The Approximation Theorem for the $\Lambda\mu$-Calculus

Ugo de'Liguoro

ICTCS'13 - September 2013, Palermo

- Parigot's $\lambda\mu$-calculus (1992) is an extension of the λ-calculus to compute with classical proofs
- However there are some defects:
  - (de Groote 1994) the type $\neg\neg A \to A$ is not inhabited by a closed term; in particular Felleisen's operator $\mathcal{C}$, that Griffin proved to be typeable by $\neg\neg A \to A$, is undefinable in $\lambda\mu$
  - (Py 1998): adding η-rule makes reduction in $\lambda\mu$ non confluent
  - (Py, David 2001): Böhm theorem on separability of distinct $\beta\eta$-normal forms does not extend to normal forms in $\lambda\mu$
- The $\Lambda\mu$-calculus (de Groote-Saurin) is an extension of $\lambda\mu$ that preserves fundamental properties of λ, like reduction confluence, standardization and Böhm theorem (Saurin 2010)

The $\Lambda\mu$-calculus (de Groote - Saurin)

Terms:

$$M, N ::= x \mid MN \mid \lambda x.M \mid M\alpha \mid \mu\alpha.M$$

The reduction $M \longrightarrow_{\Lambda\mu} N$ is the compatible closure of the axioms:

$$
\begin{array}{llcll}
(\beta_T): & (\lambda x.M)N & \longrightarrow & M[x := N] & \\
(\eta_T): & \lambda x.Mx & \longrightarrow & M & \text{if } x \notin fv(M) \\
(\beta_S): & (\mu\alpha.M)\beta & \longrightarrow & M[\alpha := \beta] & \\
(\eta_S): & \mu\alpha.M\alpha & \longrightarrow & M & \text{if } \alpha \notin fv(M) \\
(fst): & \mu\alpha.M & \longrightarrow & \lambda x.\mu\alpha.M[P\alpha := (Px)\alpha] & \text{if } x \notin fv(M)
\end{array}
$$

where

$$M[P\alpha := (PN)\alpha] \equiv \text{replacement of any } P\alpha \text{ by } (PN)\alpha \text{ in } M$$

The μ-rule

Parigot's rule:

$$(\mu):\ (\mu\alpha.M)N \longrightarrow \mu\alpha.M[P\alpha := (PN)\alpha]$$

it has critical pairs w.r.t. the η_T-rule:

$$\mu\alpha.x \ {}_{\eta_T}\!\!\longleftarrow \lambda y.(\mu\alpha.x)y \longrightarrow_\mu \lambda y.\mu\alpha.(x[P\alpha := (Py)\alpha]) \equiv \lambda y.\mu\alpha.x$$

The problem is fixed by rule (fst):

$$\mu\alpha.x \longrightarrow_{fst} \lambda y.\mu\alpha.x$$

Now the μ-rule is derivable:

$$\begin{array}{rcll} (\mu\alpha.M)N & \longrightarrow_{fst} & (\lambda x.\mu\alpha.M[P\alpha := (Px)\alpha])N & \\ & \longrightarrow_{\beta_T} & (\mu\alpha.M[P\alpha := (Px)\alpha])[x := N] & \\ & \equiv & \mu\alpha.M[P\alpha := (PN)\alpha] & \text{since } x \notin fv(M) \end{array}$$

Rule *fst* causes problems

Any term beginning by a μ-abstraction has no normal form:

$$\begin{array}{rcl} \mu\alpha.M & \longrightarrow & \lambda x_0 \mu\alpha.M[P\alpha := (Px_0)\alpha] \\ & \longrightarrow & \lambda x_0 x_1 \mu\alpha.M[P\alpha := (Px_0)\alpha][P\alpha := (Px_1)\alpha] \\ & \equiv & \lambda x_0 x_1 \mu\alpha.M[P\alpha := (Px_0 x_1)\alpha] \\ & \longrightarrow & \cdots \end{array}$$

This adds to difficulties caused by the structural substitution ...

Is there a more abstract theory of $\Lambda\mu$?

