Characterising Strong Normalisation for Explicit Substitutions

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- the foundations
- the implementations

of logic and programming languages

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- the substitution is no longer a meta-operation on terms
  it is an operation of the calculus itself
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This is important:

- from the foundational viewpoint since the explicit substitution is first-order
- from the implementation viewpoint since it leads to a more pertinent analysis of the correctness and efficiency of compilers, theorem provers, and proof-checkers.

The substitution calculus Λx

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(B) $\quad (\lambda x.M)N \qquad M\langle x = N\rangle$

(App) $\quad (MP)\langle x = N\rangle \quad \to \quad (M\langle x = N\rangle)(P\langle x = N\rangle)$

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(B)	$(\lambda x.M)N$		$M\langle x = N\rangle$
(App)	$(MP)\langle x = N\rangle$	$\rightarrow$	$(M\langle x = N\rangle)(P\langle x = N\rangle)$
(Abs)	$(\lambda y.M)\langle x = N\rangle$	$\rightarrow$	$\lambda y.(M\langle x = N\rangle)$

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(VarI)	$x\langle x = N\rangle$	$\longrightarrow$	N

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(VarI)	$x\langle x = N\rangle$	$\rightarrow$	N
(VarK)	$y\langle x = N\rangle$	$\rightarrow$	y

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(VarI)	$x\langle x = N\rangle$	$\rightarrow$	N
(VarK)	$y\langle x = N\rangle$	$\rightarrow$	y

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow ((\lambda y.x(xy))a)\langle x = f\rangle \rightarrow$$

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$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle$$

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$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle \rightarrow (\lambda y.(x(xy)\langle x = f\rangle))a\langle x = f\rangle \rightarrow$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow ((\lambda y.x(xy))a)\langle x = f\rangle \rightarrow$$
$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle \rightarrow (\lambda y.(x(xy)\langle x = f\rangle))a\langle x = f\rangle \rightarrow$$
$$(x(xy))\langle x = f\rangle \langle y = a\langle x = f\rangle\rangle$$

$$(\lambda x.\lambda y.x(xy))fa \to (\lambda y.f(fy))a \to f(fa))$$

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$$(x(xy))\langle x = f\rangle \langle y = a\langle x = f\rangle\rangle \to (x(xy))\langle x = f\rangle \langle y = a\rangle \to$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

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$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle \rightarrow (\lambda y.(x(xy)\langle x = f\rangle))a\langle x = f\rangle \rightarrow$$

$$(x(xy))\langle x = f\rangle\langle y = a\langle x = f\rangle\rangle \rightarrow (x(xy))\langle x = f\rangle\langle y = a\rangle \rightarrow$$

$$(x\langle x = f\rangle(xy)\langle x = f\rangle)\langle y = a\rangle$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow ((\lambda y.x(xy))a)\langle x=f\rangle \rightarrow$$

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$$(x(xy))\langle x=f\rangle\langle y=a\langle x=f\rangle\rangle \rightarrow (x(xy))\langle x=f\rangle\langle y=a\rangle \rightarrow$$

$$({\color{blue}x}{\color{red}\langle x=f\rangle}(xy)\langle x=f\rangle)\langle y=a\rangle \rightarrow ({\color{blue}f}(xy)\langle x=f\rangle)\langle y=a\rangle \rightarrow$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow ((\lambda y.x(xy))a)\langle x = f\rangle \rightarrow$$

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$$(x(xy))\langle x = f\rangle\langle y = a\langle x = f\rangle\rangle \rightarrow (x(xy))\langle x = f\rangle\langle y = a\rangle \rightarrow$$

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$$(f(x\langle x = f\rangle y\langle x = f\rangle))\langle y = a\rangle$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

