A game interpretation of retractable contracts

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COORDINATION - Heraklion, June 2016

- In the theory of contracts we have proposed *retractable contracts* to relax binary contract compliance using backtracking [Barbanera et al. 2016]
- Compliance up to *orchestrators* has been exploited for similar purposes [Padovani 2010]
- Using *contracts as games on event structures* [Bartoletti et al. 2016] we prove that session contracts are retractable compliant if and only if they are such up to orchestrators.

Session contracts

$$
\begin{array}{rcll}
\sigma, \rho & := & \mid \mathbf{1} & \text{success} \\
& & \mid \sum_{i \in I} a_i.\sigma_i & \text{input} \\
& & \mid \bigoplus_{i \in I} \overline{a}_i.\sigma_i & \text{output} \\
& & \mid x & \text{variable} \\
& & \mid \mathrm{rec}\, x.\sigma & \text{recursion}
\end{array}
$$

$$
\sum_{i \in I} a_i.\sigma_i \xrightarrow{a_k} \sigma_k \qquad \bigoplus_{i \in I} \overline{a}_i.\sigma_i \rightarrow \overline{a}_k.\sigma_k \xrightarrow{\overline{a}_k} \sigma_k \qquad (k \in I)
$$

$$
\frac{\rho \rightarrow \rho'}{\rho \parallel \sigma \longrightarrow \rho' \parallel \sigma} \qquad \frac{\rho \xrightarrow{\alpha} \rho' \qquad \sigma \xrightarrow{\overline{\alpha}} \sigma'}{\rho \parallel \sigma \longrightarrow \rho' \parallel \sigma'}
$$

Definition (Compliance)

The *client* ρ is compliant with the *server* σ, $\rho \dashv \sigma$, if

$$\rho \parallel \sigma \xrightarrow{*} \rho' \parallel \sigma' \not\rightarrow \quad \text{implies} \quad \rho' = \mathbf{1}$$

Examples (without trailing $\mathbf{1}$'s):

- $\mathbf{1} \dashv \sigma$
- $\bar{a} \dashv a.b$ but $\bar{a}.\bar{b} \not\dashv a$
- $\bar{a} \dashv a + b$ and $a + b \dashv \bar{a}$ but $\bar{a} \oplus \bar{b} \not\dashv a$ and $a \not\dashv \bar{a} \oplus \bar{b}$
- $\mathrm{rec}\, x.\ \bar{a}.x \oplus \bar{b} \dashv \mathrm{rec}\, x.\ a.x + b$ but $\mathrm{rec}\, x.\ \bar{a}.x \oplus \bar{b} \not\dashv \mathrm{rec}\, x.\ a.x$

$\text{Buyer} = \overline{\texttt{belt}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \oplus \overline{\texttt{bag}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}})$

$\text{Seller} = \texttt{belt}.\overline{\texttt{price}}.\texttt{cash} + \texttt{bag}.\overline{\texttt{price}}.(\texttt{card} + \texttt{cash}) + \cdots$

then $\text{Buyer} \not\dashv \text{Seller}$:

$$\begin{aligned} & \text{Buyer} \parallel \text{Seller} \\ \to\quad & \overline{\texttt{belt}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \parallel \texttt{belt}.\overline{\texttt{price}}.\texttt{cash} \\ \to\quad & \overline{\texttt{card}} \oplus \overline{\texttt{cash}} \parallel \texttt{cash} \\ \to\quad & \overline{\texttt{card}} \parallel \texttt{cash} \qquad ✗ \end{aligned}$$

$\text{Buyer} = \overline{\texttt{belt}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) + \overline{\texttt{bag}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}})$

