THE SEMANTICS OF ENTAILMENT OMEGA

MARIANGIOLA DEZANI-CIANCAGLINI,
DIPARTIMENTO DI INFORMATICA, UNIVERSITÀ DI TORINO
DEZANI@DI.UNITO.IT

ROBERT K. MEYER,
AUTOMATED REASONING, RSISE, AUSTRALIAN NATIONAL UNIVERSITY
RKM@ARP.ANU.EDU.AU

YOKO MOTOHAMA,
DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI UDINE
MOTOHAMA@DIMI.UNIUD.IT

ABSTRACT. This paper discusses the relation between the minimal positive relevant logic $\mathbf{B}_+$ and intersection and union type theories. There is a marvellous coincidence between these very differently motivated research areas. First, we show a perfect fit between the Intersection Type Discipline **ITD** and the tweaking $\mathbf{B} \wedge \mathbf{T}$ of $\mathbf{B}_+$, which saves implication $\rightarrow$ and conjunction $\wedge$ but drops disjunction $\vee$. The *filter models* of the λ-calculus (and its intimate partner Combinatory Logic **CL**) of the first author and her co-authors then become *theory models* of these calculi. (The logician's *Theory* is the algebraist's *Filter*.) The coincidence extends to a dual interpretation of key particles – the subtype $\leq$ translates to provable $\rightarrow$, type intersection $\cap$ to conjunction $\wedge$, function space $\rightarrow$ to implication and whole domain ω to the (trivially added but trivial) truth $\mathbf{T}$. This satisfying ointment contains a fly. For it is right, proper and to be expected that type union $\cup$ should correspond to the logical disjunction $\vee$ of $\mathbf{B}_+$. But the simulation of functional application by a fusion (or modus ponens product) operation $\circ$ on theories leaves the key *Bubbling lemma* of work on **ITD** unprovable for the $\vee$-*prime theories* now appropriate for the modelling. The focus of the present paper lies in an appeal to *Harrop theories* which are (a) prime and (b) closed under fusion. A version of the Bubbling lemma is then proved for Harrop theories, which accordingly furnish a model of λ and **CL**.

Keywords: Minimal relevant logic, Intersection type theory, Lambda model, Curry-Howard isomorphism, Harrop Formulas.
MSC: primary 03B47, 03B40 secondary 68N18

INTRODUCTION

This paper receives the ordinal ω for a couple of reasons. Its predecessors in Meyer's "semantics of entailment" series (mainly with Routley) were called 1, 2, etc. It's time for a summing up at the limit. A second reason has to do with the role of the *constant* ω in the *filter models* of λ developed by Dezani and her colleagues (mainly at Torino). ω is transmuted in various respects here – logically to a "Church constant" $\mathbf{T}$, and functionally to a space $\mathbf{T} \rightarrow \mathbf{T}$. But the pun remains.

One had ventured to hope that the rise of computer science would bring with it a bright new day for logic. Or at least it might bring back some good old days, beginning with those

Partially supported by EU within the FET - Global Computing initiative, project DART ST-2001-33477 and by MURST Cofin'01 project COMETA, MURST Cofin'02 project McTati. The funding bodies are not responsible for any use that might be made of the results presented here.

in which Aristotle founded logic in order to give an account of how people reason, when they are reasoning correctly. For if our computing machines are to do most of our thinking in the present millennium (as is not unlikely), then some improvement in our start-of-the-millennium logical theories is desirable. In particular Anderson, Belnap, Dunn *et al.* in [1][2] and Routley *et al.* in [20] have proposed systems of relevant logic and entailment as vehicles for this improvement. In this paper we build on our previous studies of the semantics of entailment on the one hand and of models for the λ-calculus on the other to delve more deeply into what relevant logics are about.

