Retractions in Intersection Types*

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This paper deals with retraction - intended as isomorphic embedding - in intersection types building left and right inverses as terms of a λ-calculus with a $\bot$ constant. The main result is a necessary and sufficient condition two strict intersection types must satisfy in order to assure the existence of two terms showing the first type to be a retract of the second one. Moreover, the characterisation of retraction in the standard intersection types is discussed.

1 Introduction

Isomorphism of types has been first discussed in the seminal paper [7] and then studied in various type disciplines [6, 8–11, 14–17, 23, 24]. Two types σ and τ in some typed calculus are isomorphic if there are two terms L and R of types $\tau \to \sigma$ and $\sigma \to \tau$, respectively, such that the composition $L \circ R$ is equal to the identity at type σ and the composition $R \circ L$ is equal to the identity at type τ.

We claim that in programming practice and theory the notion of *retraction* between types plays a central role, and it is more widespread than that of type isomorphism. Type σ is a retract of type τ in some typed calculus if there are two terms L and R of types $\tau \to \sigma$ and $\sigma \to \tau$, respectively, such that the composition $L \circ R$ is equal to the identity at type σ. Clearly L is *right invertible* and R is *left invertible*. The terms R and L are injective and surjective, respectively, on their domains. They are also called the *coder* and the *decoder* of type σ in type τ [20, 22]. In fact, the term R encodes the values in σ as elements of τ, while the term L decodes back from τ to σ, by returning the original values.

To the best of our knowledge, type retraction as defined above has been discussed for Curry types and higher-order types [7, 13, 20–22, 25, 26], but not for intersection types. Aim of this paper is to make a first step toward the filling of this gap. We consider the $\lambda\bot$-calculus as given in [2, Definition 14.3.1]. For this calculus the terms having left and right inverses have been characterised [19]. We reformulate this characterisation in order to simplify the study of the types which can be derived for these terms. In particular we identify a class of right inverses (the *simple right inverses*) such that if a term has a right inverse it has also a right inverse belonging to this class. All results in this paper are given for the $\lambda\bot$-calculus but they hold, as well, for the λ-calculus.

We choose to investigate retraction for the essential intersection type assignment system introduced in [1]. This system has the same typeability power of the standard system [3] and a less permissive type syntax. This restriction better fits the technical development of the present paper, as discussed in the Conclusion. Nevertheless, we also provide a result for standard intersection types in the case where the right inverse is assumed to be a simple right inverse.

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The first contribution of this paper is the characterisation of the strict intersection types which can be derived for terms having left or right inverses. Building on this result we give a necessary and sufficient condition for the existence of a retraction between two strict intersection types. This condition is a generalisation of the one defined in [7] for Curry types. We show that each retraction can be witnessed by a simple right inverse. We also prove that if μ is a retract of ν and ν is a retract of μ, then μ and ν are equivalent with respect to the usual subtyping relation of intersection types. Then we discuss retraction in standard intersection types. Finally we define *semantic retraction* as the natural adaptation to models and we show that retraction and semantic retraction coincide. This proof uses the completeness of the filter model given in [3].

Outline Section 2 introduces the $\lambda\bot$-calculus and characterises terms having left and right inverses. The essential intersection type assignment system is defined in Section 3 together with the characterisation of the types derivable for terms with left or right inverses. The results on retraction in strict intersection types are the content of Section 4. In Section 5 we extend the characterisation of retractions to standard intersection types assuming that right inverses are simple. The notion of retraction is shown equivalent to that of semantic retraction in Section 6. Related work is overviewed in Section 7. Section 8 discusses our choices and future work.

2 Left/Right Invertible Terms

Following [2, Definition 14.3.1], $\Lambda\bot$ is the set of terms obtained by adding a constant $\bot$ to the formation rules of λ-terms. The terms of $\Lambda\bot$ are generated by the syntax

$$M ::= x \mid \bot \mid \lambda x.M \mid MM$$

where x ranges over a denumerable set of term variables.

The reduction rules of the $\lambda\bot$-calculus include the β-rule and two rules for $\bot$ prescribing that both application and abstraction of $\bot$ reduce to $\bot$.

$$(\lambda x.M)N \longrightarrow M\{N/x\} \qquad \bot M \longrightarrow \bot \qquad \lambda x.\bot \longrightarrow \bot$$

The equality between terms is defined as $\beta\bot$-conversion, i.e. $M = N$ means that there is a term P such that both M and N reduce to P.

Let $\mathbf{I} = \lambda x.x$ and $M \circ N = \mathbf{B}MN$, where $\mathbf{B} = \lambda xyz.x(yz)$. We are interested in investigating the monoid of terms with the combinator $\mathbf{I}$ as identity element and $\circ$ as binary operation. This naturally leads to the following definition of left/right invertibility.

Definition 2.1 (Left/right invertibility)**.**

1. *A term M is* left invertible *if there exists a term L such that $L \circ M = \mathbf{I}$. We say that L is a* left inverse *of M.*
2. *A term M is* right invertible *if there exists a term R such that $M \circ R = \mathbf{I}$. We say that R is a* right inverse *of M.*

The left/right invertibility has been studied since the seventies. In particular the sets of terms having at least one left or one right inverse have been characterised in [5] and [19].

The characterisation of left invertible terms resorts to a set of head normal forms (hnfs for short). We recall that a hnf is a term of the shape $\lambda x_1 \dots x_n.x_j M_1 \dots M_m$ [2, Definition 2.2.11]. We define a set Ξ of hnfs and we show that a term is left invertible iff it reduces to a hnf belonging to Ξ (Theorem 2.8).

Definition 2.2. *Let Ξ be the set of hnfs inductively defined as follows:*

- $\lambda t x_1 \dots x_n.t \in \Xi$ *for* $n \geq 0$;
- *if* $\lambda t x_1 \dots x_n.M_i \in \Xi$, *then* $\lambda t x_1 \dots x_n.x_j M_1 \dots M_i \dots M_m \in \Xi$, *where* $1 \leq j \leq n$ *and* $1 \leq i \leq m$.

Example 2.3.

i) The term $\lambda t x.xx(xt) \in \Xi$ because $\lambda t x.xt \in \Xi$, and in turn $\lambda t x.xt \in \Xi$ since $\lambda t x.t \in \Xi$.

ii) The term $M = \lambda t x_1 x_2.x_2(\lambda x_3.t)(x_1 t) \in \Xi$ since $\lambda t x_1 x_2 x_3.t \in \Xi$. We can also show that $M \in \Xi$ because $\lambda t x_1 x_2.x_1 t \in \Xi$ and in turn $\lambda t x_1 x_2.x_1 t \in \Xi$ since $\lambda t x_1 x_2.t \in \Xi$.

Point *ii)* of Example 2.3 shows that there is not, in general, a unique way to derive that a term belongs to Ξ. In the following, when we write $M \in \Xi$, we refer to a particular proof, chosen according to Definition 2.2.

