega \vdash 1^{\mathsf{I}} : \mathsf{O} \qquad \dfrac{\overline{x^A : \langle \ell_0 : \omega \to \mathsf{O} \rangle \vdash x^A : \langle \ell_0 : \omega \to \mathsf{O} \rangle} \qquad \overline{x^A : \langle \ell_0 : \omega \to \mathsf{O} \rangle \vdash x^A : \omega}}{x^A : \langle \ell_0 : \omega \to \mathsf{O} \rangle \vdash (x.\ell_0)^{\mathsf{I}} : \mathsf{O}}}{\vdash a^A : \langle \ell_0 : \omega \to \mathsf{O}, \ell_1 : \langle \ell_0 : \omega \to \mathsf{O} \rangle \to \mathsf{O} \rangle}$$

Note that ℓ_0 is a field; ℓ_1 is the method $\mathtt{get}\ell_0$.

Example (2)

By rule $($*ValUpdate*$)$ one might derive

$$\dfrac{\vdash a^A : \langle \ell_0 : \omega \rightarrow \mathsf{O}, \ell_1 \rangle \langle \ell_0 : \omega \rightarrow \mathsf{O} \rangle \rightarrow \mathsf{O} \qquad \dfrac{}{y^A : \omega \vdash 2^{\mathsf{I}} : \mathsf{E}}}{\vdash (a.\ell_0 \Leftarrow \varsigma(y^A)2)^A : \langle \ell_0 : \omega \rightarrow \mathsf{E}, \ell_1 : \langle \ell_0 : \omega \rightarrow \mathsf{O} \rangle \rangle \rightarrow \mathsf{O}}$$

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Now $(a.\ell_0 \Leftarrow \varsigma(y^A)2).\ell_1 \longrightarrow^* 2$, and we assume $\nvdash\ 2 : \mathsf{O}$, but $\nvdash (a.\ell_0 \Leftarrow \varsigma(y^A)2).\ell_1 : \mathsf{O}$ because rule $($*ValSelect*$)$ does not apply since

$$\nvdash (a.\ell_0 \Leftarrow \varsigma(y^A)2) : \langle \ell_0 : \omega \to \mathsf{O} \rangle.$$

Example (3)

On the other hand, the following is legal as well, by rule (*TypeObject*):

$$\dfrac{x^A : \omega \vdash 1^{\mathsf{I}} : \mathsf{O} \qquad \dfrac{\dfrac{}{x^A : \langle \ell_0 : \omega \rightarrow \mathsf{E} \rangle \vdash x^A : \langle \ell_0 : \omega \rightarrow \mathsf{E} \rangle} \qquad \dfrac{}{x^A : \langle \ell_0 : \omega \rightarrow \mathsf{E} \rangle \vdash x^A : \omega}}{x^A : \langle \ell_0 : \omega \rightarrow \mathsf{E} \rangle \vdash (x.\ell_0)^{\mathsf{I}} : \mathsf{E}}}{\vdash a^A : \langle \ell_0 : \omega \rightarrow \mathsf{O}, \ell_1 \langle \ell_0 : \omega \rightarrow \mathsf{E} \rangle \rightarrow \mathsf{E} \rangle}$$

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$$

Therefore, by rule (*ValUpdate*), we have the assignment:

$$
\dfrac{
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Predicate Assignment (2)

The assignment system is completed by the following 'logical' rules:

$$(\omega)\ \frac{E \vdash a^B}{\Gamma \vdash a^B : \omega}\ (E \lhd \Gamma) \qquad (\wedge I)\ \frac{\Gamma \vdash a^B : \sigma \quad \Gamma \vdash a^B : \tau}{\Gamma \vdash a^B : \sigma \wedge \tau}$$

$$(\leq)\ \frac{\Gamma \vdash a^B : \sigma \quad \sigma \leq \tau}{\Gamma \vdash a^B : \tau}$$

If E is a context and Γ a basis, we say that E *fits into* Γ, written $E \lhd \Gamma$, if $x^A : \sigma \in \Gamma$ implies $x^A \in E$.

Invariance under conversion

- If $\Gamma \vdash a^A : \rho$, and $a \longrightarrow a'$, then $\Gamma \vdash a'^A : \rho$.
- If $\Gamma \vdash a^A : \rho$ and $a' \longrightarrow a$ where $E \vdash a'^A$ for $E \lhd \Gamma$, then $\Gamma \vdash a'^A : \rho$.

