Characterisation of Strongly Normalising $\lambda\mu$-Terms

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Introduction

- Parigot's $\lambda\mu$-calculus is an extension of the pure λ-calculus to compute with classical proofs
- Parigot proved that all typeable $\lambda\mu$-terms (both in first and second order logic) are strongly normalizing, but the larger set of *SN* terms in his calculus was not considered
- based on the domain-theoretic model by Streicher and Reus, we have proposed an intersection type assignment system which is invariant under reduction and expansion (TLCA'11)
- we prove here that Pottinger's characterisation of *SN* λ-terms extends to $\lambda\mu$ by suitably restricting our system

Parigot's $\lambda\mu$-calculus

$\lambda\mu$ syntax:

$$\begin{array}{rcll} M, N & ::= & x \mid \lambda x.M \mid MN \mid \mu\alpha.Q & \text{(terms)} \\ Q & ::= & [\alpha]M & \text{(commands)} \end{array}$$

Structural substitution:

$$T[\alpha \Leftarrow L] \equiv \text{the replacement of all } [\alpha]N \text{ by } [\alpha]NL \text{ in } T$$

Reduction:

$$\begin{array}{llcl} (\beta): & (\lambda x.M)N & \longrightarrow & M[N/x] \\ (\mu): & (\mu\beta.Q)N & \longrightarrow & \mu\beta.Q[\beta \Leftarrow N] \end{array}$$

Example (where $\alpha \notin M, N$):

$$(\mu\alpha.[\alpha](M(\mu\beta.[\alpha]N)))L \longrightarrow \mu\alpha.[\alpha](M(\mu\beta.[\alpha]NL)L)$$

Continuation Semantics (Streicher-Reus)

Continuation domain equations

$$\begin{cases} R & & & \textit{results} \\ D & = & [C \to R] & \textit{denotations} \\ C & = & (D \times C) & \textit{continuations} \end{cases}$$

If (R, D, C) are a solution then D is an extensional λ-model:

$$D \simeq [C \to R] \simeq [(D \times C) \to R] \simeq [D \to [C \to R]] \simeq [D \to D]$$

while C is the infinite product:

$$C \simeq D \times D \times \cdots$$

Term Interpretation

$$\text{Env} = (Var \to D) + (Name \to C)$$

$$[\![\cdot]\!]^D : \text{Trm} \to \text{Env} \to D$$

$$[\![\cdot]\!]^C : \text{Cmd} \to \text{Env} \to C$$

$$\begin{array}{rcll}
[\![x]\!]^D\, e\, k & = & e\, x\, k & \\
[\![\lambda x.M]\!]^D\, e\, k & = & [\![M]\!]^D\, e[x := d]\, k' & \langle d, k' \rangle = k \\
[\![MN]\!]^D\, e\, k & = & [\![M]\!]^D\, e\, \langle [\![N]\!]^D e, k \rangle & \\
[\![\mu\alpha.Q]\!]^D\, e\, k & = & d\, k' & \langle d, k' \rangle = [\![Q]\!]^C\, e[\alpha := k] \\
[\![[\alpha]M]\!]^C\, e & = & \langle [\![M]\!]^D\, e, e\, \alpha \rangle &
\end{array}$$

where $e \in \text{Env}$, $d \in D$ and $k, k' \in C$.

Types and type inclusion

$$\mathcal{L}_R : \quad \rho ::= \nu \mid \rho \wedge \rho \mid \omega$$

$$\mathcal{L}_C : \quad \kappa ::= \delta \times \kappa \mid \kappa \wedge \kappa \mid \omega$$

$$\mathcal{L}_D : \quad \delta ::= \rho \mid \kappa \to \rho \mid \delta \wedge \delta \mid \omega$$

Some axioms for the preorders $\leq_C$ and $\leq_D$ ($=$ is $\leq \cap \leq^{-1}$):

$$\omega \times \omega =_C \omega, \qquad \omega \to \omega =_D \omega, \qquad \rho =_D \omega \to \rho$$

Filter Domains solving Continuation Equations

Let $\mathcal{F}^A = \mathrm{Filt}(\mathcal{L}_A / \leq_A)$ for $A = R, D, C$:

$$\mathcal{F}^D \simeq [\mathcal{F}^C \to \mathcal{F}^R] \quad \text{and} \quad \mathcal{F}^C \simeq \mathcal{F}^D \times \mathcal{F}^C$$