Streams (syntax)

A stream is an applicative context of the shape ($k \geqslant 0$):

$$S[\ ] \equiv [\ ]N_1 \dots N_k\beta$$

Putting $\lambda x.M$ in the hole of $S[\ ]$ (for $k > 0$) results into the reduction:

$$(\lambda x.M)N_1N_2 \dots N_k\beta \longrightarrow M[x := N_1]N_2 \dots N_k\beta$$

Instead, putting $\mu\alpha.M$ in $S[\ ]$ results into the reduction:

$$\begin{array}{ll}
 & (\mu\alpha.M)N_1N_2N_3 \dots N_k\beta \\
\xrightarrow{*} & (\mu\alpha.M[P\alpha := (PN_1)\alpha])N_2N_3 \dots N_k\beta \\
\xrightarrow{*} & (\mu\alpha.M[P\alpha := (PN_1)\alpha][P\alpha := (PN_2)\alpha])N_3 \dots N_k\beta \\
\dots & \\
\xrightarrow{*} & (\mu\alpha.M[P\alpha := (PN_1)\alpha] \dots [P\alpha := (PN_k)\alpha])\beta \\
\longrightarrow & (M[P\alpha := (PN_1)\alpha] \dots [P\alpha := (PN_k)\alpha])[\alpha := \beta] \\
\equiv & (M[\alpha := \beta])[P\beta := (PN_1)\beta] \dots [P\beta := (PN_k)\beta]
\end{array}$$

Continuation Models (Streicher, Reus, Nakazawa, Katsumata)

- Streicher and Reus (1998) have proposed a model of $\lambda\mu$ based on the solution of the domain equations:

$$D = S \to R \qquad\qquad S = D \times S$$

(*R* is a parametric domain of "results"); however the interpretation map $[\![\cdot]\!]$ doesn't extend to a model of $\Lambda\mu$

- Nakazawa and Katsumata (2012) have considered the equations

$$D = S \to D \qquad\qquad S = D \times S$$

(where *D* itself is the domain of results) such that the interpretation $[\![M]\!]\, e \in D$ *gives a model of* $\Lambda\mu$

Models of $\Lambda\mu$

A stream model (called "extensional models of untyped $\Lambda\mu$-calculus") is a pair (D, S) that are solutions of the domain equations:

$$D = [S \to D], \qquad S = D \times S.$$

Then for all environment e we define $[\![M]\!]\,e \in D = [S \to D]$ by:

$$\begin{array}{lcll}
[\![x]\!]\,e\,s & = & e(x) & (d \in D, s \in S) \\
[\![\lambda x.M]\!]\,e\,(d :: s) & = & [\![M]\!]\,e[x \mapsto d]\,s & \\
[\![MN]\!]\,e\,s & = & [\![M]\!]\,e\,(([\![N]\!]\,e) :: s) & \\
[\![\mu\alpha.M]\!]\,e\,s & = & [\![M]\!]\,e[\alpha \mapsto s] & \\
[\![M\alpha]\!]\,e\,s & = & [\![M]\!]\,e\,e(\alpha) &
\end{array}$$

where $e(x) \in D$ and $e(\alpha) \in S$.

Remark (Berardi)

Let us call D^ω the denumerable product $D \times D \times \cdots$. Then for any R we have

$$D \simeq R^{D^\omega} \Rightarrow D \simeq D^{D^\omega}$$

In fact $D^\omega \simeq D^\omega \times D^\omega$ e.g. by the isomorphism

$$(d_0, d_1, d_2, d_3, \ldots) \mapsto \langle (d_0, d_2, \ldots), (d_1, d_3, \ldots) \rangle$$

whose inverse is

$$\langle (d_0, d_1, \ldots), (d'_0, d'_1, \ldots) \rangle \mapsto (d_0, d'_0, d_1, d'_1 \ldots)$$

Therefore

$$D \simeq R^{D^\omega} \simeq R^{D^\omega \times D^\omega} \simeq (R^{D^\omega})^{D^\omega} \simeq D^{D^\omega}$$

We conclude that any model a la Streicher-Reus can be seen (though not in a canonical way) as a model a la Nakazawa-Katsumata (and vice versa: take $R = D$).