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$$(x(xy))\langle x = f\rangle \langle y = a\langle x = f\rangle\rangle \rightarrow (x(xy))\langle x = f\rangle \langle y = a\rangle \rightarrow$$

$$(x\langle x = f\rangle (xy)\langle x = f\rangle)\langle y = a\rangle \rightarrow (f(xy)\langle x = f\rangle)\langle y = a\rangle \rightarrow$$

$$(f({\color{blue}x}{\color{red}\langle x = f\rangle}y\langle x = f\rangle))\langle y = a\rangle \rightarrow (f({\color{blue}f}y\langle x = f\rangle))\langle y = a\rangle \rightarrow$$

$$(\lambda x.\lambda y.x(xy))fa \rightarrow (\lambda y.f(fy))a \rightarrow f(fa))$$

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$$(f(x\langle x = f\rangle y\langle x = f\rangle))\langle y = a\rangle \rightarrow (f(f{\color{blue}y}{\color{red}\langle x = f\rangle}))\langle y = a\rangle \rightarrow$$
$$(f(f{\color{blue}y}))\langle y = a\rangle$$

$$(\lambda x.\lambda y.x(xy))fa \to (\lambda y.f(fy))a \to f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \to ((\lambda y.x(xy))a)\langle x = f\rangle \to$$

$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle \to (\lambda y.(x(xy)\langle x = f\rangle))a\langle x = f\rangle \to$$

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$$(f(fy))\langle y = a\rangle \to f\langle y = a\rangle(fy)\langle y = a\rangle$$

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$$(f(fy))\langle y = a\rangle \to f\langle y = a\rangle (fy)\langle y = a\rangle \to f(fy)\langle y = a\rangle \to$$

$$f(f\langle y = a\rangle y\langle y = a\rangle) \to f(fy\langle y = a\rangle)$$

$$(\lambda x.\lambda y.x(xy))fa \to (\lambda y.f(fy))a \to f(fa))$$

$$(\lambda x.\lambda y.x(xy))fa \to ((\lambda y.x(xy))a)\langle x=f\rangle \to$$

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$$(x(xy))\langle x=f\rangle\langle y=a\langle x=f\rangle\rangle \to (x(xy))\langle x=f\rangle\langle y=a\rangle \to$$

$$(x\langle x=f\rangle(xy)\langle x=f\rangle)\langle y=a\rangle \to (f(xy)\langle x=f\rangle)\langle y=a\rangle \to$$

$$(f(x\langle x=f\rangle y\langle x=f\rangle))\langle y=a\rangle \to (f(fy\langle x=f\rangle))\langle y=a\rangle \to$$

$$(f(fy))\langle y=a\rangle \to f\langle y=a\rangle(fy)\langle y=a\rangle \to f(fy)\langle y=a\rangle \to$$

$$f(f\langle y=a\rangle y\langle y=a\rangle) \to f(f{\color{blue}y}{\color{red}\langle y=a\rangle}) \to f(f{\color{blue}a})$$

we can apply more than one reduction rule

$$((\lambda y.x(xy))a)\langle x = f\rangle$$

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$$(\lambda y.x(xy))\langle x = f\rangle\, a\langle x = f\rangle$$

we can apply more than one reduction rule

$$((\lambda y.x(xy))a)\langle x=f\rangle$$

$$\rightarrow ((x(xy))\langle y=a\rangle)\langle x=f\rangle$$

we can apply more than one reduction rule

$$((\lambda y.x(xy))a)\langle x = f\rangle$$

$$(\lambda y.x(xy))\langle x = f\rangle a\langle x = f\rangle \quad \longleftarrow \quad ((\lambda y.x(xy))a)\langle x = f\rangle \quad \longrightarrow \quad ((x(xy))\langle y = a\rangle)\langle x = f\rangle$$

