A Domain Logic Approach to Models of $\lambda\mu$ and Related Calculi

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- Intersection type assignment system have been successfully used in the study of the pure λ-calculus, and also of term rewriting, object calculi, etc.
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Previous work (not all successful) by: Dougherty, Ghilezan, Lescanne (on $\overline{\lambda}\mu\tilde{\mu}$) and van Bakel (on $\mathcal{X}$, $\overline{\lambda}\mu\tilde{\mu}$ and $\lambda\mu$).

Overview

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continuation semantics	(Streicher, Reus)	+
domain logic	(Abramsky)	=

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We eventually show that simply typed $\lambda\mu$-terms (in Parigot's system) have non trivial types in our system.

Are SN $\lambda\mu$-terms characterizable in our system?

Continuation Semantics (Streicher-Reus [SR98])

Continuation domain equations

$$\begin{cases} R & & & \textit{results} \\ D & = & [C \to R] & \textit{denotations} \\ C & = & (D \times C) & \textit{continuations} \end{cases}$$

If (R, D, C) are a solution then D is an extensional λ-model:

$$D \simeq [C \to R] \simeq [(D \times C) \to R] \simeq [D \to [C \to R]] \simeq [D \to D]$$

while C is the infinite product:

$$C \simeq D \times D \times \cdots$$

Intersection Type Representation of ω-Algebraic Lattices

Given $A \in \omega\mathbf{ALG}$

$\mathcal{L}_A: \quad \sigma, \tau ::= \cdots \mid \sigma \wedge \tau \mid \omega$

$\leq_A$: a preorder s.t.

$\sigma \leq_A \omega$ — ω is the top

$\sigma \wedge \tau \leq \sigma, \tau$

$\rho \leq \sigma, \rho \leq \tau \Rightarrow \rho \leq \sigma \wedge \tau$ — the meet of σ, τ

Let

$$\Theta : \mathcal{L}_A \to \mathcal{K}(A) \text{ be surjective and s.t. } \sigma \leq_A \tau \Leftrightarrow \Theta(\sigma) \sqsupseteq \Theta(\tau)$$

then

$$\mathcal{F}^A := \mathrm{Filt}(\mathcal{L}_A) \simeq \mathrm{Filt}(\mathcal{L}_A/_{\leq_A}) \simeq \mathrm{Filt}(\mathcal{K}^{op}(A)) \simeq A$$

via the continuous extension of $\uparrow\sigma \mapsto \uparrow\Theta(\sigma)$

Type Theory $\leq_R$

Fix $R \in \omega\mathbf{ALG}$:

$$\mathcal{L}_R : \quad \rho ::= \nu_a \mid \omega \mid \rho \wedge \rho \qquad\qquad (a \in \mathcal{K}(R))$$

$$\leq_R : \quad \nu_\perp =_R \omega$$

$$\nu_{a \sqcup b} =_R \nu_a \wedge \nu_b$$

$$b \sqsubseteq a \in \mathcal{K}(R) \Rightarrow \nu_a \leq_R \nu_b$$

Proposition

Define $\mathcal{F}^R = \mathrm{Filt}(\mathcal{L}_R/_{\leq_R})$; then $R \simeq \mathcal{F}^R$ because of $\mathcal{L}_R/_{\mathcal{L}_R} \simeq \mathcal{K}^{op}(R)$ via

$$\Theta(\nu_a) = a, \quad \Theta(\omega) = \perp, \quad \Theta(\rho \wedge \rho') = \Theta(\rho) \sqcup \Theta(\rho')$$

Type Theories $\leq_C$ and $\leq_D$

Type languages (constructed following the approximants of the inverse-limit construction):

$$
\begin{array}{lll}
C_0 & = & \{\bot\} \\
D_n & = & [C_n \to R] \\
C_{n+1} & = & D_n \times C_n \\
\\
D & = & \lim_n D_n \\
C & = & \lim_n C_n
\end{array}
\quad \Longrightarrow \quad
\begin{array}{llll}
\mathcal{L}_{C_0} & \kappa_0 & ::= & \omega \\
\mathcal{L}_{D_n} & \delta_n & ::= & \rho \mid \kappa_n \to \rho \mid \delta_n \wedge \delta_n \mid \omega \\
\mathcal{L}_{C_{n+1}} & \kappa_{n+1} & ::= & \delta_n \times \kappa_n \mid \kappa_n \wedge \kappa_n \mid \omega \\
\\
& \mathcal{L}_D & = & \bigcup_n \mathcal{L}_{D_n} \\
& \mathcal{L}_C & = & \bigcup_n \mathcal{L}_{C_n}
\end{array}
$$

