A Filter Model for the $\lambda\mu$-Calculus

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Motivations

- Continuations are a powerful concept, but make reasoning hard
- Prigot's $\lambda\mu$-calculus interprets classical proofs as functional programs plus (some kind of) continuations
- Curry style type assignment system have been successfully used in the study of the pure λ-calculus, and also of term rewriting, object calculi, etc.
- We take $\lambda\mu$ as a significant case study to investigate how Curry systems might be of help to reason about programming with continuations.

Previous work by: Dougherty, Ghilezan, Lescanne (on $\overline{\lambda}\mu\tilde{\mu}$) and van Bakel (on $\lambda\mu$).

Overview

1. we fix a notion of $\lambda\mu$-model after Streicher and Reus
2. we describe an instance of such a model as a filter-domain using intersection types
3. from the term interpretation map we reconstruct a type assignment system which is invariant under subject reduction *and expansion* and is sound and complete w.r.t. the given notion of model
4. we eventually show that simply typed $\lambda\mu$-terms (in Parigot's system) have non trivial types in our system
5. are SN $\lambda\mu$-terms characterizable in our system?

Syntax and reduction

$\lambda\mu$ syntax:

$$\begin{array}{rcll} M, N & ::= & x \mid \lambda x.M \mid MN \mid \mu\alpha.Q & \text{(terms)} \\ Q & ::= & [\alpha]M & \text{(commands)} \end{array}$$

Structural substitution:

$$T[\alpha \Leftarrow L] \equiv \text{the replacement of all } [\alpha]N \text{ by } [\alpha]NL \text{ in } T$$

Reduction:

$$\begin{array}{rrcll} (\beta): & (\lambda x.M)N & \longrightarrow & M[N/x] & \\ (\mu): & (\mu\beta.Q)N & \longrightarrow & \mu\beta.Q[\beta \Leftarrow N] & \\ (ren): & [\alpha]\mu\beta.Q & \longrightarrow & Q[\alpha/\beta] & \\ (\mu\eta): & \mu\alpha.[\alpha]M & \longrightarrow & M & \text{if } \alpha \notin fn(M) \end{array}$$

Continuation Semantics (Streicher-Reus)

Continuation domain equations

$$\begin{cases} R & & & \textit{results} \\ D & = & [C \to R] & \textit{denotations} \\ C & = & (D \times C) & \textit{continuations} \end{cases}$$

If (R, D, C) are a solution then D is an extensional λ-model:

$$D \simeq [C \to R] \simeq [(D \times C) \to R] \simeq [D \to [C \to R]] \simeq [D \to D]$$

while C is the infinite product:

$$C \simeq D \times D \times \cdots$$

Term Interpretation

$$\mathsf{Env} = (\mathit{Var} \to D) + (\mathit{Name} \to C)$$

$$[\![\cdot]\!]^D : \mathsf{Trm} \to \mathsf{Env} \to D$$

$$[\![\cdot]\!]^C : \mathsf{Cmd} \to \mathsf{Env} \to C$$

$$\begin{array}{rcll}
[\![x]\!]^D\, e\, k & = & e\, x\, k & \\
[\![\lambda x.M]\!]^D\, e\, k & = & [\![M]\!]^D\, e[x := d]\, k' & \langle d, k'\rangle = k \\
[\![MN]\!]^D\, e\, k & = & [\![M]\!]^D\, e\, \langle [\![N]\!]^D e, k\rangle & \\
[\![\mu\alpha.Q]\!]^D\, e\, k & = & d\, k' & \langle d, k'\rangle = [\![Q]\!]^C\, e[\alpha := k] \\
[\![[\alpha]M]\!]^C e & = & \langle [\![M]\!]^D\, e, k\rangle & k = e\,\alpha
\end{array}$$

where $e \in \mathsf{Env}$, $d \in D$ and $k, k' \in C$.