With Sylvan (né Routley)[1], Meyer proposed in [19] and [18] a minimal positive relevant logic $\mathbf{B}_+$. As they conceived it, $\mathbf{B}_+$ had a role to play for relevant logics analogous to that played by the system **K** among normal modal logics with a Kripke-style "possible worlds" semantics in the style of [16]. That is, $\mathbf{B}_+$ satisfied just those semantical postulates that we took to be common to *arbitrary* positive logics in the relevant family. Thus on our semantics other positive logics arose from $\mathbf{B}_+$ on the addition of specific postulates. But the main ideas – e.g., that $B \wedge C$ is true at a "world" w iff B is true at w and C is true at w – remain through whatever additions are appropriate to get famous logics like relevant $\mathbf{R}_+$ or intuitionist **J**.

Moreover, the main candidate additions have a *combinatory* character, in the sense that they are suggested by the (so-called Curry-Howard) isomorphism between candidate implicational *theorems* and *combinators* set out in [8]. Indeed, the semantical postulates which match these theorems may be almost read off the Curry-Howard correspondence. But, as it turned out, there are *other* candidate theorems - for example, some involving *both* $\wedge$ and $\rightarrow$ in their formulation - which also seemed to match combinators. Back in the early '70's, Routley and Meyer did not know what to make of these new "types" for combinators. But they were sufficiently impressed by them to pronounce **CL** the "key to the universe" in [18].

Many years thereupon passed, in some of which Meyer sought to interest members of the **CL**-λ community in (what he took to be) this satisfying interplay between ideas from relevant and combinatory logics. But it was only when Bunder brought Hindley to Australia (and to ANU in particular) in the late 1980's that progress was made. For Meyer and Errol Martin learned from Hindley of the extension of Curry's type theory that had been developed in the work of Coppo and Dezani in Torino and set out most fully by them with Barendregt in [6]. For [6] had added $\wedge$ to the pure $\rightarrow$ Curry vocabulary; and this enabled them, near enough, to fix $((p \rightarrow q) \wedge p) \rightarrow q$ as the principal type of $\lambda x.xx$.

When Meyer saw this example in [6], he was *very pleased*. For $\lambda x.xx$ is one of the terms that has no type on Curry's scheme. Still, on the "correspondence theory" implicit in [19], with the ternary relation R to explicate $\rightarrow$ on our relational "worlds semantics", the validity of the *formula* $((p \rightarrow q) \wedge p) \rightarrow q$ enforces and is enforced by the total ternary reflexivity postulate $\mathsf{R}www$. Rightly viewed, that semantical postulate is just a way of saying that $\lambda x.xx$ (a.k.a. **WI** or **SII**, for **CL** fans) is a *good guy*. The *logical content* of the postulate is that the formula $((p \rightarrow q) \wedge p) \rightarrow q$ (which expresses *conjunctive modus ponens*) is a good guy. But it is nonetheless *optional* whether or not this formula should

[1] Sylvan died in June, 1996, while visiting Bali, Indonesia. After so much joint work with him on the semantics of relevant logics, we dedicate this further essay to his memory.

be taken as a *logical truth*. At the most fundamental relevant level (i.e., that of $\mathbf{B}_+$), the formula is a *non-theorem* (despite any logical propaganda that you may have imbibed.) [2].

Now [6] saw the *intersection type discipline* (henceforth, **ITD**) of that paper as a way of providing filter models for λ. Along with $\rightarrow$ and $\wedge$ the **ITD** introduced a new (universal) type $\mathbf{T}$, which is a type possessed by *every term*. But from the logical perspective $\mathbf{T}$ may be viewed simply as a greatest truth, which is entailed by every proposition. And once *union types* with $\vee$ are introduced as well, as they were for example in [4], we can feed our intuitions with the following table:

Symbol	**Logical sense**	**Type-theoretic sense**
$\rightarrow$	Implies	Function space
$\wedge$	And	Intersection
$\vee$	Or	Union
$\leq$	Entails	Subset
$\mathbf{T}$	True	Whole domain

