From Böhm's Theorem to Observational Equivalences: an Informal Account

Dedicated to Corrado Böhm

on the occasion of the EATCS Distinguished Service Award

joint paper with Elio Giovannetti

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Böhm's Theorem:

if M and N are two distinct $\beta\eta$-normal forms, then there is a context $C[\,]$
such that $C[M] \longrightarrow_{\beta}^{*} x$ and $C[N] \longrightarrow_{\beta}^{*} y$,
where x, y are arbitrary distinct variables

proof	$(\lambda xy.C[\,])\mathbf{I}\Omega$
Böhm-out technique	$C[M]$ converges and $C[N]$ diverges
Böhm trees	observational equivalences

to observe the computational behaviour of λ-terms we can

either reduce the terms and look at the meaningful information we obtain

or put the terms inside contexts and test the termination
(bisimulation in concurrent scenarios)

in both cases we obtain a notion of equality between λ-terms

full abstraction of λ-models

What is meaningful information?

first answer: a head normal form $\lambda x_1 \ldots x_n . y M_1 \ldots M_m$

The Böhm tree $\mathfrak{BT}(M)$ of a term M is defined by

if $M \rightarrow_\beta^* \lambda x_1 \ldots x_n . x M_1 \ldots M_m$ then

$$\mathfrak{BT}(M) = \begin{array}{ccc} & \lambda x_1 \ldots x_n . x & \\ \swarrow & & \searrow \\ \mathfrak{BT}(M_1) & \cdots & \mathfrak{BT}(M_m) \end{array}$$

otherwise $\mathfrak{BT}(M) = \bot$

local structure of P_ω and T_ω models [Barendregt, 84]

What is meaningful information?

second answer: a weak head normal form $\lambda x.M \qquad xM_1 \ldots M_m$

The Lévy-Longo tree $\mathfrak{LT}(M)$ of a term M is defined by

$$\text{if } M \to_\beta^* \lambda x.N \text{ then } \mathfrak{LT}(M) = \begin{array}{c} \lambda x \\ | \\ \mathfrak{LT}(N) \end{array}$$

$$\text{if } M \to_\beta^* xM_1 \ldots M_m \text{ then } \mathfrak{LT}(M) = \begin{array}{ccc} & x & \\ \swarrow & & \searrow \\ \mathfrak{LT}(M_1) & \cdots & \mathfrak{LT}(M_m) \end{array}$$

otherwise $\mathfrak{LT}(M) = \bot$

local structure of Plotkin-Scott-Engeler models [Longo, 83]

What is meaningful information?

third answer: a top normal form $\lambda x.M$ $\quad$ MN if $M \not\to_\beta^* \lambda x.M'$

The Berarducci tree tree $\mathfrak{BeT}(M)$ of a term M is defined by

if $M \to_\beta^* \lambda x.N$ then $\mathfrak{BeT}(M) =$

$$\begin{array}{c} \lambda x \\ | \\ \mathfrak{BeT}(N) \end{array}$$

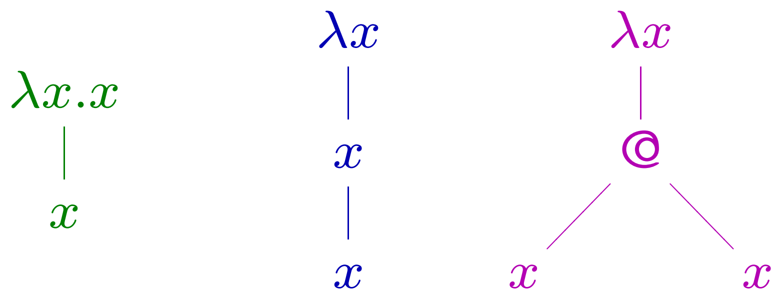
if $M \to_\beta^* M_1 M_2$ and $M_1 \not\to_\beta^* \lambda x.N$ then

$$\mathfrak{BeT}(M) = \begin{array}{ccc} & @ & \\ \swarrow & & \searrow \\ \mathfrak{BeT}(M_1) & & \mathfrak{BeT}(M_2) \end{array}$$

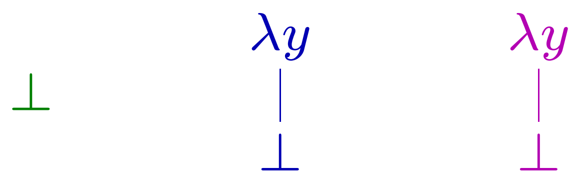
otherwise $\mathfrak{BeT}(M) = \bot$

infinite reductions [Berarducci, 94, Kennaway et al., 96]