Properties of term Interpretation

- swapping continuations:

$$[\![\mu\alpha.[\beta]M]\!]^D\, e\, k = [\![M]\!]^D\, e[\alpha := k]\ (e[\alpha := k]\ \beta)$$

- structural substitution:

$$[\![M[\alpha \Leftarrow N]]\!]^D\, e\, k = [\![M]\!]^D\, e[\alpha := \langle [\![N]\!]^D e, e\,\alpha\rangle]\, k$$

Let $=_{\beta\mu}$ be the convertibility relation induced by $\longrightarrow_{\beta\mu}$:

Soundness of term interpretation

For all $M, N \in \mathrm{Trm}$ and model (R, D, C):

$$M =_{\beta\mu} N \Rightarrow [\![M]\!]^D = [\![N]\!]^D$$

Intersection Type Representation of ω-Algebraic Lattices

Given $A \in \omega\mathbf{ALG}$

$$\mathcal{L}_A : \quad \sigma, \tau ::= \cdots \mid \sigma \wedge \tau \mid \omega$$

$\leq_A$:		a preorder s.t.
	$\sigma \leq_A \omega$	ω is the top
	$\sigma \wedge \tau \leq \sigma, \tau$ $\rho \leq \sigma, \rho \leq \tau \Rightarrow \rho \leq \sigma \wedge \tau$	the meet of σ, τ

Let

$\Theta : \mathcal{L}_A \to \mathcal{K}(A)$ be surjective and s.t. $\sigma \leq_A \tau \Leftrightarrow \Theta(\sigma) \sqsupseteq \Theta(\tau)$

then

$$\mathcal{F}^A := \mathrm{Filt}(\mathcal{L}_A) \simeq \mathrm{Filt}(\mathcal{L}_A/_{\leq_A}) \simeq \mathrm{Filt}(\mathcal{K}^{op}(A)) \simeq A$$

via the continuous extension of $\uparrow\sigma \mapsto \uparrow\Theta(\sigma)$

Type Theory $\leq_R$

Fix $R \in \omega\mathbf{ALG}$:

$$\begin{array}{lll} \mathcal{L}_R : & \rho ::= \nu_a \mid \omega \mid \rho\wedge\rho & (a \in \mathcal{K}(R)) \\ \leq_R: & \nu_\perp =_R \omega & \\ & \nu_{a\sqcup b} =_R \nu_a\wedge\nu_b & \\ & b \sqsubseteq a \in \mathcal{K}(R) \Rightarrow \nu_a \leq_R \nu_b & \end{array}$$

Proposition

Define $\mathcal{F}^R = \mathrm{Filt}(\mathcal{L}_R/_{\leq_R})$; then $R \simeq \mathcal{F}^R$ because of $\mathcal{L}_R/_{\mathcal{L}_R} \simeq \mathcal{K}^{op}(R)$ via

$$\Theta(\nu_a) = a, \quad \Theta(\omega) = \perp, \quad \Theta(\rho\wedge\rho') = \Theta(\rho) \sqcup \Theta(\rho')$$

Type Theories $\leq_C$ and $\leq_D$

Languages after the approximants in the inverse limit construction:

$$\begin{array}{lcl} C_0 & = & \{\bot\} \\ D_n & = & [C_n \to R] \\ C_{n+1} & = & D_n \times C_n \\ \\ D & = & \lim_n D_n \\ C & = & \lim_n C_n \end{array} \quad \Longrightarrow \quad \begin{array}{lcl} \mathcal{L}_{C_0} & & \kappa ::= \omega \\ \mathcal{L}_{D_n} & & \delta ::= \rho \mid \kappa \to \rho \mid \delta \wedge \delta \mid \omega \\ \mathcal{L}_{C_{n+1}} & & \kappa ::= \delta \times \kappa \mid \kappa \wedge \kappa \mid \omega \\ \\ \mathcal{L}_D & = & \bigcup_n \mathcal{L}_{D_n} \\ \mathcal{L}_C & = & \bigcup_n \mathcal{L}_{C_n} \end{array}$$

or equivalently

$$\begin{array}{llc} \mathcal{L}_D : & \delta ::= & \rho \mid \kappa \to \rho \mid \omega \mid \delta \wedge \delta \\ \mathcal{L}_C : & \kappa ::= & \delta \times \kappa \mid \omega \mid \kappa \wedge \kappa \end{array}$$