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If $\tau \in \mathcal{L}_B$ whenever $x^B : \tau \in \Gamma$ and $\Gamma \vdash a^A : \sigma$ is derivable, then $\sigma \in \mathcal{L}_A$.

Untyped ς-models (1)

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such that (writing lcond and sel in a Curryfied form):

- $\text{sel}(\text{lcond}\, x\, \ell_i\, y)\ell_i = y$,
- $i \neq j \Rightarrow \text{sel}(\text{lcond}\, x\, \ell_i\, y)\ell_j = \text{sel}\, x\, \ell_j$,
- $i \neq j \Rightarrow \text{lcond}(\text{lcond}\, x\, \ell_i\, y)\, \ell_j\, z = \text{lcond}(\text{lcond}\, x\, \ell_i\, z)\, \ell_j\, y$.

An untyped ς-model is an instance of Mitchell's λ record-combinatory structure.

Untyped ς-models (2)

To build an untyped ς-model it suffices to solve:

$$D = At + [L \to D]_\perp + [D \to D]$$

Since any $\mathcal{D}$ is a λ-model, we shall freely use abstraction notation. Moreover, we use the abbreviations:

$$\begin{aligned} \langle \cdot \rangle &= \mathsf{emp} \\ \langle \ell_i = d_i{}^{i \in \{1,\ldots,n\}} \rangle &= \mathsf{lcond}(\ldots(\mathsf{lcond}\,\mathsf{emp}\,\ell_1\,d_1)\ldots)\ell_n\,d_n \\ d \cdot \ell_i &= \mathsf{sel}\,d\,\ell_i \\ d \cdot \ell_i := e &= \mathsf{lcond}\,d\,\ell_i\,e \end{aligned}$$

Predicate Interpretation

If D is a solution of $D = At + [L \to D]_\perp + [D \to D]$, then $[\![\sigma]\!]^{\mathcal{D}}_\eta \subseteq D$:

- $[\![\omega]\!]_\eta = D$,
- $[\![\kappa]\!]_\eta = \eta(\kappa)$, where $\eta : [\![\kappa]\!] \to \wp(D)$,
- $[\![\sigma \wedge \tau]\!]_\eta = [\![\sigma]\!]_\eta \cap [\![\tau]\!]_\eta$,
- $[\![\sigma \to \tau]\!]_\eta = \{d \in D \mid \forall e \in [\![\sigma]\!]_\eta .\ de \in [\![\tau]\!]_\eta\}$,
- $[\![\langle \ell_i : \sigma_i{}^{i \in I} \rangle]\!]_\xi = \{d \in D \mid \forall i \in I .\ d \cdot \ell_i \in [\![\sigma_i]\!]_\eta\}$.

If $\sigma \leq \tau$ then, for any η, $[\![\sigma]\!]_\eta \subseteq [\![\tau]\!]_\eta$.

Type Interpretation

Let D be any domain: a *retraction* over D is a continuous function $\rho : D \to D$ such that $\rho^2 = \rho \circ \rho = \rho$.
Types can be interpreted by retractions setting (Scott):

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Proposition. Let $A \equiv [\ell_i : B_i\ ^{i \in I}]$: if ρ_{B_i} is a retraction for all $i \in I$, then there exists a retraction ρ_A such that

$$\rho_A(d) = \langle \ell_i = \rho_{A \to B_i}(d \cdot \ell_i)\ ^{i \in I} \rangle,$$

where $\rho_{A \to B}(d) = \lambda x.\rho_B(d(\rho_A(x)))$.

$\rho_A =$ *Fix* (Υ_A) where $\Upsilon_A f d = \langle \ell_i = \lambda x.\rho_{B_i}((d \cdot \ell_i)(f x))\ ^{i \in I} \rangle$.

Term Interpretation over Retractio

We say that $(\mathcal{D}, \{\rho_A\}_A)$ is a *retraction model* if $\mathcal{D}$ is an untyped ς-model and $\{\rho_A\}_A$ is a family of retractions such that

$$\rho_A(d) = \langle \ell_i = \rho_{A \to B_i}(d \cdot \ell_i) \ ^{i \in I} \rangle \text{ where } A \equiv [\ell_i : B_i \ ^{i \in I}].$$