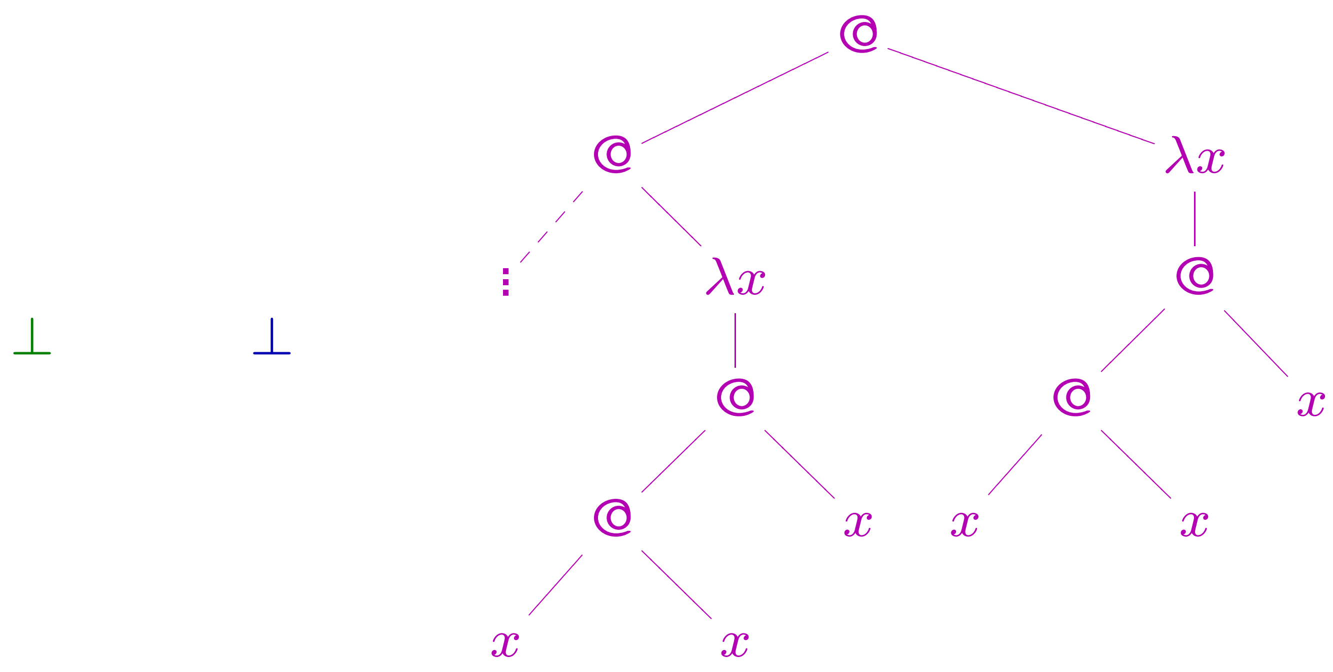
$$\lambda x.xx$$



$$\lambda y.(\lambda x.xx)(\lambda x.xx) \rightarrow_\beta \lambda y.(\lambda x.xx)(\lambda x.xx)$$

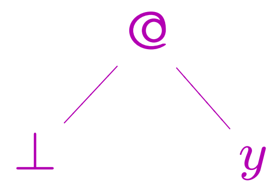


$$(\lambda x.xxx)(\lambda x.xxx) \to_\beta (\lambda x.xxx)(\lambda x.xxx)(\lambda x.xxx)$$



$$(\lambda x.xx)(\lambda x.xx)y \rightarrow_\beta (\lambda x.xx)(\lambda x.xx)y$$

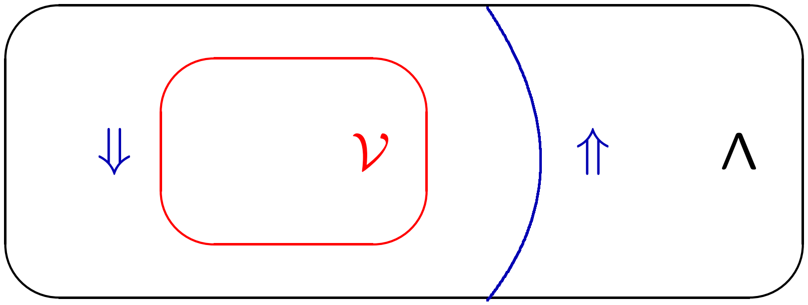
$\bot$ $\qquad$ $\bot$



$$\mathfrak{BT}(M) = \mathfrak{BT}(N) \quad \overset{\not\Rightarrow}{\Leftarrow} \quad \mathfrak{LT}(M) = \mathfrak{LT}(N) \quad \overset{\not\Rightarrow}{\Leftarrow} \quad \mathfrak{BeT}(M) = \mathfrak{BeT}(N)$$

observational equivalence

$$\mathcal{V} \subseteq \Lambda$$



$$M{\Downarrow} \quad \Leftrightarrow \quad M \to_\beta^* V \in \mathcal{V} \qquad\qquad M{\Uparrow} \quad \Leftrightarrow \quad M \not\Downarrow$$

$\mathcal{V}$ = set of head normal forms $M \simeq_h N$

local structure of D_∞ model [Wadsworth, 76]