THE STRONG NORMALISATION PROPERTY

allows us to freely apply the reduction rules

a term is strongly normalising iff every reduction terminates

$D \equiv \lambda z.zz$

a non-normalising term

$$DD$$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \rightarrow (zz)\langle z = D\rangle$$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \rightarrow (zz)\langle z = D\rangle \rightarrow z\langle z = D\rangle z\langle z = D\rangle$$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \to (zz)\langle z = D\rangle \to z\langle z = D\rangle z\langle z = D\rangle \to Dz\langle z = D\rangle$$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \rightarrow (zz)\langle z = D\rangle \rightarrow z\langle z = D\rangle z\langle z = D\rangle \rightarrow Dz\langle z = D\rangle \rightarrow DD \rightarrow \ldots$$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \to (zz)\langle z = D\rangle \to z\langle z = D\rangle z\langle z = D\rangle \to Dz\langle z = D\rangle \to DD \to \ldots$$

a non strongly normalising term

$(\lambda x.x)\langle y = DD\rangle$

$D \equiv \lambda z.zz$

a non-normalising term

$$DD \to (zz)\langle z = D\rangle \to z\langle z = D\rangle z\langle z = D\rangle \to Dz\langle z = D\rangle \to DD \to \ldots$$

a non strongly normalising term

$$(\lambda x.x)\langle y = DD\rangle \quad \to \ldots \to \quad (\lambda x.x)\langle y = DD\rangle \to \ldots$$

$$D \equiv \lambda z.zz$$

a non-normalising term

$$DD \to (zz)\langle z = D\rangle \to z\langle z = D\rangle z\langle z = D\rangle \to Dz\langle z = D\rangle \to DD \to \ldots$$

a non strongly normalising term

$$(\lambda x.x)\langle y = DD\rangle \quad \to \ldots \to \quad (\lambda x.x)\langle y = DD\rangle \to \ldots$$

$$\searrow$$

$$\lambda x.x\langle y = DD\rangle$$

$$D \equiv \lambda z.zz$$

a non-normalising term

$$DD \rightarrow (zz)\langle z = D\rangle \rightarrow z\langle z = D\rangle z\langle z = D\rangle \rightarrow Dz\langle z = D\rangle \rightarrow DD \rightarrow \ldots$$

a non strongly normalising term

$$(\lambda x.x)\langle y = DD\rangle \quad \rightarrow \ldots \rightarrow \quad (\lambda x.x)\langle y = DD\rangle \rightarrow \ldots$$

$$\downarrow$$

$$\lambda x.x\langle y = DD\rangle$$

$$\downarrow$$

$$\lambda x.x$$

How can we guarantee strong normalisation?

a possible answer is: using types!

How can we guarantee strong normalisation?

a possible answer is: using types!

SIMPLE TYPES FOR λ-CALCULUS

$$(Ax)\quad \frac{}{\Gamma \vdash x : \sigma}\ (x{:}\sigma \in \Gamma)$$

$$(\rightarrow I)\quad \frac{\Gamma, x{:}\sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau} \qquad (\rightarrow E)\quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$\Gamma = \{x{:}\sigma \to \sigma, y{:}\sigma\}$

$$\frac{\Gamma \vdash x{:}\sigma \to \sigma \quad \Gamma \vdash y{:}\sigma}{\Gamma \vdash xy{:}\sigma} \; (\to E)$$

$\Gamma = \{x{:}\sigma \to \sigma, y{:}\sigma\}$

$$\dfrac{\Gamma \vdash x{:}\sigma \to \sigma \qquad \dfrac{\Gamma \vdash x{:}\sigma \to \sigma \quad \Gamma \vdash y{:}\sigma}{\Gamma \vdash xy{:}\sigma}\,(\to E)}{\Gamma \vdash x(xy){:}\sigma}\,(\to E)$$

$$\Gamma = \{x{:}\sigma \to \sigma, y{:}\sigma\}$$

$$\dfrac{\Gamma \vdash x{:}\sigma \to \sigma \qquad \dfrac{\Gamma \vdash x{:}\sigma \to \sigma \quad \Gamma \vdash y{:}\sigma}{\Gamma \vdash xy{:}\sigma}\,(\to E)}{\dfrac{\Gamma \vdash x(xy){:}\sigma}{\{x{:}\sigma \to \sigma\} \vdash \lambda y.x(xy){:}\sigma \to \sigma}\,(\to I)}\,(\to E)$$