1. ITD = B∧T

We set out first the postulates on the intersection type theory **ITD** [6], and we relate them to the $\rightarrow\wedge$ fragment of $\mathbf{B}_+$ extended with a greatest truth $\mathbf{T}$. We call thil this fragment $\mathbf{B} \wedge \mathbf{T}$ [3]. Without loss of generality **ITD** may be assumed to be formulated with a binary predicate $\leq$, a constant $\mathbf{T}$ (a.k.a. ω), and binary function symbols $\rightarrow$ and $\wedge$. We assume a countable infinity of (type) variables, for which we use 'p', 'q', 'r', etc. As syntactical variables for (type) terms we use upper-case 'A', 'B', etc., decorating our syntactical variables as takes our fancy. We take leave of the right and good and eminently sensible syntactical conventions set out by Curry in [7] and [8] by laying it down that equal connectives shall be associated (shock, horror!) to the *right*, and that $\wedge$ shall bind more tightly than $\rightarrow$. As axiom schemes and rules of **ITD** we choose the following[4]:

Reflex.	$A \leq A$
Top.	$A \leq \mathbf{T}$
Top→.	$\mathbf{T} \leq \mathbf{T} \rightarrow \mathbf{T}$
Idem∧.	$A \leq A \wedge A$
∧E.	$A \wedge B \leq A, A \wedge B \leq B$
→∧I.	$(A \rightarrow B) \wedge (A \rightarrow C) \leq A \rightarrow B \wedge C$
Trans∧.	$A \leq B \leq C \Rightarrow A \leq C$
Mon∧.	$A \leq A', B \leq B' \Rightarrow A \wedge B \leq A' \wedge B'$
Mon→.	$A' \leq A, B \leq B' \Rightarrow A \rightarrow B \leq A' \rightarrow B'$

In a nutshell, **ITD** has $\wedge$-semilattice properties, with monotonic replacement properties for $\wedge$ and (appropriately) for $\rightarrow$, with $\mathbf{T}$ as a top element (mathematically identifiable as $\mathbf{T} \rightarrow \mathbf{T}$).

Now how did Hindley know, when he heard from Meyer about $\mathbf{B}_+$, that it was just (a somewhat tweaked version of) **ITD**?[5] Here are some axiom schemes and rules sufficient

[2] Strengthen $\mathbf{B}_+$ (e.g., to intuitionist **J** or even $\mathbf{R}_+$) and *conjunctive modus ponens* is valid!

[3] To be pronounced, "BAT".

[4] Save for notational changes these are *exactly* the postulates of [6], using $\Rightarrow$ to express *rules*.

[5] Historically the tweaking should be *vice versa*, as $\mathbf{B}_+$ anticipated **ITD** by a decade. But nobody knew that.

for $\mathbf{B}_+$, formulated in $\wedge$, $\vee$, $\rightarrow$.[6]

Reflex.	$A \rightarrow A$
$\wedge$E.	$A \wedge B \rightarrow A, A \wedge B \rightarrow B$
$\rightarrow\wedge$I.	$(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow A \rightarrow B \wedge C$
$\rightarrow\vee$E.	$(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow A \vee B \rightarrow C$
$\vee$I.	$A \rightarrow A \vee B, B \rightarrow A \vee B$
Dist$\wedge\vee$.	$A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$

As rules we choose

$\rightarrow$E.	$A \rightarrow B \Rightarrow A \Rightarrow B$
$\wedge$I.	A and $B \Rightarrow A \wedge B$
Rul**B**.	$B \rightarrow C \Rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$
Rul**B**$'$.	$A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$

Note the subtle difference between the "prefixing" Rul**B** and the "suffixing" Rul**B**$'$. Together with $\rightarrow$E either yields a derived "transitivity" rule, which we might set down as

$$\text{Rul}\mathbf{BB}'. \quad B \rightarrow C \Rightarrow A \rightarrow B \Rightarrow A \rightarrow C$$

Three moves, all trivial, suffice to transform $\mathbf{B}_+$ into **ITD**. The first is to replace $\rightarrow$ when it is the *principal* connective of a formula with $\leq$. (This has the side effect of making the formula easier to read, while it coincides with the idea that *entailment* is what logic is principally about anyway.) The second move is to drop $\vee$ and all its works. (They will be back.) And the final move is to add (the "Church constant") $\mathbf{T}$, together with the axioms