$\mathcal{V}$ = set of normal forms $M \simeq_n N$

local structure of D^* model [Coppo et al., 87]

$\mathcal{V}$ = set of weak head normal forms $M \simeq_w N$

$\mathcal{V}$ = set of top normal forms $M \simeq_t N$

$$\mathbf{I} \equiv \lambda x.x \qquad R \equiv \lambda xzy.z(xxy)$$
$$\Omega \equiv (\lambda x.xx)(\lambda x.xx) \qquad \Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$$

$$\mathbf{I} \simeq_h RR \qquad \mathbf{I} \not\simeq_n RR \qquad \mathbf{I} \not\simeq_w RR \qquad \mathbf{I} \not\simeq_t RR$$

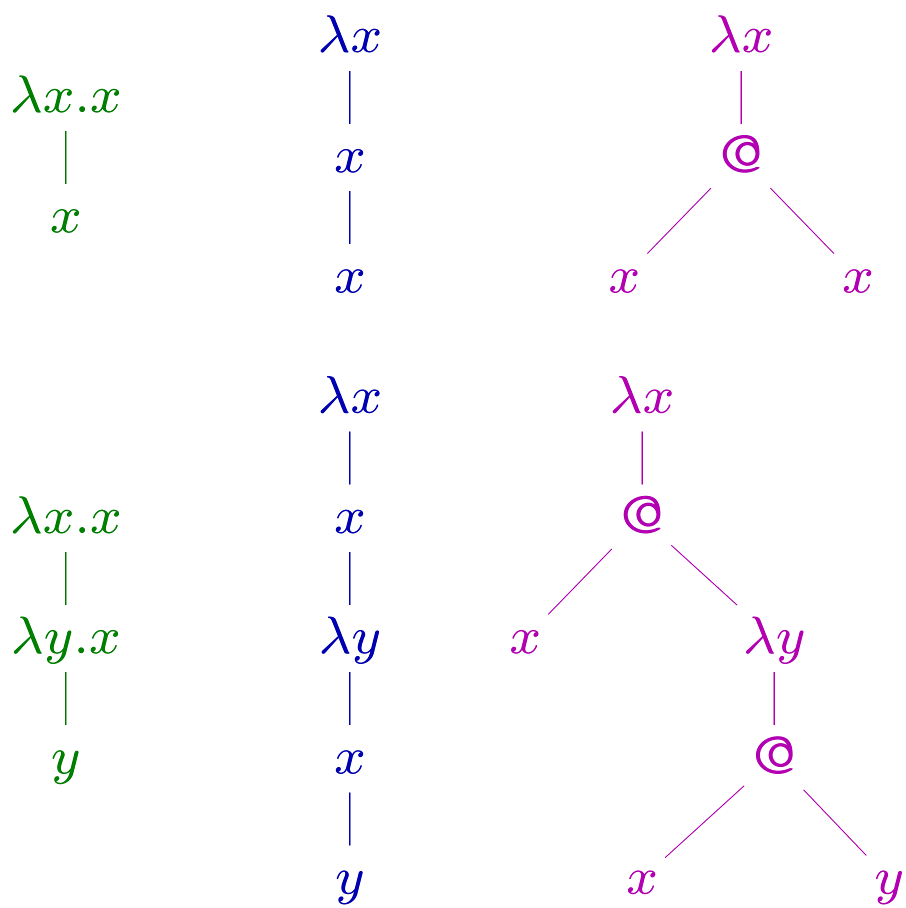
$$\Omega \simeq_h \lambda y.\Omega \qquad \Omega \simeq_n \lambda y.\Omega \qquad \Omega \not\simeq_w \lambda y.\Omega \qquad \Omega \not\simeq_t \lambda y.\Omega$$

$$\Omega \simeq_h \Omega_3 \qquad \Omega \simeq_n \Omega_3 \qquad \Omega \simeq_w \Omega_3 \qquad \Omega \not\simeq_t \Omega_3$$

$$\Omega\mathbf{I} \simeq_h \lambda y.\Omega\mathbf{I} \qquad \Omega\mathbf{I} \simeq_n \lambda y.\Omega\mathbf{I} \qquad \Omega\mathbf{I} \not\simeq_w \lambda y.\Omega\mathbf{I} \qquad \Omega\mathbf{I} \simeq_t \lambda y.\Omega\mathbf{I}$$

$$M \simeq_h N \;\; \overset{\not\Rightarrow}{\Leftarrow} \;\; M \simeq_n N \;\; \overset{\not\Rightarrow}{\Leftarrow} \;\; M \simeq_w N \;\; \overset{\not\Rightarrow}{\not\Leftarrow} \;\; M \simeq_t N$$

$$\lambda x.xx \simeq_h \lambda x.x(\lambda y.xy) \quad \lambda x.xx \simeq_n \lambda x.x(\lambda y.xy)$$
$$\lambda x.xx \simeq_w \lambda x.x(\lambda y.xy) \quad \lambda x.xx \simeq_t \lambda x.x(\lambda y.xy)$$