Soundness Theorem (Nakazawa, Katsumata)

If $=_{\Lambda\mu}$ is the convertibility relation generated by $\longrightarrow_{\Lambda\mu}$ then

$$M =_{\Lambda\mu} N \quad \Rightarrow \quad \forall e \in \text{Env. } [\![M]\!]\, e = [\![N]\!]\, e$$

But this is a result about the equational theory of $=_{\Lambda\mu}$, not about reduction:

Question

How do Nakazawa-Katsumata models relate to reduction in $\Lambda\mu$?

Our answer: the Approximation Theorem.

Approximate Normal Forms

Like for the λ-calculus we can define the "stable" part of a term under reduction as an approximate normal form in *ANF*, defined by the grammar:

$$A ::= \Omega \mid \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \ldots \mu\alpha_n\lambda\vec{x}_n.y\vec{A}_0\beta_1\vec{A}_1 \ldots \beta_m\vec{A}_m$$

Any $\Lambda\mu$-term has the shape

$$M \equiv \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \ldots \mu\alpha_n\lambda\vec{x}_n.R\vec{M}_0\beta_1\vec{M}_1 \ldots \beta_m\vec{M}_m$$

and we can define a mapping $\varphi : \Lambda\mu\text{-terms} \to ANF$ by:

$$\phi(M) = \begin{cases} \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \ldots \mu\alpha_n\lambda\vec{x}_n.y\phi(\vec{M}_0)\beta_1\phi(\vec{M}_1) \ldots \beta_m\phi(\vec{M}_m) & \text{if } R \equiv y \\ \Omega & \text{else} \end{cases}$$

The Approximants of a Term (1)

Further we define a precongruence $\preceq$ over *ANF* such that:

$$\Omega \preceq A, \qquad \text{and} \qquad \mu\alpha.A \preceq \lambda x.\mu\alpha.A[P\alpha := (Px)\alpha] \;\; (x \notin fv(A)).$$

Define the set of *approximants* of M:

$$\mathcal{A}(M) = \{A \in ANF \mid \exists N.\ M \xrightarrow{*} N \;\&\; A \preceq \phi(N)\}$$

Proposition

1. $M \xrightarrow{*} N \;\Rightarrow\; \phi(M) \preceq \phi(N)$
2. if M is closed then $\mathcal{A}(M)$ is an ideal.

The Approximants of a Term (2)

Extend the interpretation mapping to ANF by $[\![\Omega]\!]^D e = \bot$, then

Proposition

1. $A \preceq A' \Rightarrow \forall e \in \text{Env}.\ [\![A]\!]^D e \sqsubseteq [\![A']\!]^D e$
2. $A \in \mathcal{A}(M) \Rightarrow \forall e \in \text{Env}.\ [\![A]\!]^D e \sqsubseteq [\![M]\!]^D e$
3. if $\bigsqcup_{A \in \mathcal{A}(M)} [\![A]\!]^D e$ exists in D then

$$\bigsqcup_{A \in \mathcal{A}(M)} [\![A]\!]^D e \sqsubseteq [\![M]\!]^D e$$

Roadmap to the proof of the Approximation Theorem

- One way to estabilsh the approximation theorem for the λ-calculus is to prove for a suitable intersection type assignment system:

$$\Gamma \vdash M : \sigma \;\Leftrightarrow\; \exists A \in \mathcal{A}(M).\, \Gamma \vdash A : \sigma$$

- No intersection type system for $\Lambda\mu$ has been known so far, but there is one for Parigot's $\lambda\mu$-calculus, proposed in van Bakel, Barbanera, de'Liguoro [TLCA 2011]
- Proof idea: find a system for $\Lambda\mu$ satisfying the Filter-Model Theorem, and use the system to prove a similar statement as the above.