$$\Gamma = \{x{:}\sigma \to \sigma, y{:}\sigma\}$$

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash x{:}\sigma \to \sigma \qquad \cfrac{\Gamma \vdash x{:}\sigma \to \sigma \quad \Gamma \vdash y{:}\sigma}{\Gamma \vdash xy{:}\sigma}\,(\to E)}{\Gamma \vdash x(xy){:}\sigma}\,(\to E)}{\{x{:}\sigma \to \sigma\} \vdash \lambda y.x(xy){:}\sigma \to \sigma}\,(\to I)}{\vdash \lambda x.\lambda y.x(xy){:}(\sigma \to \sigma) \to \sigma \to \sigma}\,(\to I)$$

$\Gamma = \{x{:}\sigma \to \sigma, y{:}\sigma\}$

$$\dfrac{\Gamma \vdash x{:}\sigma \to \sigma \qquad \dfrac{\Gamma \vdash x{:}\sigma \to \sigma \quad \Gamma \vdash y{:}\sigma}{\Gamma \vdash xy{:}\sigma}\,(\to E)}{\dfrac{\Gamma \vdash x(xy){:}\sigma}{\dfrac{\{x{:}\sigma \to \sigma\} \vdash \lambda y.x(xy){:}\sigma \to \sigma}{\vdash \lambda x.\lambda y.x(xy){:}(\sigma \to \sigma) \to \sigma \to \sigma}\,(\to I)}\,(\to I)}\,(\to E)$$

no type for $\lambda z.zz$

INTERSECTION TYPES FOR λ-CALCULUS

$$(Ax)\quad \frac{}{\Gamma \vdash x : \sigma}\ (x{:}\sigma \in \Gamma)$$

$$(\rightarrow I)\quad \frac{\Gamma, x{:}\sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau} \qquad (\rightarrow E)\quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$(\cap I)\quad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau} \qquad (\cap E)\quad \frac{\Gamma \vdash M : \sigma_1 \cap \sigma_2}{\Gamma \vdash M : \sigma_i}\ (i \in \{1, 2\})$$

$$\Gamma = \{z{:}(\sigma{\to}\tau)\cap\sigma\}$$

$$\dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau)\cap\sigma}{\Gamma \vdash z{:}\sigma{\to}\tau}\,(\cap E) \qquad \dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau)\cap\sigma}{\Gamma \vdash z{:}\sigma}\,(\cap E)$$

$$\Gamma = \{z{:}(\sigma \to \tau) \cap \sigma\}$$

$$\dfrac{\dfrac{\Gamma \vdash z{:}(\sigma \to \tau) \cap \sigma}{\Gamma \vdash z{:}\sigma \to \tau}\,(\cap E) \qquad \dfrac{\Gamma \vdash z{:}(\sigma \to \tau) \cap \sigma}{\Gamma \vdash z{:}\sigma}\,(\cap E)}{\Gamma \vdash zz{:}\tau}\,(\to E)$$

$$\Gamma = \{z{:}(\sigma{\to}\tau)\cap\sigma\}$$

$$\dfrac{\dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau)\cap\sigma}{\Gamma \vdash z{:}\sigma{\to}\tau}\,(\cap E) \qquad \dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau)\cap\sigma}{\Gamma \vdash z{:}\sigma}\,(\cap E)}{\dfrac{\Gamma \vdash zz{:}\tau}{\vdash \lambda z.zz{:}(\sigma{\to}\tau)\cap\sigma{\to}\tau}\,(\to I)}\,(\to E)$$

$$\Gamma = \{z{:}(\sigma{\to}\tau){\cap}\sigma\}$$

$$\dfrac{\dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau){\cap}\sigma}{\Gamma \vdash z{:}\sigma{\to}\tau}\,(\cap E) \qquad \dfrac{\Gamma \vdash z{:}(\sigma{\to}\tau){\cap}\sigma}{\Gamma \vdash z{:}\sigma}\,(\cap E)}{\dfrac{\Gamma \vdash zz{:}\tau}{\vdash \lambda z.zz{:}(\sigma{\to}\tau){\cap}\sigma{\to}\tau}\,({\to}I)}\,({\to}E)$$