In order to build a left inverse of a term in Ξ we need to "reach" an occurrence of the first abstracted variable (called t in Definition 2.2). We use some machinery inspired by the Böhm-out technique [2, §10.3]. In a hnf $\lambda x_1 \dots x_n.x_j M_1 \dots M_m$ the number of initial abstractions is n, the variable x_j (bound in the j-th abstraction) is the head variable and $M_1, \dots, M_m$ are the m components. Following the definition of Ξ we can associate with each term in Ξ a list of integer triples, whose first element is the abstraction position of the head variable, whose second element is the number of components and whose third element is the position of the component used to show that the term belongs to Ξ (0 if the component is missing). More precisely, using $\frown$ to denote concatenation:

Definition 2.4. *The* path $\pi(M)$ *of the hnf $M \in \Xi$ is inductively defined by:*

- $\pi(\lambda t x_1 \dots x_n.t) = \langle 1,0,0 \rangle$;
- $\pi(\lambda t x_1 \dots x_n.x_j M_1 \dots M_m) = \langle j+1, m, i \rangle \frown \pi(\lambda t x_1 \dots x_n.M_i)$ *if* $\lambda t x_1 \dots x_n.M_i \in \Xi$ *is used to show* $\lambda t x_1 \dots x_n.x_j M_1 \dots M_m \in \Xi$.

Let p range over paths.

Example 2.5. *We get $\pi(\lambda t x.xx(xt)) = \langle 2,2,2 \rangle \frown \langle 2,1,1 \rangle \frown \langle 1,0,0 \rangle$. If M is defined as in Point ii) of Example 2.3 we get either $\pi(M) = \langle 3,2,1 \rangle \frown \langle 1,0,0 \rangle$ or $\pi(M) = \langle 3,2,2 \rangle \frown \langle 2,1,1 \rangle \frown \langle 1,0,0 \rangle$, according to the proof used to show $M \in \Xi$.*

The triple $\langle j+1, m, i \rangle$ says that the variable bound in the $j+1$-th abstraction must choose the i-th component out of m components to "reach" an occurrence of t. Then the variable bound in the $j+1$-th abstraction needs to be replaced by the term $S_i^{(m)} = \lambda y_1 \dots y_m.y_i$. We call *selectors* the terms of the shown shape. This replacement becomes problematic if we have in the same path two triples with the same first element which differ in one of the other elements. Following [19] we differentiate these occurrences using terms of the shape $P^{(m)} = \lambda z_1 \dots z_{m+1}.z_{m+1} z_1 \dots z_m$ with $m \geq 1$ that we dub *permutators*. We convene that $P^{(0)} = \bot$. We need two preliminary definitions and a technical lemma.

By $\ell(\mathsf{p})$ we denote the length, i.e. the number of triples, of the path p.

We define $\#(j+1, \mathsf{p})$ as the maximum of the second components of triples in path p whose first component is $j+1$. We assume $\#(j+1, \mathsf{p}) = 0$ if $j+1$ does not occur in p as first component. More formally:

$$\#(j+1, \langle 1,0,0 \rangle) = 0 \qquad\qquad \#(j+1, \langle h,m,i \rangle \frown \mathsf{p}) = \begin{cases} \max(m, \#(j+1, \mathsf{p})) & \text{if } h = j+1, \\ \#(j+1, \mathsf{p}) & \text{otherwise} \end{cases}$$

Lemma 2.6. *Let $M = \lambda t x_1 \dots x_n.M' \in \Xi$ and $m_j \geq \#(j+1, \pi(M))$ for $1 \leq j \leq n$. Then*

$$\lambda t.M'\{P^{(m_1)}/x_1\}\dots\{P^{(m_n)}/x_n\} = Q \in \Xi$$

and $\ell(\pi(Q)) = \ell(\pi(M))$.

Proof. The proof is by induction on the definition of Ξ. If $M' = \lambda x_{n+1} \dots x_q.t$, then $Q = \lambda t x_{n+1} \dots x_q.t$. Let $M' = \lambda x_{n+1} \dots x_q.x_r M_1 \dots M_i \dots M_m$ and $M \in \Xi$ since $\lambda t x_1 \dots x_q.M_i \in \Xi$. Because $m_j \geq \#(j+1, \pi(M))$ implies $m_j \geq \#(j+1, \pi(\lambda t x_1 \dots x_q.M_i))$ for $1 \leq j \leq n$, by induction we get

$$\lambda t x_{n+1} \dots x_q.M_i\mathsf{s} = \lambda t x_{n+1} \dots x_q.Q' \in \Xi$$

where s is the substitution $\{P^{(m_1)}/x_1\}\dots\{P^{(m_n)}/x_n\}$. If $r \geq n+1$ we can take

$$Q = \lambda t x_{n+1} \dots x_q.x_r M_1\mathsf{s} \dots M_{i-1}\mathsf{s}Q'M_{i+1}\mathsf{s}\dots M_m\mathsf{s}$$

Otherwise

$$\begin{array}{rcl} M'\mathsf{s} & = & \lambda t x_{n+1} \dots x_q.P^{(m_r)}M_1\mathsf{s}\dots M_{i-1}\mathsf{s}Q'M_{i+1}\mathsf{s}\dots M_m\mathsf{s} \\ & = & \lambda t x_{n+1} \dots x_q z_{m+1} \dots z_{m_r+1}.z_{m_r+1}M_1\mathsf{s}\dots M_{i-1}\mathsf{s}Q'M_{i+1}\mathsf{s}\dots M_m\mathsf{s}z_{m+1}\dots z_{m_r} \end{array}$$

and we can take this last hnf as Q since $\lambda t x_{n+1} \dots x_q.Q' \in \Xi$ implies $\lambda t x_{n+1} \dots x_q z_{m+1} \dots z_{m_r+1}.Q' \in \Xi$.

In all cases it is easy to verify that $\ell(\pi(Q)) = \ell(\pi(M))$. □

Example 2.7. *Let $M = \lambda t x.M'$ where $M' = xx(xt)$. We get*

$$\lambda t.M'\{P^{(2)}/x\} = \lambda t z_1.z_1(\lambda z_2 z_3 z_4.z_4 z_2 z_3)(\lambda z_5 z_6.z_6 t z_5) = Q$$

We have $\pi(Q) = \langle 2,2,2\rangle^\frown\langle 4,2,1\rangle^\frown\langle 1,0,0\rangle$.

Theorem 2.8. *A term has at least one left inverse if and only if it reduces to a hnf M in Ξ.*

Proof.

(If) Let $M \in \Xi$. The proof is by induction on $\ell(\pi(M))$. Let n be the number of initial abstractions and p be the path of the term considered in the induction step. We build a left inverse of the shape $\lambda z.zL_1 \dots L_q$, where $q \geq n$ and $L_l = \bot$ whenever $\#(l+1, \mathsf{p}) = 0$.

If $M = \lambda t x_1 \dots x_n.t$, then $\lambda z.z\underbrace{\bot\bot\cdots\bot}_{n}$ is a left inverse of M.

Let $M = \lambda t x_1 \dots x_n.x_j M_1 \dots M_i \dots M_m \in \Xi$ since $\lambda t x_1 \dots x_n.M_i \in \Xi$. We distinguish two cases. In the first case the construction of the left inverse using a selector is easy. In the second case we compose M with a term N build out of permutators. The useful property is that $N \circ M = Q \in \Xi$ and $\ell(\pi(Q)) = \ell(\pi(M))$ and Q satisfies the condition of case 1. We can then build a left inverse L of Q and $L \circ N$ is a left inverse of M.