Intersection Types for Streams

Let (D, S) be a stream model in the category of ω-algebraic lattices

$$\begin{array}{lcll} \mathcal{L}_T : & \delta & ::= & \varphi \mid \sigma \to \delta \mid \delta \wedge \delta \mid \omega_T \\ \mathcal{L}_S : & \sigma & ::= & \delta \times \sigma \mid \sigma \wedge \sigma \mid \omega_S \end{array}$$

whose intenterpretations are $[\![\delta]\!]^D \subseteq D$ and $[\![\sigma]\!]^S \subseteq S$:

$$\begin{array}{rcll} [\![\varphi]\!]^D & \subseteq & D & \text{fixed for all } \varphi \\ [\![\sigma \to \delta]\!]^D & = & \{d \in D \mid \forall s \in [\![\sigma]\!]^S.\ d(s) \in [\![\delta]\!]^D\} & \\ [\![\delta \times \sigma]\!]^S & = & \{(d :: s) \in S \mid d \in [\![\delta]\!]^D \ \&\ s \in [\![\sigma]\!]^S\} & \end{array}$$

plus

$$\begin{array}{ll} [\![\omega_T]\!]^D = D, & [\![\delta_1 \wedge \delta_2]\!]^D = [\![\delta_1]\!]^D \cap [\![\delta_2]\!]^D \\ [\![\omega_S]\!]^S = S, & [\![\sigma_1 \wedge \sigma_2]\!]^S = [\![\sigma_1]\!]^S \cap [\![\sigma_2]\!]^S \end{array}$$

Intersection Type Assignment for $\Lambda\mu$ (1)

Bases Γ and contexts Δ are defined:

$$\Gamma ::= \varnothing \mid x : \delta, \Gamma \qquad \Delta ::= \varnothing \mid \alpha : \sigma, \Delta$$

Then define the type assignment rules:

$$\frac{}{\Gamma, x : \delta \vdash x : \delta \mid \Delta}\ (\mathrm{Ax})$$

$$\frac{\Gamma, x : \delta_1 \vdash M : \sigma \to \delta_2 \mid \Delta}{\Gamma \vdash \lambda x.M : \delta_1 \times \sigma \to \delta_2 \mid \Delta}\ (\lambda) \qquad \frac{\Gamma \vdash M : \delta_1 \times \sigma \to \delta_2 \mid \Delta \quad \Gamma \vdash N : \delta_1 \mid \Delta}{\Gamma \vdash MN : \sigma \to \delta_2 \mid \Delta}\ (\mathrm{TApp})$$

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \sigma, \Delta}{\Gamma \vdash \mu\alpha.M : \sigma \to \delta \mid \Delta}\ (\mu) \qquad \frac{\Gamma \vdash M : \sigma \to \delta \mid \alpha : \sigma, \Delta}{\Gamma \vdash M\alpha : \delta \mid \alpha : \sigma, \Delta}\ (\mathrm{SApp})$$

Intersection Type Assignment for $\Lambda\mu$ (2)

We can introduce the preorders $(\mathcal{L}_T, \leqslant_T)$ and $(\mathcal{L}_S, \leqslant_S)$ such that:

1. $\wedge$ is the meet, ω_T and ω_S are the tops
2. $\rightarrow$ is axiomatised as in an EATS, plus $\varphi = \omega_S \rightarrow \varphi$, $\omega_T = \omega_S \rightarrow \omega_T$
3. $\times$ is ordered componentwise plus $\omega_T \times \omega_S = \omega_S$

Then we add to the system the rules:

$$\frac{}{\Gamma \vdash M : \omega_T \mid \Delta}\ (\omega)$$

$$\frac{\Gamma \vdash M : \delta_1 \mid \Delta \quad \Gamma \vdash M : \delta_2 \mid \Delta}{\Gamma \vdash M : \delta_1 \wedge \delta_2 \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash M : \delta_1 \mid \Delta \quad \delta_1 \leqslant_T \delta_2}{\Gamma \vdash M : \delta_2 \mid \Delta}\ (\leqslant)$$

The Filter-Model construction

A *filter* over $\mathcal{L}_T$ is a subset $F \subseteq \mathcal{L}_T$ s.t.