Type Theories $\leq_C$ and $\leq_D$

Theory $\leq_C$:

$$\frac{}{\omega \leq_C \omega \times \omega} \qquad \frac{}{(\delta_1 \times \kappa_1)\wedge(\delta_2 \times \kappa_2) \leq_C (\delta_1 \wedge \kappa_2) \times (\kappa_1 \wedge \kappa_2)}$$

$$\frac{\delta_1 \leq_D \delta_2 \quad \kappa_1 \leq_C \kappa_2}{\delta_1 \times \kappa_1 \leq_C \delta_2 \times \kappa_2}$$

Theory $\leq_D$:

$$\frac{}{\omega \leq_D \omega \rightarrow \omega} \qquad \frac{}{\nu =_D \omega \rightarrow \nu} \qquad \frac{}{(\kappa \rightarrow \delta_1)\wedge(\kappa \rightarrow \delta_2) \leq_D \kappa \rightarrow (\kappa_1 \wedge \kappa_2)}$$

$$\frac{\kappa_2 \leq_C \kappa_1 \quad \delta_1 \leq_D \delta_2}{\kappa_1 \rightarrow \delta_1 \leq_D \kappa_2 \rightarrow \delta_2}$$

Solution of Continuation Equations

Filter Domains solving Continuation Equations

Let $\mathcal{F}^C = \mathrm{Filt}(\mathcal{L}_C/\leq_C)$ and $\mathcal{F}^D = \mathrm{Filt}(\mathcal{L}_D/\leq_D)$:

$$\mathcal{F}^D \simeq [\mathcal{F}^C \to \mathcal{F}^R] \quad \text{and} \quad \mathcal{F}^C \simeq \mathcal{F}^D \times \mathcal{F}^C$$

Via the mappings $F : \mathcal{F}^D \to [\mathcal{F}^C \to \mathcal{F}^R]$ and $G : [\mathcal{F}^C \to \mathcal{F}^R] \to \mathcal{F}^D$

$$\begin{array}{lcl} F\,d\,k & = & \{\rho \in \mathcal{L}_R \mid \exists \kappa \to \rho \in d.\, \kappa \in k]\} \\ G\,f & = & \{\bigwedge_{i\in I} \kappa_i \to \rho_i \in \mathcal{L}_D \mid \forall i \in I.\, \rho_i \in f(\uparrow \kappa_i)\} \end{array}$$

and the mappings $H : \mathcal{F}^C \to (\mathcal{F}^D \times \mathcal{F}^C)$ and $K : (\mathcal{F}^D \times \mathcal{F}^C) \to \mathcal{F}^C$

$$\begin{array}{lcl} H\,k & = & \langle \{\delta \in \mathcal{L}_D \mid \delta \times \kappa \in k\}, \{\kappa \in \mathcal{L}_D \mid \delta \times \kappa \in k\} \rangle \\ K\langle d, k\rangle & = & \{\delta \times \kappa \in \mathcal{L}_C \mid \delta \in d \;\&\; \kappa \in k\} \end{array}$$

Typing judgements

Typing judgements are triples of a *basis*, a *term/command judgement* and a *context*:

- *basis*: $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$, with $x_i \in Var, \delta_i \in \mathcal{L}_D$
- *judgement*:
  - *term*: $M : \delta$ with $M \in \mathrm{Trm}, \delta \in \mathcal{L}_D$
  - *command*: $Q : \kappa$ with $Q \in \mathrm{Cmd}, \kappa \in \mathcal{L}_C$
- *context*: $\Delta = \{\alpha_1 : \kappa_1, \ldots, \alpha_m : \kappa_m\}$, with $\alpha_i \in Name, \kappa_i \in \mathcal{L}_C$

A *typing judgement* has either forms:

$$\Gamma \vdash M : \delta \mid \Delta \quad \text{or} \quad \Gamma \vdash Q : \kappa \mid \Delta$$