Typing judgements

Typing judgements are triples of a *basis*, a *term/command judgement* and a *context*:

- *basis*: $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$, with $x_i \in \mathit{Var}, \delta_i \in \mathcal{L}_D$
- *judgement*:
  - *term*: $M : \delta$ with $M \in \mathrm{Trm}, \delta \in \mathcal{L}_D$
  - *command*: $Q : \kappa$ with $Q \in \mathrm{Cmd}, \kappa \in \mathcal{L}_C$
- *context*: $\Delta = \{\alpha_1 : \kappa_1, \ldots, \alpha_m : \kappa_m\}$, with $\alpha_i \in \mathit{Name}, \kappa_i \in \mathcal{L}_C$

A *typing judgement* has either forms:

$$\Gamma \vdash M : \delta \mid \Delta \quad \text{or} \quad \Gamma \vdash Q : \kappa \mid \Delta$$

The full type assignment system (TLCA'11)

$$\frac{}{\Gamma, x:\delta \vdash x:\delta \mid \Delta}\ (Var)$$

$$\frac{\Gamma, x:\delta \vdash M:\kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \mid \Delta}\ (\to I) \qquad \frac{\Gamma \vdash M:\delta \times \kappa \to \rho \mid \Delta \quad \Gamma \vdash N:\delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \rho \mid \Delta}\ (\to E)$$

$$\frac{\Gamma \vdash M:\delta \mid \alpha:\kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha:\kappa, \Delta}\ (\times) \qquad \frac{\Gamma \vdash Q:(\kappa' \to \rho) \times \kappa' \mid \alpha:\kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta}\ (\mu)$$

$$\frac{}{\Gamma \vdash M:\omega \mid \Delta}\ (\omega) \qquad \frac{\Gamma \vdash M:\delta \mid \Delta \quad \Gamma \vdash M:\delta' \mid \Delta}{\Gamma \vdash M:\delta \wedge \delta' \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash M:\delta \mid \Delta \quad \delta \leq \delta'}{\Gamma \vdash M:\delta' \mid \Delta}\ (\leq)$$

Pottinger's Theorem

Theorem (Pottinger)

A λ-term M is strongly normalisable if and only if there exist Γ, σ such that $\Gamma \vdash M : \sigma$ is derivable in an intersection type system without the type ω.

Can we get rid of ω in the system for $\lambda\mu$?

Of course we have to abandon rule:

$$\frac{}{\Gamma \vdash M : \omega \mid \Delta}\,(\omega)$$

but what about the types?

The meaning of ω

The type

$$\kappa = \delta_1 \times \cdots \times \delta_k \times \omega \in \mathcal{L}_C$$

is semantically inhabited by any infinite tuple:

$$\langle d_1, \ldots, d_k, d_{k+1}, \ldots \rangle \in C$$

such that $d_i \in [\![\delta_i]\!] \subseteq D$ for $i = 1, \ldots, k$.
We then restrict the occurrences of ω to the end of a product type κ and add the axiom:

$$\delta_1 \times \delta_2 \times \omega \leq \delta_1 \times \omega$$

By eliminating rule (ω) the meaning of ω changes:

"lack of information" $\rightsquigarrow$ "partial information about a total object"

The computability interpretation of types

$$\mathcal{L}_R^{-\omega}: \quad \rho ::= \nu \mid \rho \wedge \rho \mid \omega$$

$$\mathcal{L}_C^{-\omega}: \quad \kappa ::= \delta \times \omega \mid \delta \times \kappa \mid \kappa \wedge \kappa \mid \omega$$

$$\mathcal{L}_D^{-\omega}: \quad \delta ::= \rho \mid \omega \to \rho \mid \kappa \to \rho \mid \delta \wedge \delta \mid \omega$$

The computability interpretation of types

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$$\mathcal{L}_D^{-\omega} : \quad \delta ::= \rho \mid \omega \to \rho \mid \kappa \to \rho \mid \delta \wedge \delta$$

A *stack* is a tuple $L_1 :: \cdots :: L_k$ for some $k \in \mathbb{N}$. SN^* is the set of stacks of terms in SN.

$$\begin{array}{rcl} [\![\omega \to \rho]\!] = [\![\rho]\!] & = & SN \\ [\![\kappa \to \rho]\!] & = & \{M \in \mathrm{Trm} \mid \forall \vec{L} \in [\![\kappa]\!].\ M\vec{L} \in [\![\rho]\!]\} \\ [\![\delta \times \omega]\!] & = & \{N :: \vec{L} \mid N \in [\![\delta]\!], \vec{L} \in SN^*\} \\ [\![\delta \times \kappa]\!] & = & \{N :: \vec{L} \mid N \in [\![\delta]\!], \vec{L} \in [\![\kappa]\!]\} \\ [\![\sigma \wedge \tau]\!] & = & [\![\sigma]\!] \cap [\![\tau]\!] \end{array}$$

where $\vec{L}$ is a vector or a stack according to the context.