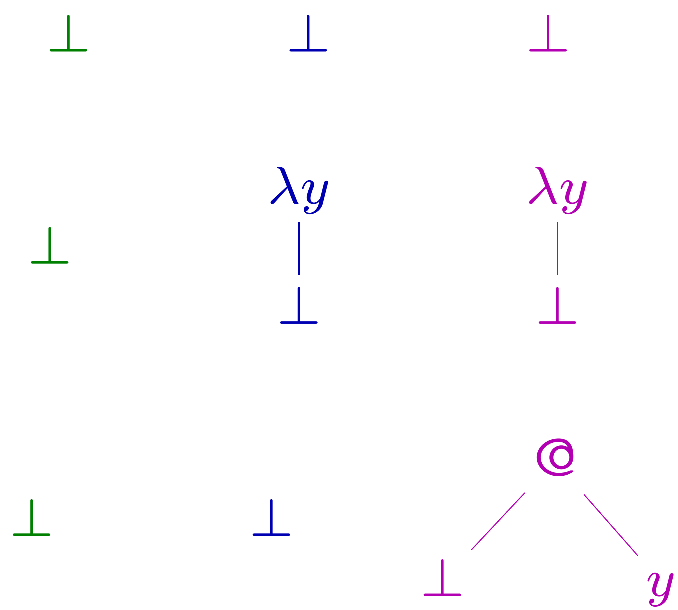


$(\lambda x.xx)(\lambda x.xx) \simeq_h \lambda y.(\lambda x.xx)(\lambda x.xx)$ $(\lambda x.xx)(\lambda x.xx) \simeq_n \lambda y.(\lambda x.xx)(\lambda x.xx)$

$(\lambda x.xx)(\lambda x.xx) \not\simeq_w \lambda y.(\lambda x.xx)(\lambda x.xx)$ $(\lambda x.xx)(\lambda x.xx) \not\simeq_t \lambda y.(\lambda x.xx)(\lambda x.xx)$

$(\lambda x.xx)(\lambda x.xx) \simeq_h (\lambda x.xx)(\lambda x.xx)y$ $(\lambda x.xx)(\lambda x.xx) \simeq_n (\lambda x.xx)(\lambda x.xx)y$

$(\lambda x.xx)(\lambda x.xx) \simeq_w (\lambda x.xx)(\lambda x.xx)y$ $(\lambda x.xx)(\lambda x.xx) \not\simeq_t (\lambda x.xx)(\lambda x.xx)y$



$$\mathfrak{BT}(M) = \mathfrak{BT}(N) \quad \underset{\not\Leftarrow}{\Rightarrow} \quad M \simeq_n N \quad \underset{\not\Leftarrow}{\Rightarrow} \quad M \simeq_h N$$

$$\mathfrak{LT}(M) = \mathfrak{LT}(N) \quad \underset{\not\Leftarrow}{\Rightarrow} \quad M \simeq_w N \qquad \mathfrak{BeT}(M) = \mathfrak{BeT}(N) \quad \underset{\not\Leftarrow}{\Rightarrow} \quad M \simeq_t N$$

$$\mathfrak{LT}(M) = \mathfrak{LT}(N) \;\Leftrightarrow\; M \simeq_w N \;\&\; \mathfrak{BT}(M) = \mathfrak{BT}(N)$$