$\text{Seller} = \texttt{belt}.\overline{\texttt{price}}.\texttt{cash} + \texttt{bag}.\overline{\texttt{price}}.(\texttt{card} + \texttt{cash}) + \cdots$

where $+$ is a backtracking point:

$$
\begin{array}{lll}
 & \text{Buyer} \parallel \text{Seller} & \\
\rightarrow & \overline{\texttt{belt}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \parallel \texttt{belt}.\overline{\texttt{price}}.\texttt{cash} & \\
\rightarrow & \overline{\texttt{card}} \oplus \overline{\texttt{cash}} \parallel \texttt{cash} & \\
\rightarrow & \overline{\texttt{card}} \parallel \texttt{cash} & \text{backtrack!} \\
\rightarrow & \overline{\texttt{bag}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \parallel \texttt{bag}.\overline{\texttt{price}}.(\texttt{card} + \texttt{cash}) + \cdots & \\
\rightarrow & \texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \parallel \overline{\texttt{price}}.(\texttt{card} + \texttt{cash}) & \\
\xrightarrow{*} & \mathbf{1} \parallel \mathbf{1} & \checkmark
\end{array}
$$

Affectible contracts

σ, ρ	$:=$	$\mid \mathbf{1}$	success
		$\mid \sum_{i \in I} a_i.\sigma_i$	affectible input
		$\mid \sum_{i \in I} \overline{a}_i.\sigma_i$	affectible output
		$\mid \bigoplus_{i \in I} \overline{a}_i.\sigma_i$	unaffectible output
		$\mid x$	variable
		$\mid \mathrm{rec}\, x.\sigma$	recursion

Let $k \in I$:

$$\sum_{i \in I} a_i.\sigma_i \xrightarrow{a_k} \sigma_k \qquad \bigoplus_{i \in I} \overline{a}_i.\sigma_i \rightarrow \overline{a}_k.\sigma_k \xrightarrow{\overline{a}_k} \sigma_k \qquad \sum_{i \in I} \overline{a}_i.\sigma_i \xrightarrow{\overline{a}_k^+} \sigma_k$$

$$\frac{\rho \rightarrow \rho'}{\rho \parallel \sigma \longrightarrow \rho' \parallel \sigma} \qquad \frac{\rho \xrightarrow{\alpha} \rho' \qquad \sigma \xrightarrow{\overline{\alpha}} \sigma'}{\rho \parallel \sigma \xrightarrow{\tau} \rho' \parallel \sigma'} \qquad \frac{\rho \xrightarrow{\overline{a}^+} \rho' \qquad \sigma \xrightarrow{a} \sigma'}{\rho \parallel \sigma \xrightarrow{+} \rho' \parallel \sigma'}$$

Let $\stackrel{\tau}{\Longrightarrow} \; = \; \stackrel{*}{\longrightarrow} \stackrel{\tau}{\longrightarrow} \stackrel{*}{\longrightarrow}$ and $\stackrel{+}{\Longrightarrow} \; = \; \stackrel{*}{\longrightarrow} \stackrel{+}{\longrightarrow} \stackrel{*}{\longrightarrow}$

Definition

The relation $\rho \dashv^{\mathsf{A}} \sigma$ is coinductively defined by:

i) $\rho \parallel \sigma \stackrel{\tau}{\not\Longrightarrow}$ & $\rho \parallel \sigma \stackrel{+}{\not\Longrightarrow}$ then $\rho = \mathbf{1}$

ii) $\rho \parallel \sigma \stackrel{\tau}{\Longrightarrow} \rho' \parallel \sigma'$ then $\rho' \dashv^{\mathsf{A}} \sigma'$ for any ρ', σ'

iii) $\rho \parallel \sigma \stackrel{+}{\Longrightarrow}$ then $\rho \parallel \sigma \stackrel{+}{\Longrightarrow} \rho' \parallel \sigma'$ & $\rho' \dashv^{\mathsf{A}} \sigma'$ for some ρ', σ'