Case 1: $\#(j+1, \pi(\lambda t x_1 \dots x_n.M_i)) = 0$. By induction hypothesis $\lambda t x_1 \dots x_n.M_i \in \Xi$ has a left inverse $\lambda z.zL'_1 \dots L'_q$ and in this case $L'_j = \bot$. Then $\lambda z.zL'_1 \dots L'_{j-1} S_i^{(m)} L'_{j+1} \dots L'_q$ is a left inverse of M.

Case 2: $\#(j+1, \pi(\lambda t x_1 \dots x_n.M_i)) = m_j \neq 0$. Let $m_l = \#(l+1, \pi(M))$ for $1 \leq l \leq j$ and

$$N = \lambda z.zP^{(m_1)} \dots P^{(m_j)}$$

By the proof of Lemma 2.6 $N \circ M = Q \in \Xi$, where Q is the hnf

$$\lambda t x_{j+1} \dots x_n z_{m+1} \dots z_{m_j+1}.z_{m_j+1}M_1\mathsf{s}\dots M_m\mathsf{s}z_{m+1}\dots z_{m_j}$$

and s is the substitution $\{P^{(m_1)}/x_1\}\dots\{P^{(m_j)}/x_j\}$. Since z_{m_j+1} does not occur in $M_i\mathsf{s}$, i.e.

$$\#(n-j+m_j-m+2,\pi(\lambda t x_{j+1}\dots x_n z_{m+1}\dots z_{m_j+1}.M_i\mathsf{s}))=0$$

and $\ell(\pi(Q))=\ell(\pi(M))$ we can build a left inverse L of Q according to previous case. Then a left inverse of M is $L\circ N$.

(Only if) Let us suppose, ad absurdum, that a term has a left inverse and it is unsolvable or its hnf doesn't belong to Ξ. The first case is obvious. In the second case the hnf M has no path which satisfies Definition 2.4. Therefore, if $M=\lambda t.N$, then there is no occurrence of t in N which is not applied and such that it is always in components whose head variables are bound. The arguments of t cannot be erased by reduction and a free variable cannot be replaced in order to get t. So we conclude that M has no left inverse. □

Example 2.9. *Let M,Q be as in Example 2.7. The left inverse of Q built according to the lemma is $L=\lambda z.z(\lambda y_1y_2.y_2)\bot(\lambda y_1y_2.y_1)$. According to the proof of previous theorem we get $N=\lambda z.zP^{(2)}$. Then a left inverse of M is $L\circ N=\lambda z.zP^{(2)}(\lambda y_1y_2.y_2)\bot(\lambda y_1y_2.y_1)$.*

The characterisations of terms having right inverses is easy.

Theorem 2.10. *A term has at least one right inverse if and only if its hnf is of the shape: $\lambda z.zM_1\dots M_m$.*

Proof.

(If) A right inverse is $\lambda t x_1\dots x_m.t$.

(Only if) An unsolvable term has no left inverse. Let suppose the hnf of a term be not of the shape $\lambda z.zM_1\dots M_m$. Then it must have more than one abstraction and/or the head variable must be a free variable. In the first case the initial abstractions and in the second case the head free variable cannot be eliminated using reductions. □

Example 2.11. *The term M of Example 2.7 is a right inverse of the term $L\circ N$ of Example 2.9. The right inverse of $L\circ N$ built by the theorem is $\lambda t x_1x_2x_3.t$.*

From the proof of Theorem 2.10 it is clear that if a term has a right inverse, then it has also a right inverse of the shape $\lambda t x_1\dots x_n.t$, i.e. a selector $S_1^{(n+1)}$. We call *simple right inverses* the hnfs of this shape.

3 Strict Intersection Types

The type system considered in this paper is a notational variant of the essential intersection assignment introduced in [1].

The set of *strict intersection types* is defined by:

$$\begin{array}{rcl}\mu & := & \varphi \mid \omega \mid \sigma\to\mu\\ \sigma & := & \mu \mid \sigma\wedge\sigma\end{array}$$

where φ ranges over type variables and ω is a constant. We convene that μ,ν range over strict intersection types (either atomic or arrow types), while σ,τ,ρ range over intersections.
Conventionally, we omit parentheses according to the precedence rule "$\wedge$ over $\to$" and we assume that $\to$ associates to the right. Intersections are considered modulo idempotence, commutativity and associativity of $\wedge$. In this section and in the following one we use type as short for strict intersection type.

A preorder relation $\leq$, representing set inclusion, is assumed between types and intersections.

Definition 3.1. *Let $\leq$ be the minimal reflexive and transitive relation such that:*

$$\sigma \leq \omega \qquad \omega \leq \omega \to \omega \qquad \sigma_1 \wedge \sigma_2 \ \leq \ \sigma_i \quad (i = 1,2)$$
$$\sigma_1 \leq \tau_1 \textit{ and } \sigma_2 \leq \tau_2 \textit{ imply } \sigma_1 \wedge \sigma_2 \ \leq \ \tau_1 \wedge \tau_2$$
$$\sigma_2 \leq \sigma_1 \textit{ and } \mu_1 \leq \mu_2 \textit{ imply } \sigma_1 \to \mu_1 \leq \sigma_2 \to \mu_2$$

We write $\sigma \sim \tau$ if $\sigma \leq \tau$ and $\sigma \geq \tau$.

A key property of this subtyping is the content of the following lemma, for a proof see [1].

Lemma 3.2. *If $\sigma \to \mu \leq \tau \to \nu$, then $\tau \leq \sigma$ and $\mu \leq \nu$.*

The essential intersection type assignment system is defined by the typing rules of Table 1. We assume that an environment associates intersections with a finite number of term variables. Let Γ range over environments. The subsumption rule uses the preorder of Definition 3.1. We write $\Gamma \vdash N : \sigma$ with $\sigma = \bigwedge_{i \in I} \mu_i$ as short for $\Gamma \vdash N : \mu_i$ for all $i \in I$.

$$(Ax) \quad \Gamma, x : \textstyle\bigwedge_{i \in I} \mu_i \vdash x : \mu_j \quad j \in I \qquad (\omega) \quad \Gamma \vdash M : \omega \qquad (\leq) \quad \frac{\Gamma \vdash M : \mu \quad \mu \leq \nu}{\Gamma \vdash M : \nu}$$

$$(\to I) \quad \frac{\Gamma, x : \sigma \vdash M : \mu}{\Gamma \vdash \lambda x.M : \sigma \to \mu} \qquad (\to E) \quad \frac{\Gamma \vdash M : \sigma \to \mu \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \mu}$$

Table 1: Typing Rules

The inversion lemma is as expected, for a proof see [1].

Lemma 3.3 (Inversion Lemma)**.**

1. *If $\Gamma \vdash x : \mu$, then either $\mu \sim \omega$ or $x : \sigma \in \Gamma$ and $\sigma \leq \mu$.*
2. *If $\Gamma \vdash \bot : \mu$, then $\mu \sim \omega$.*
3. *If $\Gamma \vdash MN : \mu$, then $\Gamma \vdash M : \sigma \to \mu$ and $\Gamma \vdash N : \sigma$.*
4. *If $\Gamma \vdash \lambda x.M : \mu$, then $\mu \sim \sigma \to \nu$ and $\Gamma, x : \sigma \vdash M : \nu$.*

In this system types are preserved by $\beta\bot$-conversion [1]:

Theorem 3.4 (Subject Conversion)**.** *If $\Gamma \vdash M : \mu$ and $M = N$, then $\Gamma \vdash N : \mu$.*

We say that *a term inhabits a type* if we can derive the type for the term starting from the empty environment. A term inhabits a set of types if it inhabits all the types belonging to the set. Let $\tau = \bigwedge_{i \in I} \mu_i$: we say that $\sigma \to \tau$ is inhabited if there exists a term which inhabits all the types $\sigma \to \mu_i$ for $i \in I$.