Type interpretation

$$
\begin{array}{lcll}
[\![\nu_a]\!]^R & = & \{r \in R \mid a \sqsubseteq r\} & \\
[\![\delta \times \kappa]\!]^R & = & [\![\delta]\!]^D \times [\![\kappa]\!]^C & \\
[\![\kappa \to \rho]\!]^D & = & \{d \in D \mid \forall k \in [\![\kappa]\!].\ d(k) \in [\![\rho]\!]^R\} & \\
[\![\rho]\!]^D & = & [\![\omega \to \rho]\!]^D & \\
[\![\omega]\!]^A & = & A & \text{for } A = R, D, C \\
[\![\sigma \wedge \tau]\!]^A & = & [\![\sigma]\!]^A \cap [\![\tau]\!]^A &
\end{array}
$$

Let $\sigma, \tau \in \mathcal{L}_A$

Soundness of Type Interpretation

1. $[\![\sigma]\!]^A \subseteq A$ and $[\![\sigma]\!]^A = \uparrow\Theta(\sigma)$
2. $\sigma \leq_A \tau \Rightarrow [\![\sigma]\!]^A \subseteq [\![\tau]\!]^A$

Models and validity

A *model* is a triple $\mathcal{M} = (R, D, C)$ satisfying the continuation equations, together with the maps $[\![\cdot]\!]^R$, $[\![\cdot]\!]^D$ and $[\![\cdot]\!]^C$

Let $e \in \text{Env}$:

- $e \models \Gamma, \Delta \;\Leftrightarrow\; \forall x : \delta \in \Gamma.\, e(x) \in [\![\delta]\!]^D \;\&\; \forall \alpha : \kappa \in \Delta.\, e(\alpha) \in [\![\kappa]\!]^C$
- $\Gamma \models_{\mathcal{M}} M : \delta \mid \Delta \;\Leftrightarrow\; \forall e \in \text{Env}.\, e \models \Gamma, \Delta \Rightarrow [\![M]\!]^D e \in [\![\delta]\!]^D$ (and similarly $\mathcal{M} \models \Gamma \vdash Q : \kappa \mid \Delta$)
- $\Gamma \models M : \delta \mid \Delta$ $(\Gamma \models Q : \kappa \mid \Delta) \Leftrightarrow$
  $\forall \mathcal{M}.\, \Gamma \models_{\mathcal{M}} M : \delta \mid \Delta$ $(\Gamma \models_{\mathcal{M}} Q : \kappa \mid \Delta)$

Say that $\Gamma \vdash M : \delta \mid \Delta$ $(\Gamma \vdash Q : \kappa \mid \Delta)$ is *valid* if $\Gamma \models M : \delta \mid \Delta$ $(\Gamma \models Q : \kappa \mid \Delta)$

The filter model

Consider $\mathcal{F} = (\mathcal{F}^R, \mathcal{F}^D, \mathcal{F}^C)$ which is the model of filters isomorphic to a model $\mathcal{M} = (R, D, C)$:

Inspired to Abramsky's *exogenous logic*, we derive the inference rules for judgements $\Gamma \vdash M : \delta \mid \Delta$ $(\Gamma \vdash Q : \kappa \mid \Delta)$ from the clauses defining the maps $[\![\cdot]\!]^{\mathcal{F}^R}, [\![\cdot]\!]^{\mathcal{F}^D}$ and $[\![\cdot]\!]^{\mathcal{F}^C}$

Rules from interpretation in $\mathcal{F}$

$$\frac{\textit{defininens}\text{, right hand side of defining clause} = \text{premises}}{\textit{definiendum}\text{, left hand side of defining clause} = \text{conclusion}}$$

Observing that

$$[\![M]\!]^{\mathcal{F}^D} e \in [\![\delta]\!]^{\mathcal{F}^D} \quad \Leftrightarrow \quad \delta \in [\![M]\!]^{\mathcal{F}^D} e$$

(similarly for $[\![Q]\!]^{\mathcal{F}^C} e \in [\![\kappa]\!]^{\mathcal{F}^C}$)

Rules from interpretation clauses: $(\to I)$ and $(\to E)$

Consider the interpretation of $\lambda x.M$ in $\mathcal{F}^D$:

$$[\![\lambda x.M]\!]^{\mathcal{F}^D} e\, \langle \uparrow\delta, \uparrow\kappa\rangle = [\![M]\!]^{\mathcal{F}^D} e[x := \uparrow\delta]\ \uparrow\kappa$$

where $\langle \uparrow\delta, \uparrow\kappa\rangle \simeq \uparrow(\delta \times \kappa)$. This reads as the inference rule:

$$\frac{\Gamma, x : \delta \vdash M : \kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \mid \Delta}\ (\to I)$$