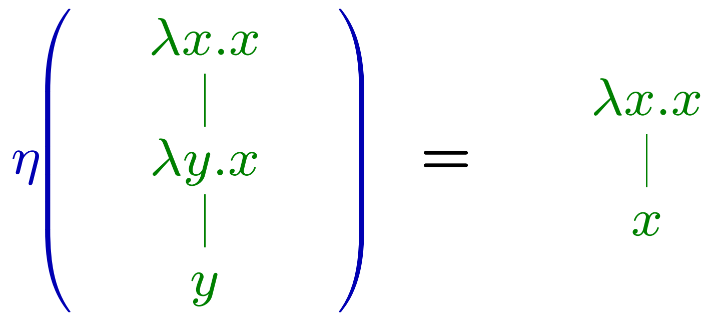
$\simeq_n$ on $\mathfrak{BT}$

$$(\eta) \qquad \lambda x.Mx \rightarrow M \quad \text{if } x \notin FV(M)$$

$$\eta(\bot) = \bot$$

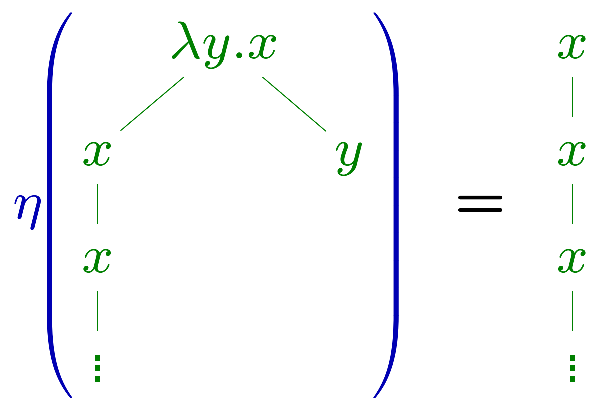
$$\eta\left(\begin{array}{ccc} & \lambda x_1 \ldots x_n.x & \\ \diagup & & \diagdown \\ T_1 & \ldots & T_m \end{array}\right) = \begin{cases} \eta\left(\begin{array}{ccc} & \lambda x_1 \ldots x_{n-1}.x & \\ \diagup & & \diagdown \\ T_1 & \ldots & T_{m-1} \end{array}\right) & \begin{array}{l}\text{if } T_m \text{ is finite,} \\ \eta(T_m) = x_n \neq x \\ \text{and } x_n \notin FV(T_i) \\ \text{for } 1 \leq i \leq m-1. \end{array} \\ \begin{array}{ccc} & \lambda x_1 \ldots x_n.x & \\ \diagup & & \diagdown \\ \eta(T_1) & \ldots & \eta(T_m) \end{array} & \text{otherwise} \end{cases}$$

$$\eta(\mathfrak{BT}(M)) = \eta(\mathfrak{BT}(N)) \quad \Longleftrightarrow \quad M \simeq_n N \text{ [Hyland, 75]}$$



$P_x \equiv A_x\, A_x$ where $A_x \equiv \lambda zy.x(zzy)$

$\lambda y.(P_x\; y)y \rightarrow_\beta \lambda y.x(P_x\; y)y \rightarrow_\beta \lambda y.x(x(P_x\; y))y \rightarrow_\beta \lambda y.x(x(x(P_x\; y)))y \rightarrow_\beta \ldots$



$\simeq_h$ on $\mathfrak{BT}$

$$\eta_\infty(\bot) = \bot$$

$$\eta_\infty\left(\begin{array}{ccc} & \lambda x_1 \ldots x_n.x & \\ / & & \backslash \\ T_1 & \ldots & T_m \end{array}\right) = \begin{cases} \eta_\infty\left(\begin{array}{ccc} & \lambda x_1 \ldots x_{n-1}.x & \\ / & & \backslash \\ T_1 & \ldots & T_{m-1} \end{array}\right) & \text{if } T_m \geq_\eta x_n,\ x_n \neq x \text{ and } x_n \notin FV(T_i) \text{ for } 1 \leq i \leq m-1. \\ \begin{array}{ccc} & \lambda x_1 \ldots x_n.x & \\ / & & \backslash \\ \eta_\infty(T_1) & \ldots & \eta_\infty(T_m) \end{array} & \text{otherwise} \end{cases}$$

$$\eta_\infty(\mathfrak{BT}(M)) = \eta_\infty(\mathfrak{BT}(N)) \iff M \simeq_h N$$ [Barendregt, 81]

$$x \quad \le_\eta \quad \begin{array}{c}\lambda y_0.x\\ | \\ y_0\end{array} \quad \le_\eta \quad \begin{array}{c}\lambda y_0.x\\ |\\ \lambda y_1.y_0\\ |\\ y_1\end{array} \quad \le_\eta \quad \begin{array}{c}\lambda y_0.x\\ |\\ \lambda y_1.y_0\\ |\\ \lambda y_2.y_1\\ |\\ y_2\end{array} \quad \le_\eta \quad \cdots \quad \le_\eta \quad \begin{array}{c}\lambda y_0.x\\ |\\ \lambda y_1.y_0\\ |\\ \lambda y_2.y_1\\ |\\ \vdots\end{array}$$

$Q \equiv RR$ where $R \equiv \lambda zxy.x(zzy)$
$Q \to_\beta \lambda xy_0.x(Qy_0) \to_\beta \lambda xy_0.x(\lambda y_1.y_0(Qy_1)) \to_\beta \ldots$

$$\eta_\infty \left(\begin{array}{c}\lambda xy_0.x\\ |\\ \lambda y_1.y_0\\ |\\ \lambda y_2.y_1\\ |\\ \vdots\end{array} \right) \quad = \quad \lambda x.x$$

scenarios for Böhm trees

non-determinism and bisimulation

Sangiorgi encoding of λ-calculus into the π-calculus $()^\pi$
divergence sensitive bisimulation between processes $\simeq^\pi$

$$\mathfrak{BT}(M) = \mathfrak{BT}(N) \iff M \simeq^\pi N$$ [Sangiorgi, 98]

non-determinism and observational equivalence

non-deterministic choice $M + N \rightarrow M \qquad M + N \rightarrow N$

adequate numeral system $\mathsf{0}$ s p zero?