Inhabitation for intersection types has been shown undecidable in general [27], but decidable for types with *rank* less than or equal to 2 [28], when the rank of types and intersections is defined by:

$$\begin{aligned} \mathrm{rank}(\mu) &= \begin{cases} \max(\mathrm{rank}(\sigma)+1, \mathrm{rank}(\nu)) & \text{if } \mu = \sigma \to \nu \text{ and } \wedge \text{ occurs in } \mu, \\ 0 & \text{otherwise} \end{cases} \\ \mathrm{rank}(\sigma \wedge \tau) &= \max(1, \mathrm{rank}(\sigma), \mathrm{rank}(\tau)) \end{aligned}$$

In the following we characterise the types of left/right invertible terms. These characterisations require inhabitation of some types and therefore they are effective only when these types are of rank at most 2.

We start defining inductively the set Θ of left types which mimic the set Ξ of hnfs, that is the construction of Θ follows the construction of Ξ. In the following definition $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \tau \in \Theta$, where $\tau = \bigwedge_{i\in I} \mu_i$, is used as short for $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \mu_i \in \Theta$ for all $i \in I$.

Definition 3.5. *The set Θ of* left types *is inductively defined by:*

- *if $\sigma \leq \nu$, then $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \nu \in \Theta$;*
- *if $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_i \in \Theta$ and $\sigma_j \leq \rho_1 \to \ldots \to \rho_m \to \nu$ for some $j \leq n$ and $i \leq m$ and $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_l$ is inhabited for $1 \leq l \leq m$, then $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \nu \in \Theta$.*

Example 3.6.

i) Let $\tau = \psi \wedge (\varphi \to \varphi') \wedge (\psi \to \varphi' \to \psi')$. We have $\varphi \to \tau \to \psi' \in \Theta$ since:

- *$\varphi \to \tau \to \varphi' \in \Theta$*
- *$\tau \leq \psi \to \varphi' \to \psi'$ and*
- *both $\varphi \to \tau \to \psi$ and $\varphi \to \tau \to \varphi'$ are inhabited.*

Moreover $\varphi \to \tau \to \varphi' \in \Theta$ since:

- *$\varphi \to \tau \to \varphi \in \Theta$*
- *$\tau \leq \varphi \to \varphi'$ and*
- *$\varphi \to \tau \to \varphi$ is inhabited.*

The type $\varphi \to \tau \to \psi'$ can be derived for the term M of Example 2.7.

ii) Let $\mu = (\varphi \to \psi) \to \psi \to \psi'$ and $\nu = \varphi \to \psi$. We have $\varphi \to \mu \to \nu \to \psi' \in \Theta$ since:

- *$\varphi \to \mu \to \nu \to \psi \in \Theta$*
- *$\mu \leq \mu$ and*
- *both $\varphi \to \mu \to \nu \to \nu$ and $\varphi \to \mu \to \nu \to \psi$ are inhabited.*

Moreover $\varphi \to \mu \to \nu \to \psi \in \Theta$ since:

- *$\varphi \to \mu \to \nu \to \varphi \in \Theta$*
- *$\nu \leq \nu$ and*
- *$\varphi \to \mu \to \nu \to \varphi$ is inhabited.*

The type $\varphi \to \mu \to \nu \to \psi'$ can be derived for the left invertible term $\lambda t x_1 x_2 . x_1 x_2 (x_2 t)$.

We define the number of top arrows of a type as expected:

$$\flat(\varphi) = \flat(\omega) = 0 \qquad\qquad \flat(\sigma \to \mu) = 1 + \flat(\mu)$$

It is useful to observe that if a type with at least n top arrows has an inhabitant, then this type has also an inhabitant with at least n initial abstraction.

Lemma 3.7. *If type μ is inhabited and $\flat(\mu) \geq n$, then there is M with at least n initial abstractions such that $\vdash M : \mu$.*

Proof. If $\mu \sim \omega$ it is trivial. Otherwise the inhabitants of μ must have hnfs, see [1] for a proof. Let $\mu = \sigma_1 \to \ldots \to \sigma_n \to \nu$ and $\lambda x_1 \ldots x_{n'}.x_j M_1 \ldots M_m$ be an inhabitant of μ with $n' < n$. It is easy to check that we get M by η-expansion (see [2, Definition 3.3.1])

$$\vdash \lambda x_1 \ldots x_{n'} y_1 \ldots y_{n-n'}.x_j M_1 \ldots M_m y_1 \ldots y_{n-n'} : \sigma_1 \to \ldots \to \sigma_n \to \nu \qquad \square$$

Lemma 3.8 (Characterisation of Types for Left Invertible Terms)**.**

1. *Left invertible terms inhabit only types which are left types.*
2. *Each left type is inhabited by a left invertible term.*

Proof.

1. By Definition 2.2 a left invertible term M is a λ-abstraction, then its type is of the shape $\sigma \to \nu$ by Lemma 3.3(4). It is enough to show that if the head normal form of M belongs to Ξ, then $\sigma \to \nu \in \Theta$. The proof is by induction on Ξ.
   Case $\lambda t x_1 \ldots x_n.t$. By the invariance of types under $\beta\bot$-conversion (Theorem 3.4) we get
   $$\vdash \lambda t x_1 \ldots x_n.t : \sigma \to \nu$$
   which implies by repeated application of Lemma 3.3(4) $\nu \sim \sigma_1 \to \ldots \to \sigma_n \to \mu$ and
   $$t : \sigma, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \mu$$
   Then $\sigma \leq \mu$ by Lemma 3.3(1) and we can conclude $\sigma \to \nu \in \Theta$.
   Case $\lambda t x_1 \ldots x_n.x_j M_1 \ldots M_m \in \Xi$ with $j \leq n$ since $\lambda t x_1 \ldots x_n.M_i \in \Xi$ with $i \leq m$. As in previous case we get $\nu \sim \sigma_1 \to \ldots \to \sigma_n \to \mu$ and
   $$t : \sigma, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_j M_1 \ldots M_m : \mu$$
   Let $\Gamma = t : \sigma, x_1 : \sigma_1, \ldots, x_n : \sigma_n$. By repeated application of Lemma 3.3(3) we have
   $$\Gamma \vdash x_j : \rho_1 \to \ldots \rho_m \to \mu \text{ and } \Gamma \vdash M_l : \rho_l \text{ for } 1 \leq l \leq m.$$
   Lemma 3.3(1) implies $\sigma_j \leq \rho_1 \to \ldots \rho_m \to \mu$. Moreover
   $$\lambda t x_1 \ldots x_n.M_l \text{ inhabits } \sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_l \text{ for } 1 \leq l \leq m.$$
   Lastly $\lambda t x_1 \ldots x_n.M_i \in \Xi$ implies by induction
   $$\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_i \in \Theta$$
   We can then conclude $\sigma \to \nu \in \Theta$.
2. The proof is by induction on Θ. If $\sigma \leq \nu$ we can derive $\vdash \lambda t x_1 \ldots x_n.t : \sigma \to \sigma_1 \to \ldots \to \sigma_n \to \nu$. Otherwise by Lemma 3.7 we can assume that the inhabitants of $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_l$ for $1 \leq l \leq m$ have at least $n+1$ initial abstractions. Let
   $$\lambda t x_1 \ldots x_n.M_l \text{ be an inhabitant of } \sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_l \text{ for } 1 \leq l \leq m \text{ and}$$
   $$\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \rho_i \in \Theta \text{ for some } i \leq m.$$
   By induction $\lambda t x_1 \ldots x_n.M_i \in \Xi$, then $\lambda t x_1 \ldots x_n.x_j M_1 \ldots M_i \ldots M_m \in \Xi$. Moreover

if $\sigma_j \leq \rho_1 \to \ldots \to \rho_m \to \nu$, then $\lambda t x_1 \ldots x_n . x_j M_1 \ldots M_i \ldots M_m$ inhabits $\sigma \to \sigma_1 \to \ldots \to \sigma_n \to \nu$.