Similarly in case of MN, from

$$[\![MN]\!]^{\mathcal{F}^D} e\ \uparrow\kappa = [\![M]\!]^{\mathcal{F}^D} e\, \langle [\![N]\!]^{\mathcal{F}^D} e, \uparrow\kappa\rangle$$

and for any $\delta \in [\![N]\!]^{\mathcal{F}^D} e$ we get:

$$\frac{\Gamma \vdash M : \delta \times \kappa \to \rho \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \rho \mid \Delta}\ (\to E)$$

Rules from interpretation clauses: $(\times)$

Consider:

$$[\![[\alpha]M]\!]^{\mathcal{F}^C} e = \langle [\![M]\!]^{\mathcal{F}^D} e, e\,\alpha \rangle$$

then for any $\delta \in [\![M]\!]^{\mathcal{F}^D} e$ and $\kappa \in (e\,\alpha)$:

$$\delta \times \kappa \in [\![[\alpha]M]\!]^{\mathcal{F}^C} e$$

from which we obtain the rule

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \kappa, \Delta}{\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \alpha : \kappa, \Delta}\,(\times)$$

Rules from interpretation clauses: (μ)

Recall that

$$[\![\mu\alpha.Q]\!]^{\mathcal{F}^D} e \uparrow \kappa = d \cdot k' \quad \text{where} \quad [\![Q]\!]^{\mathcal{F}^C} e[\alpha :=\uparrow \kappa] = \langle d, k' \rangle$$

Filter application is $d \cdot k' = \{\rho \mid \exists \kappa' \to \rho.\ \kappa' \in d\}$ hence

$$(\kappa' \to \rho) \times \kappa' \in [\![Q]\!]^{\mathcal{F}^C} e[\alpha :=\uparrow \kappa] \quad \Rightarrow \quad \rho \in [\![\mu\alpha.Q]\!]^{\mathcal{F}^D} e \uparrow \kappa$$

and we obtain the rule

$$\frac{\Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta}\ (\mu)$$

The type assignment system $\vdash_\wedge$

$$\frac{}{\Gamma, x:\delta \vdash x:\delta \mid \Delta}\ (Var)$$

$$\frac{\Gamma, x:\delta \vdash M:\kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M:\delta\times\kappa \to \rho \mid \Delta}\ (\to I) \qquad \frac{\Gamma \vdash M:\delta\times\kappa \to \rho \mid \Delta \quad \Gamma \vdash N:\delta \mid \Delta}{\Gamma \vdash MN:\kappa \to \rho \mid \Delta}\ (\to E)$$

$$\frac{\Gamma \vdash M:\delta \mid \alpha:\kappa, \Delta}{\Gamma \vdash [\alpha]M:\delta\times\kappa \mid \alpha:\kappa, \Delta}\ (\times) \qquad \frac{\Gamma \vdash Q:(\kappa' \to \rho)\times\kappa' \mid \alpha:\kappa, \Delta}{\Gamma \vdash \mu\alpha.Q:\kappa \to \rho \mid \Delta}\ (\mu)$$

$T = M, Q$; $A = D, C$; $\sigma, \tau \in \mathcal{L}_A$:

$$\frac{}{\Gamma \vdash T:\omega \mid \Delta}\ (\omega) \qquad \frac{\Gamma \vdash T:\sigma \mid \Delta \quad \Gamma \vdash T:\tau \mid \Delta}{\Gamma \vdash T:\sigma\wedge\tau \mid \Delta}\ (\wedge) \qquad \frac{\Gamma \vdash T:\sigma \mid \Delta \quad \sigma \leq_A \tau}{\Gamma \vdash T:\tau \mid \Delta}\ (\leq)$$