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The *term interpretation* $[\![a^A]\!]^{\mathcal{D}}{}_\xi \in D$, where ξ is an environment, is defined:

$$\begin{aligned}
[\![x^A]\!]_\xi &= \xi(x) \\
[\![[\ell_i = \varsigma(x_i^A)b_i^{B_i}\ ^{i \in I}]]\!]_\xi &= \langle \ell_i = \lambda d.[\![b_i^{B_i}]\!]_{\xi[x_i := \rho_A(d)]}\ ^{i \in I} \rangle \\
[\![(a^A.\ell_i)^{B_i}]\!]_\xi &= ([\![a^A]\!]_\xi \cdot \ell_i)[\![a^A]\!]_\xi \\
[\![a^A.\ell_i \Leftarrow \varsigma(x^A)b^{B_i}]\!]_\xi &= [\![a^A]\!]_\xi \cdot \ell_i := \lambda d.[\![b^{B_i}]\!]_{\xi[x := \rho_A(d)]}.
\end{aligned}$$

Soundness

Set

- $\xi \models E$ if $\xi(x^A) \in [\![A]\!]^{\mathcal{D}}$ whenever $x^A \in E$;
- $E \models a^A$ if for all ξ s.t. $\xi \models E$, $[\![a^A]\!]^{\mathcal{D}}{}_{\xi} \in [\![A]\!]^{\mathcal{D}}$;
- $\xi \models \Gamma$ if $x^A : \sigma \in \Gamma$ implies $\xi(x^A) \in [\![\sigma]\!]_{\eta} \subseteq [\![A]\!]^{\mathcal{D}}$;
- $\Gamma \models a^A : \sigma$ if for all ξ s.t. $\xi \models \Gamma$,
  $[\![a^A]\!]^{\mathcal{D}}{}_{\xi} \in [\![\sigma]\!]_{\eta} \subseteq [\![A]\!]^{\mathcal{D}}$.

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Then

1. If $E \vdash a^A$ then $E \models a^A$;
2. if $\Gamma \vdash a^A : \sigma$ then $\Gamma \models a^A : \sigma$.

The Filter Model

If $\mathcal{F}$ is the set of *filters* over $\mathcal{L}$ (nonempty, upward closed sets of predicates closed under $\wedge$), the it is a λ-model such that:

- emp $= \; \uparrow\langle\cdot\rangle$;
- $F \cdot \ell_i = \{\sigma \mid \langle \ell_i : \sigma\rangle \in F\}$;
- $(F \cdot \ell_i :=$

$$G) = \left\{ \langle \ell_j : \sigma_j{}^{j\in J}\rangle \mid \begin{array}{c} (j \neq i \,\&\, \langle \ell_j : \sigma\rangle \in F) \vee \\ (j = i \,\&\, \sigma_i \in G) \end{array} \right.$$

$\mathcal{F}$ is an untyped ς-model.

Languages: retractions over $\mathcal{F}$

The family $\{\rho_A\}_A$ where $\rho_A(F) = F \cap \mathcal{L}_A$, is a family of retractions turning $\mathcal{F}$ into a retraction model. Indeed, for all $F \in \mathcal{F}$ and type $A \equiv [\ell_i : B_i \ {}^{i \in I}]$:

$$F \cap \mathcal{L}_A = \langle \ell_i = \lambda X.(F \cdot \ell_i)(X \cap \mathcal{L}_A) \cap \mathcal{L}_{B_i} \ {}^{i \in I} \rangle,$$

where $\lambda X.e[X] = \{\sigma \rightarrow \tau \mid \tau \in e[\uparrow\sigma]\}$ represents a continuous function over $\mathcal{F}$.

Moreover, $[\![\sigma]\!]_\eta = \{F \in \mathcal{F} \mid \sigma \in F\}$ is a predicate interpretation such that, if $\eta(\kappa) \subseteq [\![K]\!]$ whenever $\kappa \in \mathcal{L}_K$, then $[\![\sigma]\!]_\eta \subseteq [\![A]\!]$ if $\sigma \in \mathcal{L}_A$.

Completeness

For all a^A such that $E \vdash a^A$, for some E, and all environment ξ s.t. $\xi \models E$:

$$[\![a^A]\!]^{\mathcal{F}}{}_{\xi} = \{\sigma \mid \exists \Gamma.\ \xi \models \Gamma \ \&\ \Gamma \vdash a^A : \sigma\}$$

$$\Gamma \vdash a^A : \sigma \iff \Gamma \models a^A : \sigma.$$

Further steps

- Account for (first order) subtyping (some ideas, many problems).
- Study higher order calculi, possibly including classes and class inheritance (mixins).
- Consider hybrid calculi, like objects and MA (work in progress with Barbanera).