□

The types of right invertible terms are easy to define, as expected.

Definition 3.9. *A type $\tau \to \mu$ is a* right type *if $\tau \leq \rho_1 \to \ldots \to \rho_m \to \mu$ and $\tau \to \rho_i$ is inhabited for $1 \leq i \leq m$.*

Example 3.10. *A right type is $((\psi_1 \to \psi_1) \wedge (\psi_2 \to \psi_2) \to \omega \to \varphi) \wedge \psi \to \varphi$. Another right type is*

$$((\varphi_1 \to \varphi_2 \to (\varphi_1 \to \varphi_2 \to \varphi_3) \to \varphi_3) \to (\psi_1 \to \psi_2 \to \psi_2) \to \omega \to (\psi_1 \to \psi_2 \to \psi_1) \to \varphi) \to \varphi$$

This last type can be derived for the term $L \circ N$ of Example 2.9.

Lemma 3.11 (Characterisation of Types for Right Invertible Terms)**.**

1. *Right invertible terms inhabit only types which are right types.*
2. *Each right type is inhabited by a right invertible term.*

Proof.

1. By Theorem 2.10 is it enough to show that $\vdash \lambda z.zM_1 \ldots M_m : \tau \to \mu$ implies that $\tau \to \mu$ is a right type. By Lemma 3.3(4) and (3) we get: $z : \tau \vdash z : \rho_1 \to \ldots \to \rho_m \to \mu$ and $z : \tau \vdash M_i : \rho_i$ for $1 \leq i \leq m$. By Lemma 3.3(1) $\tau \leq \rho_1 \to \ldots \to \rho_m \to \mu$. Moreover $\lambda z.zM_i$ inhabits $\tau \to \rho_i$ for $1 \leq i \leq m$. Therefore $\tau \to \mu$ is a right type.
2. Let M_i be an inhabitant of $\tau \to \rho_i$ for $1 \leq i \leq m$ and $\tau \leq \rho_1 \to \ldots \to \rho_m \to \mu$. Then we can derive $\vdash \lambda z.zM_1 \ldots M_m : \tau \to \mu$. □

It is easy to verify that ω is both a left and a right type.

4 Characterisation of Retraction in Strict Intersection Types

We can discuss now retractions, i.e. isomorphic embeddings, in strict types using terms of $\Lambda\bot$.

Definition 4.1. *Type μ is a* retract *of type ν (notation $\mu \lhd \nu$) if there exist terms L and R such that:*

1. $\vdash L : \nu \to \mu$;
2. $\vdash R : \mu \to \nu$;
3. $L \circ R = \mathbf{I}$.

We say that L, R witness the retraction.

Example 4.2. *$L = \lambda z.z\mathbf{I}(\lambda y.yy)z$ and $R = \lambda t x_1 x_2 x_3 . x_2 x_1 t$ witness the retraction*

$$\varphi \lhd (\varphi \to \varphi) \wedge ((\varphi \to \varphi) \to \varphi \to \varphi) \to \sigma \to \omega \to \varphi$$

where $\sigma = (\varphi \to \varphi) \wedge ((\varphi \to \varphi) \to \varphi \to \varphi) \to \varphi \to \varphi$. The same retraction is witnessed by L and $R' = \lambda t x_1 x_2 x_3 . t$, which is a simple right inverse. Notice that L cannot be typed with Curry types.

It easy to prove that the retraction relation enjoys the transitivity property. In fact if L, R witness $\mu \lhd \mu'$ and L', R' witness $\mu' \lhd \nu$, then $L \circ L', R' \circ R$ witness $\mu \lhd \nu$.

Retraction can be fully characterised.

Theorem 4.3 (Characterisation of Retraction)**.** *$\mu \lhd \nu$ if and only if $\nu \sim \rho_1 \to \ldots \to \rho_m \to \mu$ and $\nu \to \rho_i$ is inhabited for $1 \le i \le m$. Moreover, each retraction can be witnessed by a simple right inverse.*

Proof.

(If) Let M_i be an inhabitant of $\nu \to \rho_i$ for $1 \le i \le m$. We can choose $L = \lambda z.z(M_1 z)\ldots(M_m z)$ and $R = \lambda t x_1 \ldots x_m.t$. Notice that R is a simple right inverse. It it easy to verify that L and R satisfy the conditions of Definition 4.1.

(Only if) By Theorem 2.10 $L = \lambda z.zM_1 \ldots M_m$. Then applying Lemma 3.3(4) to $\vdash L : \nu \to \mu$ we get $z : \nu \vdash zM_1 \ldots M_m : \mu$. By repeated applications of Lemma 3.3(3) this implies

$$z : \nu \vdash z : \sigma_1 \to \ldots \to \sigma_m \to \mu \text{ and } z : \nu \vdash M_i : \sigma_i \text{ for } 1 \le i \le m.$$

From $z : \nu \vdash z : \sigma_1 \to \ldots \to \sigma_m \to \mu$ we have $\nu \le \sigma_1 \to \ldots \to \sigma_m \to \mu$ by Lemma 3.3(1). We can assume $\nu = \rho_1 \to \ldots \to \rho_m \to \mu'$, which implies $\sigma_i \le \rho_i$ for $1 \le i \le m$ and $\mu' \le \mu$ by Lemma 3.2. Observe that $L \circ R = \lambda x.RxM'_1 \ldots M'_m$ where $M'_i = M_i\{Rx/z\}$ for $1 \le i \le m$. From $\vdash R : \mu \to \nu$ and $z : \nu \vdash M_i : \sigma_i$ and $\sigma_i \le \rho_i$ we can derive $x : \mu \vdash M'_i : \rho_i$ for $1 \le i \le m$. This together with $\nu = \rho_1 \to \ldots \to \rho_m \to \mu'$ implies $x : \mu \vdash RxM'_1 \ldots M'_m : \mu'$. From $L \circ R = \mathbf{I}$ we get $RxM'_1 \ldots M'_m = x$. Subject Conversion derives $x : \mu \vdash x : \mu'$, so by Lemma 3.3(1) $\mu \le \mu'$. We conclude $\mu \sim \mu'$. □

As an easy consequence of this theorem if $\mu \lhd \nu$, then $\mu \lhd \rho \to \nu$ for any intersection ρ such that $\nu \to \rho$ is inhabited. Moreover if $\mu \not\sim \omega$, then neither $\mu \lhd \omega$ nor $\omega \lhd \mu$ can hold.