Some admissible rules

if $\alpha \notin fn(L)$ then the following rules are admissible:

$$\frac{\Gamma \vdash M : \delta \mid \alpha : \delta' \times \kappa', \Delta \quad \Gamma \vdash L : \delta' \mid \Delta}{\Gamma \vdash M[\alpha \Leftarrow L] : \delta \mid \alpha : \kappa', \Delta}$$

$$\frac{\Gamma \vdash [\beta]M : \kappa \mid \alpha : \delta' \times \kappa', \Delta \quad \beta \neq \alpha \quad \Gamma \vdash L : \delta' \mid \Delta}{\Gamma \vdash ([\beta]M)[\alpha \Leftarrow L] \equiv [\beta](M[\alpha \Leftarrow L]) : \kappa \mid \alpha : \kappa', \Delta}$$

$$\frac{\Gamma \vdash [\alpha]M : (\delta \times \kappa \to \rho) \times (\delta' \times \kappa') \mid \alpha : \delta' \times \kappa', \Delta \quad \Gamma \vdash L : \delta \wedge \delta' \mid \Delta}{\Gamma \vdash ([\alpha]M)[\alpha \Leftarrow L] \equiv [\alpha](M[\alpha \Leftarrow L]L) : (\kappa \to \rho) \times \kappa' \mid \alpha : \kappa', \Delta}$$

Subject reduction (1)

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash M : \kappa' \to \rho \mid \alpha : \delta \times \kappa, \Delta \qquad \Delta(\beta) = \kappa'}{\Gamma \vdash [\beta]M : (\kappa' \to \rho) \times \kappa' \mid \alpha : \delta \times \kappa, \Delta}}{\Gamma \vdash \mu\alpha.[\beta]M : \color{blue}{\delta} \times \kappa \to \rho \mid \Delta} \qquad \Gamma \vdash N : \color{blue}{\delta} \mid \Delta}{\Gamma \vdash (\mu\alpha.[\beta]M)N : \kappa \to \rho \mid \Delta}$$

$$(\mu\alpha.[\beta]M)N \longrightarrow \mu\alpha.[\beta](M[\alpha \Leftarrow N]) \qquad \text{if } \alpha \neq \beta$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash M : \kappa' \to \rho \mid \alpha : \color{red}{\delta} \times \kappa, \Delta \quad \Gamma \vdash N : \color{red}{\delta} \mid \Delta}{\Gamma \vdash M[\alpha \Leftarrow N] : \kappa' \to \rho \mid \alpha : \kappa, \Delta}}{\Gamma \vdash M[\alpha \Leftarrow N] : \kappa' \to \rho \mid \alpha : \kappa, \Delta \qquad \Delta(\beta) = \kappa'}}{\Gamma \vdash [\beta](M[\alpha \Leftarrow N]) : (\kappa' \to \rho) \times \kappa' \mid \alpha : \kappa, \Delta}}{\Gamma \vdash \mu\alpha.[\beta](M[\alpha \Leftarrow N]) : \kappa \to \rho \mid \Delta}$$

Subject reduction (2)

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash M : \delta \times \kappa \to \rho \mid \alpha : \delta \times \kappa, \Delta}{\Gamma \vdash [\alpha]M : (\delta \times \kappa \to \rho) \times (\delta \times \kappa) \mid \alpha : \delta \times \kappa, \Delta}}{\Gamma \vdash \mu\alpha.[\alpha]M : \delta \times \kappa \to \rho \mid \Delta} \qquad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash (\mu\alpha.[\alpha]M)N : \kappa \to \rho \mid \Delta}$$

$$(\mu\alpha.[\alpha]M)N \longrightarrow \mu\alpha.[\alpha](M[\alpha \Leftarrow N])N$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash M : \color{blue}{\delta} \times \kappa \to \rho \mid \alpha : \color{red}{\delta} \times \kappa, \Delta \qquad \Gamma \vdash N : \color{red}{\delta} \mid \Delta}{\Gamma \vdash M[\alpha \Leftarrow N] : \color{blue}{\delta} \times \kappa \to \rho \mid \alpha : \kappa, \Delta} \qquad \Gamma \vdash N : \color{blue}{\delta} \mid \Delta}{\Gamma \vdash (M[\alpha \Leftarrow N])N : \kappa \to \rho \mid \alpha : \kappa, \Delta}}{\Gamma \vdash [\alpha](M[\alpha \Leftarrow N])N : (\kappa \to \rho) \times \kappa \mid \alpha : \kappa, \Delta}}{\Gamma \vdash \mu\alpha.[\alpha](M[\alpha \Leftarrow N])N : \kappa \to \rho \mid \Delta}$$