1. $\omega_T \in F$
2. $\delta \in F$ & $\delta \leqslant_T \delta' \Rightarrow \delta' \in F$
3. $\delta, \delta' \in F \Rightarrow \delta \wedge \delta' \in F$

The set $\mathcal{F}_T$ of filters over $\mathcal{L}_T$, ordered by $\subseteq$ is an ω-algebraic lattice.
Similarly the set $\mathcal{F}_S$ of filters over $\mathcal{L}_S$ (w.r.t. $\leqslant_S$) is an ω-algebraic lattice.
Moreover $(\mathcal{F}_T, \mathcal{F}_S)$ is a stream model:

Proposition

1. $\mathcal{F}_T \simeq [\mathcal{F}_S \to \mathcal{F}_T]$ and $\mathcal{F}_S \simeq \mathcal{F}_T \times \mathcal{F}_S$
2. any stream model (D, S) in the category of ω-algebraic lattices is isomorphic to some $(\mathcal{F}_T, \mathcal{F}_S)$.

The Filter-Model Theorem

Let $e \models \Gamma, \Delta$ iff $x : \delta \in \Gamma$ implies $e(x) \in [\![\delta]\!]^{D}$ and $\alpha : \sigma \in \Delta$ implies $e(\alpha) \in [\![\sigma]\!]^{S}$; we define:

$$\Gamma \models M : \delta \mid \Delta \Leftrightarrow \forall e \in \text{Env}.\ e \models \Gamma, \Delta \Rightarrow [\![M]\!]^{D} e \in [\![\delta]\!]^{D}$$

Theorem: Filter-Model

$$[\![M]\!]^{\mathcal{F}_T} e = \{\delta \in \mathcal{L}_T \mid \exists\, \Gamma, \Delta.\ e \models \Gamma, \Delta \ \ \&\ \ \Gamma \vdash M : \delta \mid \Delta\}$$

The Approximation Theorem

Lemma (Approximation Theorem for Deductions)

$$\Gamma \vdash M : \delta \mid \Delta \Leftrightarrow \exists A \in \mathcal{A}(M).\ \Gamma \vdash A : \delta \mid \Delta$$

By the Filter-Model Theorem we have:

$$\begin{aligned}
\delta \in [\![M]\!]^{\mathcal{F}_\mathcal{T}} e & \Leftrightarrow & \exists \Gamma, \Delta.\ e \models \Gamma, \Delta \ \&\ \Gamma \vdash M : \delta \mid \Delta \\
& \Leftrightarrow & \exists \Gamma, \Delta.\ e \models \Gamma, \Delta \ \&\ \exists A \in \mathcal{A}(M).\ \Gamma \vdash A : \delta \mid \Delta \\
& \Leftrightarrow & \exists A \in \mathcal{A}(M).\ \delta \in [\![A]\!]^{\mathcal{F}_\mathcal{T}} e
\end{aligned}$$

and we conclude that

Approximation Theorem for ω-algebraic Lattices

$$[\![M]\!]^{\mathcal{F}_\mathcal{T}} e = \bigcup_{A \in \mathcal{A}(M)} [\![A]\!]^{\mathcal{F}_\mathcal{T}} e = \bigsqcup_{A \in \mathcal{A}(M)} [\![A]\!]^{\mathcal{F}_\mathcal{T}} e$$

Conclusions and future work

1. We have established the approximation theorem for the $\Lambda\mu$-calculus w.r.t. stream models in the category of ω-algebraic lattices
2. this result should extend to any stream model in a category of algebraic domains with a countable basis, admitting a logical description in the sense of Abramsky
3. stream models are of interest on their own in the study of control operators and effects, that are central in the constructive analysis of classical proofs: we hope that intersection type systems like ours will be a useful tool in the investigation of sophisticated control structures like delimited continuations.

Reference

Ugo de'Liguoro,
"The Approximation Theorem for the $\Lambda\mu$-Calculus",
to appear in *MSCS*;
draft version:
`http://www.di.unito.it/~deligu/papers/ApproxLM.pdf`