or equivalently

$$
\begin{array}{llll}
\mathcal{L}_D : & \delta ::= & \rho \mid \kappa \to \rho \mid \omega \mid \delta \wedge \delta \\
\mathcal{L}_C : & \kappa ::= & \delta \times \kappa \mid \omega \mid \kappa \wedge \kappa
\end{array}
$$

Type Theories $\leq_C$ and $\leq_D$

Theory $\leq_C$:

$$\frac{}{\omega \leq_C \omega \times \omega} \qquad \frac{}{(\delta_1 \times \kappa_1) \wedge (\delta_2 \times \kappa_2) \leq_C (\delta_1 \wedge \kappa_2) \times (\kappa_1 \wedge \kappa_2)}$$

$$\frac{\delta_1 \leq_D \delta_2 \quad \kappa_1 \leq_C \kappa_2}{\delta_1 \times \kappa_1 \leq_C \delta_2 \times \kappa_2}$$

Theory $\leq_D$:

$$\frac{}{\omega \leq_D \omega \to \omega} \qquad \frac{}{\nu =_D \omega \to \nu} \qquad \frac{}{(\kappa \to \delta_1) \wedge (\kappa \to \delta_2) \leq_D \kappa \to (\kappa_1 \wedge \kappa_2)}$$

$$\frac{\kappa_2 \leq_C \kappa_1 \quad \delta_1 \leq_D \delta_2}{\kappa_1 \to \delta_1 \leq_D \kappa_2 \to \delta_2}$$

Solution of Continuation Equations

Filter Domains solving Continuation Equations

Let $\mathcal{F}^C = \mathrm{Filt}(\mathcal{L}_C/\leq_C)$ and $\mathcal{F}^D = \mathrm{Filt}(\mathcal{L}_D/\leq_D)$:

$$\mathcal{F}^D \simeq [\mathcal{F}^C \to \mathcal{F}^R] \quad \text{and} \quad \mathcal{F}^C \simeq \mathcal{F}^D \times \mathcal{F}^C$$

Via the mappings $F : \mathcal{F}^D \to [\mathcal{F}^C \to \mathcal{F}^R]$ and $G : [\mathcal{F}^C \to \mathcal{F}^R] \to \mathcal{F}^D$

$$\begin{array}{lcl} F\,d\,k & = & \{\rho \in \mathcal{L}_R \mid \exists \kappa \to \rho \in d.\, \kappa \in k]\} \\ G\,f & = & \{\bigwedge_{i\in I} \kappa_i \to \rho_i \in \mathcal{L}_D \mid \forall i \in I.\, \rho_i \in f(\uparrow \kappa_i)\} \end{array}$$

and the mappings $H : \mathcal{F}^C \to (\mathcal{F}^D \times \mathcal{F}^C)$ and $K : (\mathcal{F}^D \times \mathcal{F}^C) \to \mathcal{F}^C$

$$\begin{array}{lcl} H\,k & = & \langle \{\delta \in \mathcal{L}_D \mid \delta \times \kappa \in k\}, \{\kappa \in \mathcal{L}_D \mid \delta \times \kappa \in k\} \rangle \\ K\langle d, k\rangle & = & \{\delta \times \kappa \in \mathcal{L}_C \mid \delta \in d \;\&\; \kappa \in k\} \end{array}$$

The λ-calculus: syntax and continuation semantics

Syntax:

$$M, N \quad ::= \quad x \mid \lambda x.M \mid MN \qquad \text{(terms)}$$

Semantics [SR98]:

$$[\![\cdot]\!]^D : \mathrm{Trm} \to \mathrm{Env} \to D \quad \text{where} \quad D = C \to R$$

$$\begin{array}{rcl} [\![x]\!]^D\, e\, k & = & e\, x\, k \\ [\![\lambda x.M]\!]^D\, e\, k & = & [\![M]\!]^D\, e[x := d]\, k' \qquad \langle d, k' \rangle = k \\ [\![MN]\!]^D\, e\, k & = & [\![M]\!]^D\, e\, \langle [\![N]\!]^D e, k \rangle \end{array}$$

where $e \in \mathrm{Env} = Var \to D$, $d \in D$ and $k, k' \in C$ so that $[\![M]\!]^D\, e\, k \in R$.