$\mathsf{p}\,\mathsf{0} \rightarrow \mathsf{0} \qquad \mathsf{p}(\mathsf{s}\mathsf{n}) \rightarrow \mathsf{n} \qquad \text{zero?}\,\mathsf{0} \rightarrow \mathbf{T} \qquad \text{zero?}(\mathsf{s}\mathsf{n}) \rightarrow \mathbf{F}$

where n stands for $\underbrace{\mathsf{s}(\mathsf{s}\ldots(\mathsf{s}\,\mathsf{0})\ldots)}_{n \text{ times}}$, and $\mathbf{T} \equiv \lambda xy.x$, $\mathbf{F} \equiv \lambda xy.y$.

$\mathcal{V}$ = set of numerals

$M \Downarrow$ is "may"

observational equivalence $\simeq_{\mathbb{N}}$

$\mathfrak{BT}(M) = \mathfrak{BT}(N) \iff M \simeq_{\mathbb{N}} N$ [Dezani et al., 98]

$$(\lambda x.xx)(\mathsf{s}+0)$$

$$\downarrow$$

$$(\mathsf{s}+0)(\mathsf{s}+0)$$

$$\mathsf{s}(\mathsf{s}+0) \qquad 0(\mathsf{s}+0)$$

$$\mathsf{ss} \qquad \mathsf{s}\,0 \qquad 0\,\mathsf{s} \qquad 00$$

$$(\lambda x.x(\lambda y.xy))(\mathsf{s}+0)$$

$$\downarrow$$

$$(\mathsf{s}+0)(\lambda y.(\mathsf{s}+0)y)$$

$$\mathsf{s}(\lambda y.(\mathsf{s}+0)y) \qquad 0(\lambda y.(\mathsf{s}+0)y)$$

$$\mathsf{s}(\lambda y.\mathsf{s}y) \qquad \mathsf{s}(\lambda y.0y) \qquad 0(\lambda y.\mathsf{s}y) \qquad 0(\lambda y.0y)$$

$$(\lambda x.xx)(\mathsf{s}+0)\Downarrow \qquad (\lambda x.x(\lambda y.xy))(\mathsf{s}+0)\Uparrow$$

$$(\lambda x.xx) \not\simeq_{\mathbb{N}} (\lambda x.x(\lambda y.xy))$$

scenarios for Lévy-Longo trees

non-determinism and bisimulation

Milner encoding of lazy λ-calculus into the π-calculus $(\,)^{\pi}_{l}$

weak bisimulation between processes $\simeq^{\pi}_{l}$

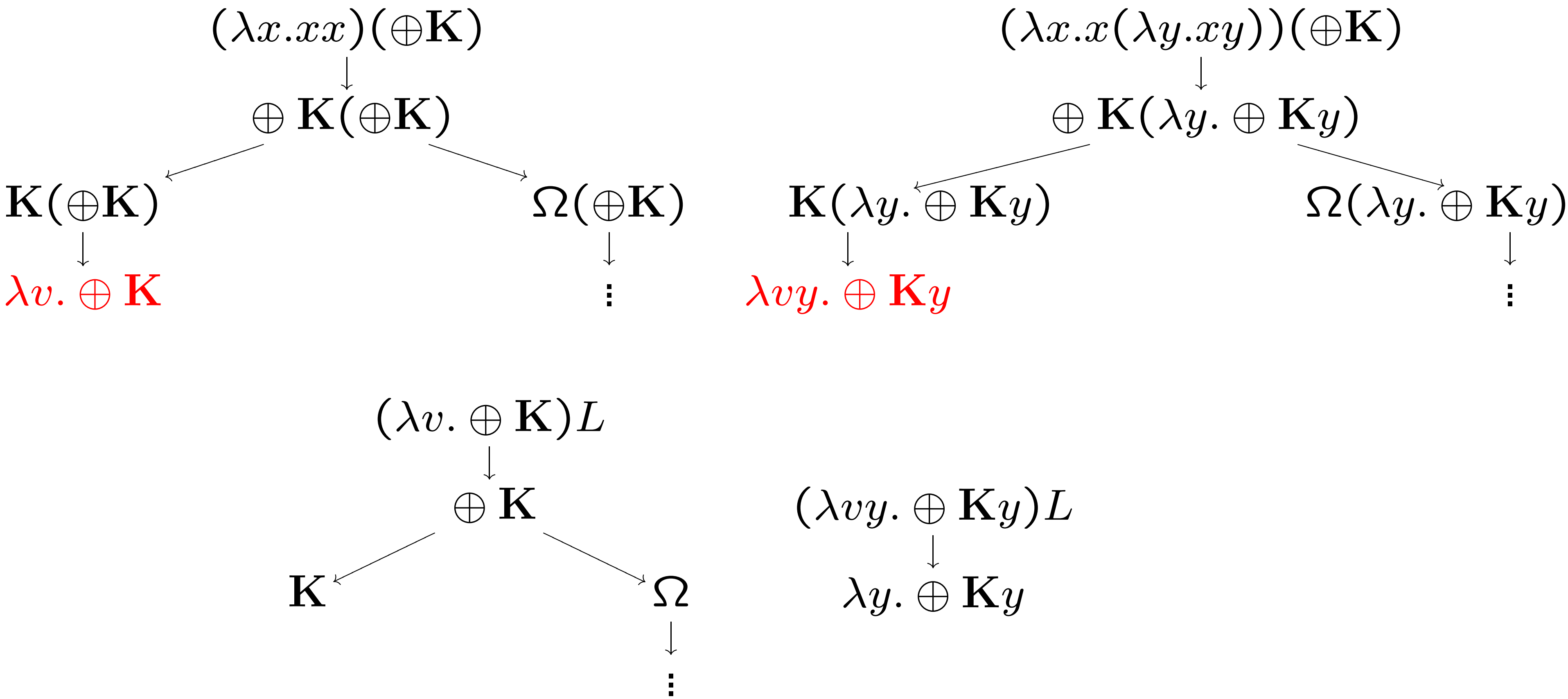
$$\mathfrak{LT}(M) = \mathfrak{LT}(N) \iff M \simeq^{\pi}_{l} N \qquad \text{[Sangiorgi, 92]}$$