Reciprocal retraction implies equivalence.

Corollary 4.4. *If $\mu \lhd \nu$ and $\nu \lhd \mu$, then $\mu \sim \nu$.*

Proof. By previous theorem $\nu \sim \rho_1 \to \ldots \to \rho_m \to \mu$ and $\mu \sim \sigma_1 \to \ldots \to \sigma_n \to \nu$, which imply $m = n = 0$ and then $\mu \sim \nu$. □

Given two terms L and R such that $L \circ R = \mathbf{I}$, they do not witness a retraction for all the pairs of their types. For instance, the terms $L \circ N$, M in Examples 2.9 and 2.7 do not witness a retraction using their types shown in Examples 3.10 and 3.6 *i)*. However $L \circ N$, M witness the following retraction:

$$\mu \ \lhd \ (\mu \to \nu_1) \wedge (\omega \to \nu_1 \to \nu_2) \to \nu_2$$

where $\nu_1 = \omega \to (\mu \to \omega \to \mu) \to \mu$ and $\nu_2 = (\omega \to \nu_1 \to \nu_1) \to \nu_1$.

Notice that $L \circ R = \mathbf{I}$ implies that for all types μ there is an intersection σ such that L has type $\sigma \to \mu$ and R has type $\mu \to \sigma$. The proof is easy using the inversion lemma and the invariance of types under β-expansion. This does not mean that L, R always witness a retraction between strict types, since σ can be an intersection of strict types, as shown in the following example.

Example 4.5. *Let $L = \lambda u.u\bot(u\bot(\lambda z.\boldsymbol{I}))$ and $R = \lambda t x_1 x_2.x_2 t$. We derive*

$$\vdash L : (\omega \to (\varphi \to \varphi) \to \varphi) \wedge (\omega \to (\varphi \to \varphi \to \varphi) \to \varphi \to \varphi) \to \varphi$$

$$\vdash R : \varphi \to \omega \to (\varphi \to \varphi) \to \varphi \text{ and } \vdash R : \varphi \to \omega \to (\varphi \to \varphi \to \varphi) \to \varphi \to \varphi$$

When R is a simple right inverse instead we always get a set of retractions.

Theorem 4.6. *If $L \circ R = \mathbf{I}$ and R is a simple right inverse, then for each type μ we find types ν such that L, R witness the retraction $\mu \lhd \nu$.*

Proof. By Theorem 2.10 $L = \lambda z.zM_1 \ldots M_m$. By definition of simple right inverse $R = \lambda t x_1 \ldots x_m.t$. We can then choose $\nu \sim \rho_1 \to \ldots \to \rho_m \to \mu$ for each $\rho_1, \ldots, \rho_m$ such that $\vdash \lambda z.M_i : \nu \to \rho_i$ for $1 \le i \le m$. In particular we can always take $\rho_i = \omega$ for $1 \le i \le m$. □

5 Characterisation of Simple Retraction in Standard Intersection Types

In this section we extend the characterisation of retraction (Theorem 4.3), which holds for strict types, to the case of standard intersection types, by considering only simple right inverses.

The set of standard intersection types (simply *intersection types*) is defined by:

$$\sigma := \varphi \mid \omega \mid \sigma \to \sigma \mid \sigma \wedge \sigma$$

where, as before, φ ranges over type variables and ω is a constant. In this section, we convene that σ, τ, ρ range over intersection types.

The preorder of Definition 3.1 is extended to intersection types by adding the rule

$$(\sigma \to \tau) \wedge (\sigma \to \rho) \leq \sigma \to \tau \wedge \rho$$

The following lemma gives a crucial property of this subtyping, which is shown in [3].

Lemma 5.1. *If $\bigwedge_{i\in I}(\sigma_i \to \tau_i) \leq \sigma \to \tau$, then there is $J \subseteq I$ such that $\sigma \leq \bigwedge_{i\in J}\sigma_i$ and $\bigwedge_{i\in J}\tau_i \leq \tau$.*

The type assignment system is defined by the typing rules of Table 2, where environments are finite mappings from term variables to intersection types.

$$(Ax)\quad \Gamma, x:\sigma \vdash x:\sigma \qquad (\omega)\quad \Gamma \vdash M:\omega \qquad (\leq)\quad \frac{\Gamma \vdash M:\sigma \quad \sigma \leq \tau}{\Gamma \vdash M:\tau}$$

$$(\to I)\quad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma\to\tau} \qquad (\to E)\quad \frac{\Gamma \vdash M:\sigma\to\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau} \qquad (\wedge I)\quad \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma\wedge\tau}$$

Table 2: Typing Rules of Standard Intersection Types

The inversion lemma is as expected, for a proof see [4, Theorem 12.1.13].

Lemma 5.2 (Inversion Lemma)**.**

1. *If $\Gamma \vdash x:\tau$, then either $\tau \sim \omega$ or $x:\sigma \in \Gamma$ for $\sigma \leq \tau$.*
2. *If $\Gamma \vdash \bot:\tau$, then $\tau \sim \omega$.*
3. *If $\Gamma \vdash MN:\tau$, then $\Gamma \vdash M:\sigma\to\tau$ and $\Gamma \vdash N:\sigma$.*
4. *If $\Gamma \vdash \lambda x.M:\tau$, then $\tau \sim \bigwedge_{i\in I}(\sigma_i \to \rho_i)$ and $\Gamma, x:\sigma_i \vdash M:\rho_i$ for all $i \in I$.*

The notion of retraction (Definition 4.1) can be extended to standard intersection types. It is useful here to add the notion of simple retraction.

Definition 5.3.

1. *Type σ is a* retract *of type τ (notation $\sigma \lhd \tau$) if there exist terms L and R such that:*
   (a) $\vdash L:\tau\to\sigma$;
   (b) $\vdash R:\sigma\to\tau$;
   (c) $L \circ R = \mathbf{I}$.
2. *Type σ is a* simple retract *of type τ (notation $\sigma \lhd_s \tau$) if $\sigma \lhd \tau$ and R is a simple right inverse.*

We say that L, R witness the (simple) retraction.

We can now show the desired characterisation.

Theorem 5.4 (Characterisation of Simple Retractions in Standard Intersection Types)**.** *$\sigma \lhd_s \tau$ if and only if $\sigma \sim \bigwedge_{i\in I}\sigma_i$ and $\tau \sim \bigwedge_{i\in I}(\rho_1^{(i)} \to \cdots \to \rho_m^{(i)} \to \sigma_i)$ and the types $\tau \to \bigwedge_{i\in I}\rho_k^{(i)}$ are inhabited for $1 \le k \le m$.*

Proof.

(If) Let M_k be an inhabitant of $\tau \to \bigwedge_{i\in I}\rho_k^{(i)}$ for $1 \le k \le m$. We can choose $L = \lambda z.z(M_1 z)\dots(M_m z)$ and $R = \lambda t x_1 \dots x_m.t$. It is easy to verify that L and R satisfy the conditions of Definition 5.3.

(Only if) Since R is simple, we can assume $R = \lambda t x_1 \dots x_m.t$. Lemma 5.2(4) applied to $\vdash R : \sigma \to \tau$ gives $\tau \sim \bigwedge_{i\in I}(\rho_1^{(i)} \to \cdots \to \rho_m^{(i)} \to \sigma_i)$ and $t : \sigma \vdash t : \sigma_i$ for all $i \in I$. Then $\sigma \le \sigma_i$ for all $i \in I$, which implies $\sigma \le \bigwedge_{i\in I}\sigma_i$.