ALL AND ONLY THE STRONGLY NORMALIZING λ-TERMS ARE TYPABLE

INTERSECTION TYPES FOR Λx-CALCULUS: first attempt

$$(Ax)\ \frac{}{\Gamma \vdash x:\sigma}\ (x:\sigma \in \Gamma)$$

$$(\rightarrow I)\ \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \rightarrow \tau} \qquad (\rightarrow E)\ \frac{\Gamma \vdash M:\sigma \rightarrow \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}$$

$$(\cap I)\ \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma \cap \tau} \qquad (\cap E)\ \frac{\Gamma \vdash M:\sigma_1 \cap \sigma_2}{\Gamma \vdash M:\sigma_i}\ (i \in \{1,2\})$$

$$(cut\ \mathbf{I})\ \frac{\Gamma, x:\sigma \vdash M:\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M\langle x = N\rangle:\tau}$$

$$\Gamma = \{y{:}\rho, z{:}(\sigma \to \tau) \cap \sigma\}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash z : (\sigma \to \tau) \cap \sigma}{\Gamma \vdash z : \sigma \to \tau}\,(\cap E) \quad \dfrac{\Gamma \vdash z : (\sigma \to \tau) \cap \sigma}{\Gamma \vdash z : \sigma}\,(\cap E)}{\Gamma \vdash zz : \tau}\,(\to E)}{\{y{:}\rho\} \vdash \lambda z.zz : (\sigma \to \tau) \cap \sigma \to \tau}\,(\to I)$$

$\Gamma = \{y{:}\rho, z{:}(\sigma\to\tau)\cap\sigma\}$

$$\dfrac{\{y{:}\rho, x{:}(\sigma\to\tau)\cap\sigma\}\vdash y:\rho \qquad \dfrac{\dfrac{\dfrac{\Gamma\vdash z:(\sigma\to\tau)\cap\sigma}{\Gamma\vdash z:\sigma\to\tau}\,(\cap E) \qquad \dfrac{\Gamma\vdash z:(\sigma\to\tau)\cap\sigma}{\Gamma\vdash z:\sigma}\,(\cap E)}{\Gamma\vdash zz:\tau}\,(\to E)}{\{y{:}\rho\}\vdash \lambda z.zz:(\sigma\to\tau)\cap\sigma\to\tau}\,(\to I)}{\{y{:}\rho\}\vdash y\langle x=\lambda z.zz\rangle:\rho}\,(\mathit{cut}\ \mathbf{I})$$

$$\Gamma = \{y{:}\rho, z{:}(\sigma\to\tau)\cap\sigma\}$$

$$\dfrac{\{y{:}\rho, x{:}(\sigma\to\tau)\cap\sigma\} \vdash y : \rho \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash z : (\sigma\to\tau)\cap\sigma}{\Gamma \vdash z : \sigma\to\tau}\,(\cap E) \qquad \dfrac{\Gamma \vdash z : (\sigma\to\tau)\cap\sigma}{\Gamma \vdash z : \sigma}\,(\cap E)}{\Gamma \vdash zz : \tau}\,(\to E)}{\{y{:}\rho\} \vdash \lambda z.zz : (\sigma\to\tau)\cap\sigma\to\tau}\,(\to I)}{\{y{:}\rho\} \vdash y\langle x = \lambda z.zz\rangle : \rho}\,(cut\ \mathbf{I})$$