lazy λ-calculus

non-deterministic operator $\quad \oplus M \to M \qquad \oplus M \to \Omega$

applicative bisimulation $\simeq_{\oplus}$

$$\mathfrak{LT}(M) = \mathfrak{LT}(N) \iff M \simeq_{\oplus} N \qquad \text{[Sangiorgi, 92]}$$



$$(\lambda x.xx) \not\simeq_{\oplus} (\lambda x.x(\lambda y.xy))$$

deadlock and observational equivalence

λ-calculus of multiplicities

$(\lambda x.M)N^m \rightarrow M < N^m/x >$ $\quad$ N is available at most m times

standard (β) $m = \infty$ $\quad$ possibility of deadlock

$\mathcal{V}$ = set of abstractions

observational equivalence $\quad \simeq_\mu$

$$\mathfrak{LT}(M) = \mathfrak{LT}(N) \iff M \simeq_\mu N$$ [Boudol and Laneve, 96]

$$(\lambda x.xx)\mathbf{I}^1 \rightarrow (xx) < \mathbf{I}^1/x > \rightarrow (\mathbf{I}x) < \mathbf{I}^0/x > \rightarrow z < x^\infty/z >< \mathbf{I}^0/x >$$
$$\rightarrow x < x^\infty/z >< \mathbf{I}^0/x > \Uparrow$$
$$(\lambda x.x(\lambda y.xy))\mathbf{I}^1 \rightarrow (x(\lambda y.xy)) < \mathbf{I}^1/x > \rightarrow (\mathbf{I}(\lambda y.xy)) < \mathbf{I}^0/x >$$
$$\rightarrow z < (\lambda y.xy)^\infty/z >< \mathbf{I}^0/x > \rightarrow (\lambda y.xy) < (\lambda y.xy)^\infty/z >< \mathbf{I}^0/x > \Downarrow$$

$$(\lambda x.xx) \not\simeq_\mu (\lambda x.x(\lambda y.xy))$$

concurrency and observational equivalence

call-by-value and call-by-name variables

parallel operator $\dfrac{M \to M' \quad N \to N'}{M \| N \to M' \| N'}$ $(\|)$

non-deterministic choice $M + N \to M \qquad M + N \to N$

$\mathcal{V}$: $\quad \xi \mid \lambda x.M \mid \lambda \xi.M \mid V \| M \mid M \| V$

$M \Downarrow$ is "must"

observational equivalence $\quad \simeq_{+\|}$

$\mathfrak{LT}(M) = \mathfrak{LT}(N) \iff M \simeq_{+\|} N$ [Dezani et al., 97]

$$(\lambda x.xx)(P+Q) \to (P+Q)(P+Q)$$

$$(P+Q)(P+Q) \to P(P+Q) \qquad (P+Q)(P+Q) \to Q(P+Q)$$

$$P(P+Q) \to PP \qquad P(P+Q) \to PQ \qquad Q(P+Q) \to \lambda x.x\Omega$$

$$PP \to P(P\mathbf{O}) \to P(\mathbf{O}(\mathbf{O}\mathbf{O})) \to P\mathbf{I} \to \mathbf{I}(\mathbf{I}\mathbf{O}) \to \mathbf{I}\mathbf{O} \to \mathbf{O}$$

$$PQ \to Q(Q\mathbf{O}) \to \lambda x.x\Omega$$

$$P \equiv \lambda\xi.\xi(\xi\mathbf{O})$$
$$\mathbf{O} \equiv \lambda zt.t$$
$$Q \equiv \lambda yx.x\Omega$$