Type interpretation

$$
\begin{array}{lcll}
[\![\nu_a]\!]^R & = & \{r \in R \mid a \sqsubseteq r\} & \\
[\![\delta \times \kappa]\!]^R & = & [\![\delta]\!]^D \times [\![\kappa]\!]^C & \\
[\![\kappa \rightarrow \rho]\!]^D & = & \{d \in D \mid \forall k \in [\![\kappa]\!].\ d(k) \in [\![\rho]\!]^R\} & \\
[\![\rho]\!]^D & = & [\![\omega \rightarrow \rho]\!]^D & \\
[\![\omega]\!]^A & = & A & \text{for } A = R, D, C \\
[\![\sigma \wedge \tau]\!]^A & = & [\![\sigma]\!]^A \cap [\![\tau]\!]^A &
\end{array}
$$

Let $\sigma, \tau \in \mathcal{L}_A$

Soundness of Type Interpretation

1. $[\![\sigma]\!]^A \subseteq A$ and $[\![\sigma]\!]^A = \uparrow\Theta(\sigma)$
2. $\sigma \leq_A \tau \Rightarrow [\![\sigma]\!]^A \subseteq [\![\tau]\!]^A$

The λ-calculus: continuation models and validity

A *continuation model* (a *model* for short) is a triple $\mathcal{M} = (R, D, C)$ satisfying the continuation equations, together with the maps $[\![\cdot]\!]^R, [\![\cdot]\!]^D$ and $[\![\cdot]\!]^C$.

A *basis* is a set $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$, with $x_i \in Var, \delta_i \in \mathcal{L}_D$.
A *judgement* is an expression $\Gamma \vdash M : \delta$, with $\delta \in \mathcal{L}_D$.

The concepts of *satisfaction* and *validity* are the standard ones:

- $e \models \Gamma \;\Leftrightarrow\; \forall x : \delta \in \Gamma.\, e(x) \in [\![\delta]\!]^D$
- $\Gamma \models_{\mathcal{M}} M : \delta \;\Leftrightarrow\; \forall e \in \mathrm{Env}.\, e \models \Gamma \Rightarrow [\![M]\!]^D e \in [\![\delta]\!]^D$
- $\Gamma \models M : \delta \Leftrightarrow \forall \mathcal{M}.\, \Gamma \models_{\mathcal{M}} M : \delta$

where $e \in \mathrm{Env} = Var \rightarrow D$ is any environment.

$\Gamma \vdash M : \delta$ is *valid* if and only if $\Gamma \models M : \delta$

The filter model

Consider the model $\mathcal{F} = (\mathcal{F}^R, \mathcal{F}^D, \mathcal{F}^C)$:

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Rules from interpretation in $\mathcal{F}$

$$\frac{\textit{defininens}\text{, right hand side of defining clause} = \text{premises}}{\textit{definiendum}\text{, left hand side of defining clause} = \text{conclusion}}$$

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Observing that

$$[\![M]\!]^{\mathcal{F}^D} e \in [\![\delta]\!]^{\mathcal{F}^D} \quad \Leftrightarrow \quad \delta \in [\![M]\!]^{\mathcal{F}^D} e$$

Rules from interpretation clauses: $(\to I)$ and $(\to E)$

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Consider the interpretation of $\lambda x.M$ in $\mathcal{F}^D$:

$$[\![\lambda x.M]\!]^{\mathcal{F}^D} e\,\langle\uparrow\delta,\uparrow\kappa\rangle = [\![M]\!]^{\mathcal{F}^D} e[x := \uparrow\delta]\ \uparrow\kappa$$

where $\langle\uparrow\delta,\uparrow\kappa\rangle \simeq \uparrow(\delta\times\kappa)$. This reads as the inference rule:

$$\frac{\Gamma, x:\delta \vdash M : \kappa \to \rho}{\Gamma \vdash \lambda x.M : \delta\times\kappa \to \rho}\,(\to I)$$

Rules from interpretation clauses: $(\to I)$ and $(\to E)$

Consider the interpretation of $\lambda x.M$ in $\mathcal{F}^D$:

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where $\langle \uparrow\delta, \uparrow\kappa \rangle \simeq \uparrow(\delta \times \kappa)$. This reads as the inference rule:

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \rho}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho} \; (\to I)$$

Similarly in case of MN, from

$$[\![MN]\!]^{\mathcal{F}^D} e \uparrow\kappa = [\![M]\!]^{\mathcal{F}^D} e \, \langle [\![N]\!]^{\mathcal{F}^D} e, \uparrow\kappa \rangle$$

and for any $\delta \in [\![N]\!]^{\mathcal{F}^D} e$ we get:

$$\frac{\Gamma \vdash M : \delta \times \kappa \to \rho \quad \Gamma \vdash N : \delta}{\Gamma \vdash MN : \kappa \to \rho} \; (\to E)$$

Remark

By extending $\mathcal{L}_D$ with $\delta_1 \to \delta_2$ types and putting

$$\delta \times \kappa \to \rho =_D \delta \to (\kappa \to \rho)$$

we have by subsumption:

$$\frac{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \quad \delta \times \kappa \to \rho \leq_D \delta \to (\kappa \to \rho)}{\Gamma \vdash M : \delta \to (\kappa \to \rho)}$$

that (a form of) the standard rule $(\to I)$ is admissible:

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \rho}{\Gamma \vdash \lambda x.M : \delta \to (\kappa \to \rho)}$$

Similarly for rule $(\to E)$.

The $\lambda\mu$-calculus: syntax and reduction

$\lambda\mu$ syntax:

$$\begin{array}{rcll} M, N & ::= & x \mid \lambda x.M \mid MN \mid \mu\alpha.Q & \text{(terms)} \\ Q & ::= & [\alpha]M & \text{(commands)} \end{array}$$

Structural substitution:

$$T[\alpha \Leftarrow L] \equiv \text{the replacement of all } [\alpha]N \text{ by } [\alpha]NL \text{ in } T$$

Reduction:

$$\begin{array}{rrcll} (\beta): & (\lambda x.M)N & \longrightarrow & M[N/x] & \\ (\mu): & (\mu\beta.Q)N & \longrightarrow & \mu\beta.Q[\beta \Leftarrow N] & \\ (ren): & [\alpha]\mu\beta.Q & \longrightarrow & Q[\alpha/\beta] & \\ (\mu\eta): & \mu\alpha.[\alpha]M & \longrightarrow & M & \text{if } \alpha \notin fn(M) \end{array}$$

The $\lambda\mu$-calculus: term semantics

$$\mathrm{Env} = (\mathit{Var} \to D) + (\mathit{Name} \to C)$$

$$[\![\cdot]\!]^{D} : \mathrm{Trm} \to \mathrm{Env} \to D$$

$$[\![\cdot]\!]^{C} : \mathrm{Cmd} \to \mathrm{Env} \to C$$

$$\begin{array}{rcll}
[\![x]\!]^{D}\, e\, k &=& e\, x\, k & \\
[\![\lambda x.M]\!]^{D}\, e\, k &=& [\![M]\!]^{D}\, e[x := d]\, k' & \langle d, k' \rangle = k \\
[\![MN]\!]^{D}\, e\, k &=& [\![M]\!]^{D}\, e\, \langle [\![N]\!]^{D} e, k \rangle & \\
[\![\mu\alpha.Q]\!]^{D}\, e\, k &=& d\, k' & \langle d, k' \rangle = [\![Q]\!]^{C}\, e[\alpha := k] \\
[\![[\alpha]M]\!]^{C}\, e &=& \langle [\![M]\!]^{D}\, e, k \rangle & k = e\, \alpha
\end{array}$$

where $e \in \mathrm{Env}$, $d \in D$ and $k, k' \in C$. We immediately have:

$$M =_{\beta\mu} N \Rightarrow \forall e \in \mathrm{Env}, k \in C,\ [\![M]\!]^{D}\, e\, k = [\![N]\!]^{D}\, e\, k$$

Typing judgements

Typing judgements are triples of a *basis*, a *term/command judgement* and a *context*:

- *basis*: $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$, with $x_i \in Var, \delta_i \in \mathcal{L}_D$
- *judgement*:
  - *term*: $M : \delta$ with $M \in \mathrm{Trm}, \delta \in \mathcal{L}_D$
  - *command*: $Q : \kappa$ with $Q \in \mathrm{Cmd}, \kappa \in \mathcal{L}_C$
- *context*: $\Delta = \{\alpha_1 : \kappa_1, \ldots, \alpha_m : \kappa_m\}$, with $\alpha_i \in Name, \kappa_i \in \mathcal{L}_C$

A *typing judgement* has either forms:

$$\Gamma \vdash M : \delta \mid \Delta \quad \text{or} \quad \Gamma \vdash Q : \kappa \mid \Delta$$

The $\lambda\mu$-calculus: validity revised

Let $e \in \text{Env}$:

- $e \models \Gamma, \Delta \;\Leftrightarrow\; \forall x : \delta \in \Gamma.\, e(x) \in [\![\delta]\!]^D \;\&\; \forall \alpha : \kappa \in \Delta.\, e(\alpha) \in [\![\kappa]\!]^C$
- $\Gamma \models_{\mathcal{M}} M : \delta \mid \Delta \;\Leftrightarrow\; \forall e \in \text{Env}.\, e \models \Gamma, \Delta \Rightarrow [\![M]\!]^D e \in [\![\delta]\!]^D$

  and similarly $\mathcal{M} \models \Gamma \vdash Q : \kappa \mid \Delta$
- $\Gamma \models M : \delta \mid \Delta$ ($\Gamma \models Q : \kappa \mid \Delta$) $\Leftrightarrow$

  $$\forall \mathcal{M}.\, \Gamma \models_{\mathcal{M}} M : \delta \mid \Delta \;(\Gamma \models_{\mathcal{M}} Q : \kappa \mid \Delta)$$

Say that $\Gamma \vdash M : \delta \mid \Delta$ ($\Gamma \vdash Q : \kappa \mid \Delta$) is *valid* if $\Gamma \models M : \delta \mid \Delta$ ($\Gamma \models Q : \kappa \mid \Delta$)

Rules from interpretation clauses: $(\times)$

Consider:

$$[\![[\alpha]M]\!]^{\mathcal{F}^C} e = \langle [\![M]\!]^{\mathcal{F}^D} e, e\,\alpha \rangle$$

then for any $\delta \in [\![M]\!]^{\mathcal{F}^D} e$ and $\kappa \in (e\,\alpha)$:

$$\delta \times \kappa \in [\![[\alpha]M]\!]^{\mathcal{F}^C} e$$

from which we obtain the rule

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha : \kappa, \Delta}\ (\times)$$

Rules from interpretation clauses: (μ)

Recall that

$$[\![\mu\alpha.Q]\!]^{\mathcal{F}^D} e \uparrow \kappa = d \cdot k' \quad \text{where} \quad [\![Q]\!]^{\mathcal{F}^C} e[\alpha := \uparrow \kappa] = \langle d, k' \rangle$$

Filter application is $d \cdot k' = \{\rho \mid \exists \kappa' \to \rho \in d.\ \kappa' \in k'\}$ hence

$$(\kappa' \to \rho) \times \kappa' \in [\![Q]\!]^{\mathcal{F}^C} e[\alpha := \uparrow \kappa] \quad \Rightarrow \quad \rho \in [\![\mu\alpha.Q]\!]^{\mathcal{F}^D} e \uparrow \kappa$$

and we obtain the rule

$$\frac{\Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta} \ (\mu)$$

The type assignment system $\vdash_\wedge$

$$\frac{}{\Gamma, x:\delta \vdash x:\delta \mid \Delta}\,(Var)$$

$$\frac{\Gamma, x:\delta \vdash M:\kappa\to\rho \mid \Delta}{\Gamma \vdash \lambda x.M:\delta\times\kappa\to\rho \mid \Delta}\,(\to I) \qquad \frac{\Gamma \vdash M:\delta\times\kappa\to\rho \mid \Delta \quad \Gamma \vdash N:\delta \mid \Delta}{\Gamma \vdash MN:\kappa\to\rho \mid \Delta}\,(\to E)$$

$$\frac{\Gamma \vdash M:\delta \mid \alpha:\kappa,\Delta}{\Gamma \vdash [\alpha]M:\delta\times\kappa \mid \alpha:\kappa,\Delta}\,(\times) \qquad \frac{\Gamma \vdash Q:(\kappa'\to\rho)\times\kappa' \mid \alpha:\kappa,\Delta}{\Gamma \vdash \mu\alpha.Q:\kappa\to\rho \mid \Delta}\,(\mu)$$