Types are saturated sets

Properties of the computability interpretation of types:

- $[\![\delta]\!] \subseteq SN$ and $[\![\kappa]\!] \subseteq SN^*$ for $\delta \in \mathcal{L}_D^{-\omega}$ and $\kappa \in \mathcal{L}_C^{-\omega}$
- $M[N/x]\vec{L} \in [\![\delta]\!] \ \& \ N \in [\![\delta']\!] \ \Rightarrow \ (\lambda x.M)N\vec{L} \in [\![\delta]\!]$
- $(\mu\alpha[\alpha](M[\alpha \Leftarrow N]N))\vec{L} \in [\![\delta]\!] \ \Rightarrow \ (\mu\alpha[\alpha]M)N\vec{L} \in [\![\delta]\!]$
- $(\mu\alpha[\beta](M[\alpha \Leftarrow N]))\vec{L} \in [\![\delta]\!] \ \& \ N \in [\![\delta']\!] \ \Rightarrow$ $(\mu\alpha[\beta]M)N\vec{L} \in [\![\delta]\!]$

  if $\alpha \neq \beta$
- $\sigma \leq \tau \ \Rightarrow \ [\![\sigma]\!] \subseteq [\![\tau]\!]$

Typing judgements for commands are problematic

Toward soundness, for $\xi \in (Var \to \mathrm{Trm}) + (Name \to \mathrm{Trm}^*)$ we set:

$$M_\xi = M\,[\xi(x_1)/x_1, \ldots, \xi(x_h)/x_h, [\alpha_1 \Leftarrow \xi(\alpha_1)], \ldots, [\alpha_k \Leftarrow \xi(\alpha_k)]]$$

We expect that if $\xi(x) \in [\![\Gamma(x)]\!]$ and $\xi(\alpha) \in [\![\Delta(\alpha)]\!]$ for all $x \in dom(\Gamma)$ and $\alpha \in dom(\Delta)$ then:

$$\Gamma \vdash M : \delta \mid \Delta \;\Rightarrow\; M_\xi \in [\![\delta]\!]$$

How should we interpret a statement like $[\alpha]M : \kappa$?

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha : \kappa, \Delta}\,(\times) \qquad \frac{\Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta}\,(\mu)$$

First solution: two derived rules

We replace the rules:

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha : \kappa, \Delta}\ (\times) \qquad \frac{\Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta}\ (\mu)$$

by the admissible rules:

$$\frac{\Gamma \vdash M : \kappa \to \rho \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : \kappa \to \rho \mid \Delta}\ (\mu_1) \qquad \frac{\Gamma \vdash M : \kappa' \to \rho \mid \alpha : \kappa, \beta : \kappa', \Delta}{\Gamma \vdash \mu\alpha.[\beta]M : \kappa \to \rho \mid \beta : \kappa', \Delta}\ (\mu_2)$$

First solution: the restricted system with (μ_1) and (μ_2)

$$\frac{}{\Gamma, x : \delta \vdash x : \delta \mid \Delta}\ (Var)$$

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \mid \Delta}\ (\to I) \qquad \frac{\Gamma \vdash M : \delta \times \kappa \to \rho \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \rho \mid \Delta}\ (\to E)$$

$$\frac{\Gamma \vdash M : \kappa \to \rho \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : \kappa \to \rho \mid \Delta}\ (\mu_1) \qquad \frac{\Gamma \vdash M : \kappa' \to \rho \mid \alpha : \kappa, \beta : \kappa', \Delta}{\Gamma \vdash \mu\alpha.[\beta]M : \kappa \to \rho \mid \beta : \kappa', \Delta}\ (\mu_2)$$

$$\frac{\Gamma \vdash M : \delta \mid \Delta \quad \Gamma \vdash M : \delta' \mid \Delta}{\Gamma \vdash M : \delta \wedge \delta' \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash M : \delta \mid \Delta \quad \delta \leq \delta'}{\Gamma \vdash M : \delta' \mid \Delta}\ (\leq)$$

where types are in $\mathcal{L}^{-\omega} = \mathcal{L}_R^{-\omega} \cup \mathcal{L}_D^{-\omega} \cup \mathcal{L}_C^{-\omega}$ and $\leq$ is restricted to $\mathcal{L}^{-\omega}$.