Properties of the type assignment system

Theorem: subject reduction and expansion

If $M \longrightarrow_{\beta\mu} N$ then

$$\Gamma \vdash M : \delta \mid \Delta \quad \Leftrightarrow \quad \Gamma \vdash N : \delta \mid \Delta$$

Theorem: soundness and completeness

$$\Gamma \vdash M : \delta \mid \Delta \quad \Leftrightarrow \quad \Gamma \models M : \delta \mid \Delta$$

Proof based on the filter model construction and the fact that

$$[\![M]\!]^{\mathcal{F}^D} e = \{\delta \mid \exists \Gamma, \Delta.\, e \models \Gamma, \Delta \;\; \& \;\; \Gamma \vdash M : \delta \mid \Delta\}$$

Parigot's (first order) type assignment system $\vdash_P$

Propositional types:

$$A, B ::= \varphi \mid \bot_A \mid A \to B$$

Rules:

$$\frac{}{\Gamma, x : A \vdash x : A \mid \Delta}$$

$$\frac{\Gamma, x : A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x.M : A \to B \mid \Delta} \qquad \frac{\Gamma \vdash M : A \to B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta}$$

$$\frac{\Gamma \vdash Q : \bot_B \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.Q : A \mid \Delta} \qquad \frac{\Gamma \vdash M : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]M : \bot_A \mid \alpha : A, \Delta}$$

Translating Parigot's types into intersection types

Take $a \in R \setminus \{\bot\}$, and write $\nu \equiv \nu_a$. Consider the maps from propositional types into ours, $A^C \in \mathcal{L}_C$ and $A^D \in \mathcal{L}_D$:

- $\varphi^C = \nu \times \omega$
- $\bot_A^C = (A^C \to \nu) \times A^C$
- $(A \to B)^C = (A^C \to \nu) \times B^C$
- $A^D = A^C \to \nu$

Write $\Gamma^D = \{x : A^D \mid x : A \in \Gamma\}$ and $\Delta^C = \{\alpha : A^C \mid \alpha : A \in \Delta\}$

Theorem

1. $\Gamma \vdash_P M : A \mid \Delta \Rightarrow \Gamma^D \vdash_\wedge M : A^D \mid \Delta^C$
2. $\Gamma \vdash_P Q : A \mid \Delta \Rightarrow \Gamma^D \vdash_\wedge Q : A^C \mid \Delta^C$

Conjecture

$$\begin{array}{llll} \mathcal{L}_R^- : & \rho ::= & \nu_a \mid \rho\wedge\rho & (a \neq \bot) \\ \mathcal{L}_C^- : & \kappa ::= & \delta\times\omega \mid \delta\times\kappa \mid \kappa\wedge\kappa & \\ \mathcal{L}_D^- : & \delta ::= & \nu \mid \kappa\to\rho \mid \delta\wedge\delta & \end{array}$$

Proposition

- $\sigma \in \mathcal{L}_C^- \cup \mathcal{L}_D^- \Rightarrow \sigma \neq \omega$
- for all formula A, both A^C and A^D are in $\mathcal{L}_C^- \cup \mathcal{L}_D^-$

Conjecture: exactly all strongly normalizable terms are typeable in a subsystem $\vdash_\wedge^-$ with types in $\mathcal{L}_C^- \cup \mathcal{L}_D^-$, and without rule (ω).

Conclusions and future work

1. we have defined a type theory which describes a denotational model of $\lambda\mu$
2. the type assignment system induces a filter model of the $\lambda\mu$-calculus
3. all terms which are typeable in Parigot's (first order) assignment system have a non trivial typing in our system
4. we conjecture that non trivial typing in a proper subsystem of ours characterizes strongly normalizable terms
5. the system we have proposed could characterize other interesting sets of terms, and more interestingly might provide a tool to investigate Parigot's computational interpretation of classical proofs