ALL THE TYPABLE Λx-TERMS ARE STRONGLY NORMALIZING

$$\Gamma = \{y{:}\rho, z{:}(\sigma\to\tau)\cap\sigma\}$$

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash z:(\sigma\to\tau)\cap\sigma}{\Gamma\vdash z:\sigma\to\tau}(\cap E) \quad \dfrac{\Gamma\vdash z:(\sigma\to\tau)\cap\sigma}{\Gamma\vdash z:\sigma}(\cap E)}{\Gamma\vdash zz:\tau}(\to E)}{}$$

$$\dfrac{\{y{:}\rho, x{:}(\sigma\to\tau)\cap\sigma\}\vdash y:\rho \qquad \dfrac{\Gamma\vdash zz:\tau}{\{y{:}\rho\}\vdash \lambda z.zz:(\sigma\to\tau)\cap\sigma\to\tau}(\to I)}{\{y{:}\rho\}\vdash y\langle x=\lambda z.zz\rangle:\rho}(cut\ \mathbf{I})$$

ALL THE TYPABLE Λx-TERMS ARE STRONGLY NORMALIZING

but there are strongly normalizing Λx-terms

like $y\langle t=xx\rangle\langle x=\lambda z.zz\rangle$ which are not typable

INTERSECTION TYPES FOR Λx-CALCULUS

$$(Ax)\quad \frac{}{\Gamma \vdash x:\sigma}\ (x:\sigma \in \Gamma)$$

$$(\to I)\quad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma\to\tau}$$

$$(\to E)\quad \frac{\Gamma \vdash M:\sigma\to\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}$$

$$(\cap I)\quad \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma\cap\tau}$$

$$(\cap E)\quad \frac{\Gamma \vdash M:\sigma_1\cap\sigma_2}{\Gamma \vdash M:\sigma_i}\ (i \in \{1,2\})$$

$$(cut\ \mathbf{I})\quad \frac{\Gamma, x:\sigma \vdash M:\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M\langle x = N\rangle:\tau}$$

$$(cut\ \mathbf{K})\quad \frac{\Gamma \vdash M:\tau \quad \Delta \vdash N:\sigma}{\Gamma \vdash M\langle x = N\rangle:\tau}\ (x \notin \Gamma)$$

$$\Gamma = \{x{:}(\mu{\to}\nu)\cap\mu\}$$

$$\dfrac{\dfrac{\Gamma \vdash x{:}(\mu{\to}\nu)\cap\mu}{\Gamma \vdash x{:}\mu{\to}\nu} \qquad \dfrac{\Gamma \vdash x{:}(\mu{\to}\nu)\cap\mu}{\Gamma \vdash x{:}\mu}}{\Gamma \vdash xx{:}\nu}$$

$$\Gamma = \{x{:}(\mu{\to}\nu)\cap\mu\}$$

$$\dfrac{\{y{:}\rho\}\vdash y:\rho \qquad \dfrac{\dfrac{\Gamma\vdash x:(\mu{\to}\nu)\cap\mu}{\Gamma\vdash x:\mu{\to}\nu} \qquad \dfrac{\Gamma\vdash x:(\mu{\to}\nu)\cap\mu}{\Gamma\vdash x:\mu}}{\Gamma\vdash xx:\nu}}{\{y{:}\rho\}\vdash y\langle t = xx\rangle:\rho}\ (\textit{cut}\ \mathbf{K})$$

$$\Gamma = \{x{:}(\mu{\to}\nu)\cap\mu\}$$

$$\dfrac{\{y{:}\rho\}\vdash y:\rho \qquad \dfrac{\dfrac{\Gamma\vdash x:(\mu{\to}\nu)\cap\mu}{\Gamma\vdash x:\mu{\to}\nu} \quad \dfrac{\Gamma\vdash x:(\mu{\to}\nu)\cap\mu}{\Gamma\vdash x:\mu}}{\Gamma\vdash xx:\nu}\ (\textit{cut}\,\mathbf{K}) \qquad \dfrac{\vdots}{\vdash \lambda z.zz:(\sigma{\to}\tau)\cap\sigma{\to}\tau}}{\dfrac{\{y{:}\rho\}\vdash y\langle t = xx\rangle:\rho \qquad \vdash \lambda z.zz:(\sigma{\to}\tau)\cap\sigma{\to}\tau}{\{y{:}\rho\}\vdash y\langle t = xx\rangle\,\langle x = \lambda z.zz\rangle:\rho}\ (\textit{cut}\,\mathbf{K})}$$