We have Buyer $\dashv^{\mathsf{A}}$ Seller

Lemma

If $\rho \dashv^{\mathsf{A}} \sigma$, then one of the following conditions holds:

i) $\rho = \mathbf{1}$;

ii) $\rho = \sum_{i\in I} \alpha_i.\rho_i$, $\sigma = \sum_{j\in J} \overline{\alpha}_j.\sigma_j$ *and* $\exists h \in I \cap J.\ \rho_h \dashv^{\mathsf{A}} \sigma_h$;

iii) $\rho = \bigoplus_{i\in I} \overline{a}_i.\rho_i$, $\sigma = \sum_{j\in J} a_j.\sigma_j$, $I \subseteq J$ *and* $\forall h \in I.\ \rho_h \dashv^{\mathsf{A}} \sigma_h$;

iv) $\rho = \sum_{i\in I} a_i.\rho_i$, $\sigma = \bigoplus_{j\in J} \overline{a}_j.\sigma_j$, $I \supseteq J$ *and* $\forall h \in J.\ \rho_h \dashv^{\mathsf{A}} \sigma_h$.

Theorem

The client ρ and server σ are compliant as retractable contracts if and only if $\rho \dashv^{\mathsf{A}} \sigma$.

Orchestrators

$$\mu ::= \langle \alpha, \overline{\alpha} \rangle \qquad \mu^+ ::= \langle \alpha, \overline{\alpha} \rangle^+$$

$$f ::= 1 \mid \mu^+.f \mid \mu_1.f_1 \vee \ldots \vee \mu_n.f_n \mid x \mid \mathsf{rec}\, x.f$$

$$\mu^+.f \overset{\mu^+}{\mapsto} f \qquad (\bigvee_{i \in I} \mu_i.f_i) \overset{\mu_k}{\mapsto} f_k$$

$$\frac{\rho \xrightarrow{\alpha} \rho' \quad f \overset{\langle \overline{\alpha}, \alpha \rangle}{\mapsto} f' \quad \sigma \xrightarrow{\overline{\alpha}} \sigma'}{\rho \parallel_f \sigma \xrightarrow{\tau} \rho' \parallel_f \sigma'} \qquad \frac{\rho \xrightarrow{a} \rho' \quad f \overset{\langle \overline{a}, a \rangle^+}{\mapsto} f' \quad \sigma \xrightarrow{\overline{a}^+} \sigma'}{\rho \parallel_f \sigma \xrightarrow{+} \rho' \parallel_{f'} \sigma'}$$

$$\text{Buyer} = \overline{\texttt{belt}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) + \overline{\texttt{bag}}.\texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}})$$

$$\text{Seller} = \texttt{belt}.\overline{\texttt{price}}.\texttt{cash} + \texttt{bag}.\overline{\texttt{price}}.(\texttt{card} + \texttt{cash}) + \cdots$$

$$f = \langle \texttt{bag}, \overline{\texttt{bag}} \rangle^{+}.f'$$

$$f' = \langle \overline{\texttt{price}}, \texttt{price} \rangle.f''$$

$$f'' = \langle \texttt{cash}, \overline{\texttt{cash}} \rangle \vee \langle \texttt{card}, \overline{\texttt{card}} \rangle$$

$$\text{Buyer} \parallel_f \text{Seller}$$

$$\xrightarrow{\langle \texttt{bag}, \overline{\texttt{bag}} \rangle^{+}} \quad \texttt{price}.(\overline{\texttt{card}} \oplus \overline{\texttt{cash}}) \parallel_{f'} \overline{\texttt{price}}.(\texttt{card} + \texttt{cash})$$

$$\xrightarrow{\langle \overline{\texttt{price}}, \texttt{price} \rangle} \quad \overline{\texttt{card}} \oplus \overline{\texttt{cash}} \parallel_{f''} \texttt{card} + \texttt{cash}$$

$$\xrightarrow{\langle \texttt{card}, \overline{\texttt{card}} \rangle} \quad \mathbf{1} \parallel_1 \mathbf{1}$$

or

$$\xrightarrow{\langle \texttt{cash}, \overline{\texttt{cash}} \rangle} \quad \mathbf{1} \parallel_1 \mathbf{1}$$

Let $\Longrightarrow \; = \; \rightarrow^* \left(\xrightarrow{\tau} \cup \xrightarrow{+} \right) \rightarrow^*$

Definition

i) $f : \rho \dashV^{\text{Orch}} \sigma$ if for any ρ' and σ', the following holds:

$$\rho \parallel_f \sigma \Longrightarrow^* \rho' \parallel_{f'} \sigma' \not\Longrightarrow \quad \text{implies} \quad \rho' = \mathbf{1}.$$

ii) $\rho \dashV^{\text{Orch}} \sigma$ if $\exists f.\; [\, f : \rho \dashV^{\text{Orch}} \sigma \,]$.