By Theorem 2.10 we have $L = \lambda z.zM_1 \dots M_m$. Then applying the Inversion Lemma to $\vdash L : \tau \to \sigma$ we get $\tau \le \rho_1 \to \cdots \to \rho_m \to \sigma$ and $z : \tau \vdash M_k : \rho_k$ for $1 \le k \le m$. Then

$$\bigwedge_{i\in I}(\rho_1^{(i)} \to \cdots \to \rho_m^{(i)} \to \sigma_i) \le \rho_1 \to \cdots \to \rho_m \to \sigma$$

By Lemma 5.1 this implies $\bigwedge_{i\in J}\sigma_i \le \sigma$ and $\rho_k \le \bigwedge_{i\in J}\rho_k^{(i)}$ for some $J \subseteq I$ and for all $1 \le k \le m$. From $\sigma \le \bigwedge_{i\in I}\sigma_i$ and $\bigwedge_{i\in J}\sigma_i \le \sigma$ with $J \subseteq I$ we get $J = I$ and $\sigma \sim \bigwedge_{i\in I}\sigma_i$. From $z : \tau \vdash M_k : \rho_k$ and $\rho_k \le \bigwedge_{i\in I}\rho_k^{(i)}$ we can derive $z : \tau \vdash M_k : \bigwedge_{i\in I}\rho_k^{(i)}$ for $1 \le k \le m$. Therefore the types $\tau \to \bigwedge_{i\in I}\rho_k^{(i)}$ are inhabited for $1 \le k \le m$. □

In [1] it is proved that each intersection type is equivalent to an intersection of strict types. Owing to this property and the idempotence of the intersection type constructor, we can show that a simple retraction between intersection types implies a set of retractions between strict types.

Corollary 5.5. *If $\sigma \lhd_s \tau$ and $\tau \sim \bigwedge_{i\in I}\nu_i$, then there are strict types μ_i with $i \in I$ such that $\sigma \sim \bigwedge_{i\in I}\mu_i$, and $\mu_i \lhd \nu_i$ for $i \in I$.*

Proof. By Theorem 5.4 $\nu_i \sim \rho_1^{(i)} \to \cdots \to \rho_m^{(i)} \to \mu_i$ for some μ_i and $\sigma \sim \bigwedge_{i\in I}\mu_i$. Then $\mu_i \lhd \nu_i$ for $i \in I$ by Theorem 4.3. □

Example 5.6.

i) Consider $\sigma = \varphi \wedge \psi$ and $\tau = \omega \to ((\varphi \to \varphi \to \varphi) \to \varphi) \wedge ((\psi \to \omega \to \psi) \to \psi)$. We get $\sigma \lhd_s \tau$ by choosing $L = \lambda z.z\bot\mathbf{K}$, where $\mathbf{K} = \lambda xy.x$ and its simple right inverse $R = \lambda t x_1 x_2.t$. The terms L, R witness also the two retractions $\varphi \lhd \omega \to (\varphi \to \varphi \to \varphi) \to \varphi$ and $\psi \lhd \omega \to (\psi \to \omega \to \psi) \to \psi$ between strict types.

ii) The terms $L = \lambda z.z\mathbf{I}$ and $R = \lambda t x.t$ witness the simple retraction $\varphi \lhd_s ((\psi \to \psi) \to \varphi) \wedge (\omega \to \varphi)$. The same terms show the two retractions $\varphi \lhd (\psi \to \psi) \to \varphi$ and $\varphi \lhd \omega \to \varphi$ between strict types.

6 Retractions in Models

In this section we discuss retractions in models of $\lambda\bot$-calculus. Since standard intersection types can be seen as a conservative extension of strict intersection types, the results of this section hold for both systems.
It is easy to adapt the Hindley-Longo definition of λ-calculus models [18] to the $\lambda\bot$-calculus. We use $\mathcal{V}$ to range over mappings from term variables to elements of the domain.

Definition 6.1. *A model of the $\lambda\bot$-calculus is a structure $\mathcal{M} =< D, \cdot, [\![-]\!]^{\mathcal{M}}, d_0 >$ which satisfies the following conditions:*

1. *d_0 is a distinguished element of the domain D such that $d_0 \cdot e = d_0$ for all $e \in D$*
2. $[\![x]\!]^{\mathcal{M}}_{\mathcal{V}} = \mathcal{V}(x)$
3. $[\![MN]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![M]\!]^{\mathcal{M}}_{\mathcal{V}} \cdot [\![N]\!]^{\mathcal{M}}_{\mathcal{V}}$
4. $[\![\lambda x.M]\!]^{\mathcal{M}}_{\mathcal{V}} \cdot d = [\![M]\!]^{\mathcal{M}}_{\mathcal{V}[d/x]}$
5. *$\mathcal{V} = \mathcal{V}'$ implies* $[\![M]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![M]\!]^{\mathcal{M}}_{\mathcal{V}'}$
6. $[\![\lambda x.M]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\lambda y.M[y/x]]\!]^{\mathcal{M}}_{\mathcal{V}}$ *if y is not in M*
7. $\forall d \in D \;\; [\![M]\!]^{\mathcal{M}}_{\mathcal{V}[d/x]} = [\![N]\!]^{\mathcal{M}}_{\mathcal{V}[d/x]}$ *implies* $[\![\lambda x.M]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\lambda x.N]\!]^{\mathcal{M}}_{\mathcal{V}}$
8. $[\![\bot]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\lambda x.\bot]\!]^{\mathcal{M}}_{\mathcal{V}} = d_0$.

As usual we interpret types as subsets of the model domain.

Definition 6.2. *The standard* interpretation of types *in a model $\mathcal{M}$ with domain D is defined by:*

1. $[\![\omega]\!]^{\mathcal{M}}_{\mathcal{W}} = D$
2. $[\![\varphi]\!]^{\mathcal{M}}_{\mathcal{W}} = \mathcal{W}(\varphi)$
3. $[\![\sigma \to \tau]\!]^{\mathcal{M}}_{\mathcal{W}} = \{d \mid \forall e \in [\![\sigma]\!]^{\mathcal{M}}_{\mathcal{W}} \;\; d \cdot e \in [\![\tau]\!]^{\mathcal{M}}_{\mathcal{W}}\}$
4. $[\![\sigma \wedge \tau]\!]^{\mathcal{M}}_{\mathcal{W}} = [\![\sigma]\!]^{\mathcal{M}}_{\mathcal{W}} \cap [\![\tau]\!]^{\mathcal{M}}_{\mathcal{W}}$

where $\mathcal{W}$ ranges over mappings from type variables to subsets of the domain.

From these definitions it easy to show the soundness of the type systems of Sections 3 and 5. We say that the mappings $\mathcal{V}, \mathcal{W}$ for a model $\mathcal{M}$ respect an environment Γ if $x : \sigma \in \Gamma$ implies $[\![x]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\sigma]\!]^{\mathcal{M}}_{\mathcal{W}}$.

Theorem 6.3 (Soundness)**.** *If $\Gamma \vdash M : \sigma$, then $[\![M]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\sigma]\!]^{\mathcal{M}}_{\mathcal{W}}$ for all $\mathcal{M}$ and all $\mathcal{V}$, $\mathcal{W}$ respecting Γ.*

The semantic retraction is naturally defined as follows.