$$\Gamma = \{x{:}(\mu\to\nu)\cap\mu\}$$

$$\dfrac{\dfrac{\{y{:}\rho\}\vdash y:\rho \qquad \dfrac{\dfrac{\Gamma\vdash x:(\mu\to\nu)\cap\mu}{\Gamma\vdash x:\mu\to\nu} \qquad \dfrac{\Gamma\vdash x:(\mu\to\nu)\cap\mu}{\Gamma\vdash x:\mu}}{\Gamma\vdash xx:\nu}}{\{y{:}\rho\}\vdash y\langle t=xx\rangle:\rho}\,(\textit{cut}\ \mathbf{K}) \qquad \dfrac{\vdots}{\vdash \lambda z.zz:(\sigma\to\tau)\cap\sigma\to\tau}}{\{y{:}\rho\}\vdash y\langle t=xx\rangle\,\langle x=\lambda z.zz\rangle:\rho}\,(\textit{cut}\ \mathbf{K})$$

ALL AND ONLY THE STRONGLY NORMALIZING Λx-TERMS ARE TYPABLE

THE SET OF PROPER FREE VARIABLES

$$\begin{aligned} \textit{pfv}(x) &= \{x\} \\ \textit{pfv}(\lambda x.M) &= \textit{pfv}(M) \backslash x \\ \textit{pfv}(MN) &= \textit{pfv}(M) \cup \textit{pfv}(N) \end{aligned}$$

THE SET OF PROPER FREE VARIABLES

$$
\begin{array}{rcl}
\textit{pfv}(x) & = & \{x\} \\
\textit{pfv}(\lambda x.M) & = & \textit{pfv}(M)\backslash x \\
\textit{pfv}(M N) & = & \textit{pfv}(M) \cup \textit{pfv}(N) \\
\textit{pfv}(M\langle x = N\rangle) & = & \begin{cases} \textit{pfv}(M)\backslash x \cup \textit{pfv}(N) & \text{if } x \in \textit{pfv}(M) \\ \textit{pfv}(M) & \text{otherwise} \end{cases}
\end{array}
$$

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\end{array}
$$

$$
\textit{pfv}(y) = \{y\}
$$

THE SET OF PROPER FREE VARIABLES

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\begin{aligned}
\textit{pfv}(x) &= \{x\} \\
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\end{aligned}
$$

$$
\begin{aligned}
\textit{pfv}(y) &= \{y\} \\
\textit{pfv}(y\langle t = xx\rangle) &= \{y\}
\end{aligned}
$$

THE SET OF PROPER FREE VARIABLES

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\textit{pfv}(y\langle t = xx\rangle) & = & \{y\} \\
\textit{pfv}(y\langle t = xx\rangle\langle x = \lambda z.zz\rangle) & = & \{y\}
\end{array}
$$

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a new reduction rule: garbage collection

$$(\text{gc}_p) \qquad M\langle x = N\rangle \to M \text{ if } x \notin \textit{pfv}(M)$$

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$$

$$
\begin{array}{rcl}
\textit{pfv}(y) & = & \{y\} \\
\textit{pfv}(y\langle t = xx\rangle) & = & \{y\} \\
\textit{pfv}(y\langle t = xx\rangle\langle x = \lambda z.zz\rangle) & = & \{y\}
\end{array}
$$

a new reduction rule: garbage collection

(gc_p) $\quad M\langle x = N\rangle \to M$ if $x \notin \textit{pfv}(M)$

$$y\langle t = xx\rangle\langle x = \lambda z.zz\rangle \quad \to \quad y\langle t = xx\rangle$$

OUR RESULT: A TYPE SYSTEM CHARACTERING THE STRONGLY NORMALISING Λx-TERMS

further work:

- characterize other computational properties of substitution calculi by means of intersection types;
- build semantic models of substitution calculi by means of intersection types;
- consider substitution calculi with composition of substitutions.