$$\begin{aligned}
&(\lambda x.x(\lambda y.xy))(P+Q)\\
\to\ &(P+Q)(\lambda y.(P+Q)y)\\
\to\ &P(\lambda y.(P+Q)y)\\
\to\ &(\lambda y.(P+Q)y)((\lambda y.(P+Q)y)\mathbf{O})\\
\to\ &(P+Q)((\lambda y.(P+Q)y)\mathbf{O})\\
\to\ &P((\lambda y.(P+Q)y)\mathbf{O})\\
\to\ &P((P+Q)\mathbf{O})\\
\to\ &P(Q\mathbf{O})\\
\to\ &P(\lambda x.x\Omega)\\
\to\ &(\lambda x.x\Omega)((\lambda x.x\Omega)\mathbf{O})\\
\to\ &(\lambda x.x\Omega)\mathbf{O}\Omega\\
\to\ &\mathbf{O}\Omega\Omega\\
\to\ &\Omega\\
\to\ &\vdots
\end{aligned}$$

$$(\lambda x.xx) \not\simeq_{+\parallel} (\lambda x.x(\lambda y.xy))$$

scenarios for Berarducci trees

selectors and observational equivalence

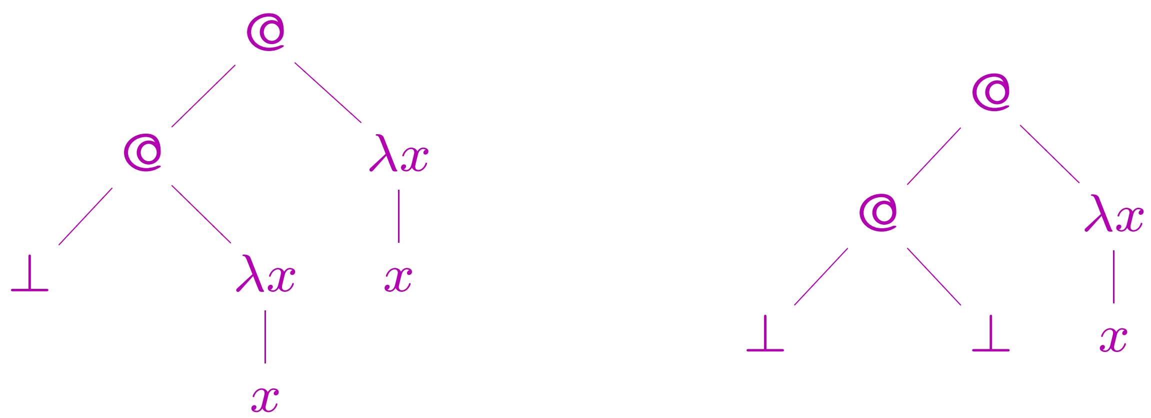
operator selector $\mathsf{O}(MN) \to M$ if M is an unsolvable of order zero

argument selector $\mathsf{A}(MN) \to N$ if M is an unsolvable of order zero

$\mathcal{V}$ = set of top normal forms

observational equivalence $\simeq_{\mathsf{OA}}$

$$\mathfrak{BeT}(M) = \mathfrak{BeT}(N) \iff M \simeq_{\mathsf{OA}} N$$ [Dezani et al., 00]



$$\mathsf{A}(\mathsf{O}(\Omega\mathbf{I}\mathbf{I})) \longrightarrow \mathsf{A}(\Omega\mathbf{I}) \longrightarrow \mathbf{I}$$
$$\mathsf{A}(\mathsf{O}(\Omega\Omega\mathbf{I})) \longrightarrow \mathsf{A}(\Omega\Omega) \longrightarrow \Omega$$

$$\Omega\mathbf{I}\mathbf{I} \not\simeq_{\mathsf{OA}} \Omega\Omega\mathbf{I}$$

$$
\begin{array}{ccccccccccc}
 & & & & & & & & \eta_\infty(\mathfrak{BT}) & \Longleftrightarrow & \simeq_h \\
 & & & & & & & & \Uparrow & & \Uparrow \\
 & & & & & & & & \eta(\mathfrak{BT}) & \Longleftrightarrow & \simeq_n \\
 & & & & & & & & \Uparrow & \nearrow\!\!\!\!\nearrow & \Uparrow \\
 & & & & \simeq^\pi & \Longleftrightarrow & \simeq_\mathbb{N} & \Longleftrightarrow & \mathfrak{BT} & & \simeq_w \\
 & & & & & & \Uparrow & & \Uparrow & \nearrow\!\!\!\!\nearrow & \\
\simeq^\pi_l & \Longleftrightarrow & \simeq_{+\|} & \Longleftrightarrow & \simeq_\mu & \Longleftrightarrow & \simeq_\oplus & \Longleftrightarrow & \mathfrak{LT} & & \simeq_t \\
 & & & & & & \Uparrow & & \Uparrow & \nearrow\!\!\!\!\nearrow & \\
 & & & & & & \simeq_{\mathsf{OA}} & \Longleftrightarrow & \mathfrak{BeT} & &
\end{array}
$$

open problems:

$$
\begin{array}{ccc}
?(\mathfrak{LT}) & \Longleftrightarrow & \simeq_w \\
?(\mathfrak{BeT}) & \Longleftrightarrow & \simeq_t
\end{array}
$$