We have Buyer $\dashV^{\text{Orch}}$ Seller

The game $\mathcal{G}_{\rho \| \sigma}$

Players and moves:

- A moves are the unaffectible client's actions
- B moves are the unaffectible server's actions
- C moves are the affectible actions of both

$$\bigoplus_{i\in I}\bar{a}_i.\rho_i \,|||\, \tilde{\sigma} \xrightarrow{\text{A}:\bar{a}_k} [\bar{a}_k]\rho_k \,|||\, \tilde{\sigma} \qquad \Sigma_{i\in I} a_i.\rho_i \,|||\, [\bar{a}_k]\sigma \xrightarrow{\text{A}:a_k} \rho_k \,|||\, \sigma$$

$$\tilde{\rho} \,|||\, \bigoplus_{i\in I}\bar{a}_i.\sigma_i \xrightarrow{\text{B}:\bar{a}_k} \tilde{\rho} \,|||\, [\bar{a}_k]\sigma_k \qquad [\bar{a}_k]\rho \,|||\, \Sigma_{i\in I} a_i.\sigma_i \xrightarrow{\text{B}:a_k} \rho \,|||\, \sigma_k$$

$$\bar{a}.\rho + \rho' \,|||\, a.\sigma + \sigma' \xrightarrow{\text{C}:a} \rho \,|||\, \sigma \qquad a.\rho + \rho' \,|||\, \bar{a}.\sigma + \sigma' \xrightarrow{\text{C}:a} \rho \,|||\, \sigma$$

$$\mathbf{1} \,|||\, \tilde{\rho} \xrightarrow{\text{C}:\checkmark} \mathbf{0} \,|||\, \tilde{\rho}$$

The game $\mathcal{G}_{\rho \| \sigma}$

Positions, plays and payoff:

- a *position* is finite sequence $\langle A_1{:}\alpha_1, \ldots, A_n{:}\alpha_n \rangle$ s.t.

$$\rho \,|||\, \sigma \xrightarrow{A_1:\alpha_1} \cdots \xrightarrow{A_n:\alpha_n} \rho' \,|||\, \sigma'$$

- a *play* is a maximal sequence $\boldsymbol{s} = \langle A_1{:}\alpha_1, A_2{:}\alpha_2, \ldots \rangle$ s.t. all its finite prefixes are positions
- a *payoff* is a partial function Φ of players and positions or infinite plays s.t.

$$\Phi A\, \boldsymbol{s} = \begin{cases} 1 & \text{if either } \boldsymbol{s} \text{ is infinite or it ends by the move } A{:}\checkmark \\ -1 & \text{otherwise} \end{cases}$$

Then $\mathcal{G}_{\rho \| \sigma} = (\rho \,\|\, \sigma, \Phi)$ is a *game*.

The game $\mathcal{G}_{\rho\|\sigma}$

Strategies:

- a *strategy* for player A is a map Σ from plays to finite sets of moves of A s.t.

$$\forall A{:}\alpha \in \Sigma(\boldsymbol{s}).\ \boldsymbol{s}\, A{:}\alpha \text{ is a play}$$

- a play $\boldsymbol{s} = \langle A_1{:}\alpha_1, A_2{:}\alpha_2, \ldots \rangle$ *conforms* to a strategy Σ for A if

$$\forall i.\ A_i{:}\alpha_i \in \boldsymbol{s} \ \& \ A = A_i \ \Rightarrow\ A_i{:}\alpha_i \in \Sigma(\langle A_1{:}\alpha_1, \ldots, A_{i-1}{:}\alpha_{i-1} \rangle)$$