Definition 6.4. *Type σ is a* semantic retract *of type τ (notation $\sigma \blacktriangleleft \tau$) if there are two terms L and R such that for all models $\mathcal{M}$ of $\lambda\bot$-calculus and for all mappings $\mathcal{V}, \mathcal{W}$:*

1. $[\![R]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\sigma \to \tau]\!]^{\mathcal{M}}_{\mathcal{W}}$
2. $[\![L]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\tau \to \sigma]\!]^{\mathcal{M}}_{\mathcal{W}}$
3. $[\![L]\!]^{\mathcal{M}}_{\mathcal{V}} \circ [\![R]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\mathbf{I}]\!]^{\mathcal{M}}_{\mathcal{V}}$.

An interesting model of λ-calculus is the *filter* model $\mathcal{F}$ defined in [3]. The domain of this model is the set of filters of types. We refer to that paper for the basic definitions and properties. The model $\mathcal{F}$ can be easily seen as a model of the $\lambda\bot$-calculus by taking the filter generated by type ω as d_0.

In [3] the filter model is proved to be complete for λ-calculus and standard intersection types. It is easy to adapt this result to $\lambda\bot$-calculus. The completeness theorem of [3] can then be reformulated as follows:

Theorem 6.5 (Completeness of the Filter Model)**.** *If* $[\![M]\!]^{\mathcal{F}}_{\mathcal{V}} \in [\![\sigma]\!]^{\mathcal{F}}_{\mathcal{W}}$ *for all* $\mathcal{V}$, $\mathcal{W}$ *respecting* Γ, *then* $\Gamma \vdash M : \sigma$.

We can show that retraction and semantic retraction coincide using the soundness and the completeness of the filter model.

Theorem 6.6. $\sigma \lhd \tau$ *if and only if* $\sigma \blacktriangleleft \tau$.

Proof.

(If) By Theorem 6.5 we immediately have that

- $[\![L]\!]^{\mathcal{F}}_{\mathcal{V}} \in [\![\tau \to \sigma]\!]^{\mathcal{F}}_{\mathcal{W}}$ for all $\mathcal{V}$, $\mathcal{W}$ implies $\vdash L : \tau \to \sigma$ and
- $[\![R]\!]^{\mathcal{F}}_{\mathcal{V}} \in [\![\sigma \to \tau]\!]^{\mathcal{F}}_{\mathcal{W}}$ for all $\mathcal{V}$, $\mathcal{W}$ implies $\vdash R : \sigma \to \tau$.

Notice that $[\![L]\!]^{\mathcal{M}}_{\mathcal{V}} \circ [\![R]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\mathbf{B}LR]\!]^{\mathcal{M}}_{\mathcal{V}}$. Since $\varphi \to \varphi \in [\![\mathbf{I}]\!]^{\mathcal{F}}_{\mathcal{V}}$, the completeness of the filter model gives $\vdash \mathbf{B}LR : \varphi \to \varphi$. It is easy to prove that this implies $\mathbf{B}LR = \mathbf{I}$, as remarked in [3].

(Only if) The soundness of the type system (Theorem 6.3) implies that

- $\vdash L : \tau \to \sigma$ gives $[\![L]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\tau \to \sigma]\!]^{\mathcal{F}}_{\mathcal{W}}$ for all $\mathcal{M}$, $\mathcal{V}$, $\mathcal{W}$ and
- $\vdash R : \sigma \to \tau$ gives $[\![R]\!]^{\mathcal{M}}_{\mathcal{V}} \in [\![\sigma \to \tau]\!]^{\mathcal{F}}_{\mathcal{W}}$ for all $\mathcal{M}$, $\mathcal{V}$, $\mathcal{W}$.

If $L \circ R = \mathbf{I}$, then by definition of model we get $[\![L]\!]^{\mathcal{M}}_{\mathcal{V}} \circ [\![R]\!]^{\mathcal{M}}_{\mathcal{V}} = [\![\mathbf{I}]\!]^{\mathcal{M}}_{\mathcal{V}}$. □

7 Related Work

The seminal paper [7] characterises for Curry types both isomorphism in the $\lambda\beta\eta$-calculus and retraction in the $\lambda\beta$-calculus.

Isomorphism in the $\lambda\beta\eta$-calculus is characterised for various type disciplines by means of equations between types [16]. Product and unit types are considered in [6,23,24], universally quantified types in [7], all the above type constructors in [15]. Characterisation of isomorphism for intersection types instead requires a notion of type similarity [9, 14]. Analogous result holds for intersection and union types [8, 10, 11]. [17] shows that isomorphism for product, arrow and sum types is not finitely axiomatizable.

Retraction witnessed by affine terms of $\lambda\beta\eta$-calculus for Curry types with only one atom is characterised in [13]. As an auxiliary result in the study of the relation between iteration and recursion, [25] gives a necessary condition for retraction considering universally quantified types and $\lambda\beta$-calculus. An algorithm to decide if a Curry type with a single atom is a retract of another one in the $\lambda\beta\eta$-calculus is given in [20]. This algorithm builds the witnesses of the retraction, when they exist. The results of [13] are extended to Curry types with many atoms in [21]. Moreover [21] gives necessary conditions for retraction witnessed by arbitrary terms of $\lambda\beta\eta$-calculus dealing with both Curry and universally quantified types. The problem of retraction solved in [13] is shown to be NP-complete in [22]. [26] gives a proof system which leads to an exponential decision procedure to characterise retraction for Curry types in the $\lambda\beta\eta$-calculus.

8 Conclusion

This paper deals with retraction for strict and standard intersection types in the $\lambda\bot$-calculus. Both the choices of the calculus and of the types can be discussed.

We considered the $\lambda\bot$-calculus following [19]. Our results are easily adapted to the λ-calculus taking an unsolvable term to play the role of $\bot$.

By conservativity the given characterisation of retraction in strict intersection types holds in Curry types [12]. In this way we obtain the result of [7]. We give also a characterisation of retractions in standard intersection types when the right inverses are simple. However the retraction of Example 4.5 cannot be shown by a simple right inverse, since it should have both the types

$$\varphi \to \omega \to (\varphi \to \varphi) \to \varphi \text{ and } \varphi \to \omega \to (\varphi \to \varphi \to \varphi) \to \varphi \to \varphi.$$

The definition of a necessary and sufficient condition for the existence of a retraction between standard intersection types is not obvious when the right inverses are arbitrary. For example, no type can be a retract of the type

$$\omega \to ((\omega \to \varphi \to \varphi) \to \varphi) \wedge ((\psi \to \omega \to \psi) \to \psi)$$

essentially since $(\omega \to \varphi \to \varphi) \wedge (\psi \to \omega \to \psi)$ is not inhabited. Notice that this type and the type τ of Example 5.6 *i)* differ only for one occurrence of ω in place of one occurrence of φ.

We plan to investigate retractions in standard intersection types without conditions on right inverses, and in intersection and union types. The problem for the $\lambda\beta\eta$-calculus is surely more difficult as shown by the papers [7, 13, 20–22, 26] and it is left for future work.

As suggested by one referee, an interesting future development is to see how the results presented here can be adapted to programming languages with richer sets of constructs. This would allow to apply automatic retraction inference in dealing with API of functional programs or proof-assistants so as to maximise code reuse.

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