Second solution: distinct interpretation of judgments

For $\xi \in (Var \to \mathrm{Trm}) + (Name \to \mathrm{Trm}^*)$ we set $\xi \models \Gamma, \Delta$ iff

$$\forall x \in dom(\Gamma), \forall \alpha \in dom(\Delta).\ \xi(x) \in [\![\Gamma(x)]\!] \ \&\ \xi(\alpha) \in [\![\Delta(\alpha)]\!]$$

hence we define:

- $\Gamma \models M : \delta \mid \Delta \ \Leftrightarrow\ \forall \xi.\ \xi \models \Gamma, \Delta \Rightarrow M_\xi \in [\![\delta]\!]$
- $\Gamma \models [\alpha]M : \kappa \mid \Delta \ \Leftrightarrow\ \forall \xi.\ \xi \models \Gamma, \Delta \Rightarrow M_\xi :: \xi(\alpha) \in [\![\kappa]\!]$

where $N :: \vec{L}$ is just the stack obtained by pushing N in front of $\vec{L}$ and recall that

$$M_\xi = M\,[\xi(x_1)/x_1, \ldots, \xi(x_h)/x_h, [\alpha_1 \Leftarrow \xi(\alpha_1)], \ldots, [\alpha_k \Leftarrow \xi(\alpha_k)]]$$

Second solution: the restricted system with $(\times)$ and (μ)

$$\frac{}{\Gamma, x : \delta \vdash x : \delta \mid \Delta}\ (Var)$$

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \mid \Delta}\ (\to I) \qquad \frac{\Gamma \vdash M : \delta \times \kappa \to \rho \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \rho \mid \Delta}\ (\to E)$$

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha : \kappa, \Delta}\ (\times) \qquad \frac{\Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta}\ (\mu)$$

$$\frac{\Gamma \vdash M : \delta \mid \Delta \quad \Gamma \vdash M : \delta' \mid \Delta}{\Gamma \vdash M : \delta \wedge \delta' \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash M : \delta \mid \Delta \quad \delta \leq \delta'}{\Gamma \vdash M : \delta' \mid \Delta}\ (\leq)$$

where types are in $\mathcal{L}^{-\omega} = \mathcal{L}_R^{-\omega} \cup \mathcal{L}_D^{-\omega} \cup \mathcal{L}_C^{-\omega}$ and $\leq$ is restricted to $\mathcal{L}^{-\omega}$.

The characterisation theorem

Theorem: soundness

1. $\Gamma \vdash^{-\omega} M : \delta \mid \Delta \Rightarrow \Gamma \models M : \delta \mid \Delta$
2. $\Gamma \vdash^{-\omega} Q : \kappa \mid \Delta \Rightarrow \Gamma \models Q : \kappa \mid \Delta$

Therefore if $\Gamma \vdash^{-\omega} M : \delta \mid \Delta$ then $M \in SN$.

Proof. By simultaneous induction on derivation. Finally take ξ_0 s.t. $\xi_0(x) = x$ and $\xi_0(\alpha) = \vec{y}$. If $\Gamma \vdash^{-\omega} M : \delta \mid \Delta$ then:

- $\xi_0 \models \Gamma, \Delta$ so that $M_{\xi_0} \in [\![\delta]\!] \subseteq SN$
- $M_{\xi_0} \in SN \;\Rightarrow\; M \in SN$

Theorem: completeness

If $M \in SN$ then $\Gamma \vdash^{-\omega} M : \delta \mid \Delta$ for some Γ, δ and Δ.

Proof. Similar to the proof for the λ-calculus.

Final remarks

- we have a type characterisation of *SN*-terms of $\lambda\mu$-calculus, that should extend smoothly to De Groote-Saurin variant, and also to Fellaisen's $\lambda\mathcal{C}$-calculus
- we plan to compare the present proof with Berger's semantic proof of strong normalisation for extended λ-calculi
- we think that intersection type machinery can be used to obtain elegant analysis of continuations as well as of other non functional aspects of extended λ-calculi, often studied by means of e.g. effect systems