- Σ is a *winning strategy* for A if it is a strategy for A s.t.

$$\forall \text{ play } \boldsymbol{s}.\ \boldsymbol{s} \text{ conforms to } \Sigma \ \Rightarrow\ \Phi\, A\, \boldsymbol{s} = 1$$

Example:

$$
\begin{aligned}
\boldsymbol{s}_1 &= (\mathsf{C}{:}\mathtt{bag})(\mathsf{B}{:}\overline{\mathtt{price}})(\mathsf{A}{:}\mathtt{price})(\mathsf{A}{:}\overline{\mathtt{cash}})(\mathsf{B}{:}\mathtt{cash})(\mathsf{C},\checkmark) \\
\boldsymbol{s}_2 &= (\mathsf{A}{:}\mathtt{bag})(\mathsf{C}{:}\overline{\mathtt{price}}) \\
\boldsymbol{s}_3 &= (\mathsf{C}{:}\mathtt{bag})(\mathsf{B}{:}\overline{\mathtt{price}})(\mathsf{A}{:}\mathtt{price})(\mathsf{A}{:}\overline{\mathtt{cash}})(\mathsf{B}{:}\mathtt{cash})
\end{aligned}
$$

$$\Phi\,\mathsf{C}\,\boldsymbol{s}_1 = 1, \qquad \Phi\,\mathsf{A}\,\boldsymbol{s}_1 = -1, \qquad \Phi\,\mathsf{B}\,\boldsymbol{s}_2 = -1, \qquad \Phi\,\mathsf{C}\,\boldsymbol{s}_3 = -1$$

$$\Sigma(\boldsymbol{s}) = \begin{cases} \{\, (\mathsf{C}{:}\mathtt{bag}) \,\} & \text{if } \boldsymbol{s} = \langle\rangle \\ \{\, (\mathsf{C},\checkmark) \,\} & \text{if } \boldsymbol{s} = \boldsymbol{s}_3 \\ \emptyset & \text{otherwise} \end{cases}$$

Σ is winning for C.

Theorem

$\rho \dashv^{\mathsf{A}} \sigma$, namely they are affectible compliant, if and only if there exists a winning strategy Σ for player C in the game $\mathcal{G}_{\rho\|\sigma}$

Theorem (Main theorem)

Given a client ρ and a server σ the following are equivalent:

i) *$\rho \dashv\!\|^{\mathsf{Orch}} \sigma$, namely they are compliant up to an orchestrator item*
$\rho \dashv^{\mathsf{rbk}} \sigma$, namely they are retractably compliant

Proof idea: if $f : \rho \dashv\!\|^{\mathsf{Orch}} \sigma$ then we can encode f into a winning strategy Σ_f for player C.
Vice versa if $\rho \dashv^{\mathsf{rbk}} \sigma$, then $\rho \dashv^{\mathsf{A}} \sigma$ so that there exists a winning strategy Σ for player C; then there exists a strategy Σ' refining Σ that is *univocal*:

$$\forall \text{ position } \boldsymbol{s}.\ \Sigma'(\boldsymbol{s}) \subseteq \Sigma(\boldsymbol{s}) \ \ \& \ \ |\Sigma'(\boldsymbol{s})| \leq 1$$

Now univocal strategies can be encoded into orchestrators.

Further results:

1. there is a sound and complete deduction system inferring whether $\rho \dashv^{\mathsf{A}} \sigma$, which is decidable
2. there exists an algorithm **Synth** computing an orchestrator f and therefore a strategy Σ_f such that $f : \rho \dashv\!\!|^{\mathsf{Orch}} \sigma$ if any, and rejecting otherwise

- Games are a natural setting to model interaction among (multiple) interacting agents and its properties
- There exists a natural relation between orchestrators and strategies in game theoretic models that reminds Abramsky's ideas on using process algebra to represent strategies
- Games on event structures are multiplayer so that moving to multiparty sessions instead of binary should be possible and natural