$T = M, Q$; $A = D, C$; $\sigma,\tau \in \mathcal{L}_A$:

$$\frac{}{\Gamma \vdash T:\omega \mid \Delta}\,(\omega) \qquad \frac{\Gamma \vdash T:\sigma \mid \Delta \quad \Gamma \vdash T:\tau \mid \Delta}{\Gamma \vdash T:\sigma\wedge\tau \mid \Delta}\,(\wedge) \qquad \frac{\Gamma \vdash T:\sigma \mid \Delta \quad \sigma \leq_A \tau}{\Gamma \vdash T:\tau \mid \Delta}\,(\leq)$$

Properties of the type assignment system

Theorem: subject reduction and expansion

If $M \longrightarrow_{\beta\mu} N$ then

$$\Gamma \vdash M : \delta \mid \Delta \quad \Leftrightarrow \quad \Gamma \vdash N : \delta \mid \Delta$$

Theorem: soundness and completeness

$$\Gamma \vdash M : \delta \mid \Delta \quad \Leftrightarrow \quad \Gamma \models M : \delta \mid \Delta$$

Proof based on the filter model construction and the fact that

$$[\![M]\!]^{\mathcal{F}^D} e = \{\delta \mid \exists \Gamma, \Delta.\, e \models \Gamma, \Delta \;\;\&\;\; \Gamma \vdash M : \delta \mid \Delta\}$$

Parigot's (first order) type assignment system $\vdash_P$

Propositional types:

$$A, B ::= \varphi \mid \bot_A \mid A \to B$$

Rules:

$$\frac{}{\Gamma, x : A \vdash x : A \mid \Delta}$$

$$\frac{\Gamma, x : A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x.M : A \to B \mid \Delta} \qquad \frac{\Gamma \vdash M : A \to B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta}$$

$$\frac{\Gamma \vdash Q : \bot_B \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.Q : A \mid \Delta} \qquad \frac{\Gamma \vdash M : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]M : \bot_A \mid \alpha : A, \Delta}$$

Translating Parigot's types into intersection types

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Take $a \in R \setminus \{\bot\}$, and write $\nu \equiv \nu_a$. Consider the maps from propositional types into ours, $A^C \in \mathcal{L}_C$ and $A^D \in \mathcal{L}_D$:

- $\varphi^C = \nu \times \omega$
- $\bot_A^C = (A^C \rightarrow \nu) \times A^C$
- $(A \rightarrow B)^C = (A^C \rightarrow \nu) \times B^C$
- $A^D = A^C \rightarrow \nu$

Write $\Gamma^D = \{x : A^D \mid x : A \in \Gamma\}$ and $\Delta^C = \{\alpha : A^C \mid \alpha : A \in \Delta\}$

Translating Parigot's types into intersection types

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- $\varphi^C = \nu \times \omega$
- $\bot_A^C = (A^C \rightarrow \nu) \times A^C$
- $(A \rightarrow B)^C = (A^C \rightarrow \nu) \times B^C$
- $A^D = A^C \rightarrow \nu$

Write $\Gamma^D = \{x : A^D \mid x : A \in \Gamma\}$ and $\Delta^C = \{\alpha : A^C \mid \alpha : A \in \Delta\}$

Theorem

1. $\Gamma \vdash_P M : A \mid \Delta \Rightarrow \Gamma^D \vdash_\wedge M : A^D \mid \Delta^C$
2. $\Gamma \vdash_P Q : A \mid \Delta \Rightarrow \Gamma^D \vdash_\wedge Q : A^C \mid \Delta^C$

Conjecture: exactly all strongly normalizable terms are typeable in a subsystem $\vdash_\wedge^-$, essentially eliminating ω.

Conclusions and future work

1. we have defined a type theory which describes a denotational model of $\lambda\mu$
2. the type assignment system induces a filter model of the $\lambda\mu$-calculus
3. all terms which are typeable in Parigot's (first order) assignment system have a non trivial typing in our system
4. we conjecture that non trivial typing in a proper subsystem of ours characterizes strongly normalizable terms
5. the system we have proposed could characterize other interesting sets of terms, and more interestingly might provide a tool to investigate Parigot's computational interpretation of classical proofs

Reference

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