Top.	$A \rightarrow \mathbf{T}$
Top$\rightarrow$.	$\mathbf{T} \rightarrow \mathbf{T} \rightarrow \mathbf{T}$

When $\mathbf{B}_+$ has been so massaged, we call it **B**$\wedge$**T**. I.e., we presuppose a translation * from the vocabulary of **ITD** to that of **B**$\wedge$**T**, such that $p^* = p$ and $\mathbf{T}^* = \mathbf{T}$ for all atoms, and otherwise $(A \wedge B)^* = A^* \wedge B^*$, $(A \rightarrow B)^* = A^* \rightarrow B^*$, and $(A \leq B)^* = A^* \rightarrow B^*$. And we now give a simple *metavaluations* argument that for all elementary statements $A \leq B$ of **ITD**, we have $A \leq B$ a theorem of **ITD** iff $A^* \rightarrow B^*$ is a theorem of **B**$\wedge$**T**.[7] Note that it is elementary that **ITD** $\subseteq$ **B**$\wedge$**T** on the * translation, since the axioms and rules of the former are readily derived in **B**$\wedge$**T**. For the converse we define a class MTR of metatruths thus:

$v\mathbf{T}$.	$\mathbf{T} \in$ MTR
vp.	$p \notin$ MTR, where p is a variable
$v \rightarrow$.	$A^* \rightarrow B^* \in$ MTR iff (i) $A \leq B$ in **ITD** and (ii) $A^* \notin$ MTR or $B^* \in$ MTR
$v \wedge$.	$A^* \wedge B^* \in$ MTR iff both $A^* \in$ MTR and $B^* \in$ MTR

Coherence lemma. $A \in$ **B**$\wedge$**T** $\Rightarrow A^* \in$ MTR.
For *proof*, show by deductive induction that all theorems of **B**$\wedge$**T** are metatruths. Whence we have the
Coincidence theorem. **ITD** = **B**$\wedge$**T** on the * translation.
Proof. Inclusion from left to right is trivial as noted. And the converse holds given the coherence lemma, in virtue of (i) under $v \rightarrow$. End of proof.

---

[6]Binary connectives are also ranked $\wedge$, $\vee$, $\rightarrow$ in order of *increasing* scope. We continue as above to use $\Rightarrow$ as a *metalogical* connective in framing rules; $\Rightarrow$ also associates here to the *right*.

[7]Venneri uses another argument in [21]. But she notes the (previously unpublished) argument set out here.

We shall now give a "worlds semantics" for **ITD**, adapting [19] and Fine's contribution to [1][2].[8] We take a *positive model structure* (henceforth, $+ms$) to be a structure $\mathbf{K} =< K, \circ >$, where K is a set (of worlds) and $\circ$ is a binary operation on K. [9] Let Var be the set of variables, and let $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ be the set of truth-values. A *valuation* v on the $+ms$ **K** is a function from $Var \times K$ to **2**. Let *Form* be the set of all formulas. A valuation v on **K** is extended to an *interpretation* $\mathcal{I}$ from $Form \times K$ to **2** as follows [10] for $w \in K$:

$$
\begin{array}{ll}
Tp. & \mathcal{I}(p, w) = v(p, w), \text{ for all } p \in Var \\
T\wedge. & \mathcal{I}(A \wedge B, w) = min[I(A, w), I(B, w)] \\
T\rightarrow. & \mathcal{I}(A \rightarrow B, w) = \mathbf{1} \text{ iff } \forall x \in K(\mathcal{I}(A, x) = \mathbf{0} \text{ or } \mathcal{I}(B, wx) = \mathbf{1}) \\
T\mathbf{T}. & \mathcal{I}(\mathbf{T}, w) = \mathbf{1}
\end{array}
$$

And we now say that A *entails* B on a valuation v in **K** (equivalently, on the associated interpretation $\mathcal{I}$) iff $\forall w \in K(\mathcal{I}(A, w) = \mathbf{0}$ or $\mathcal{I}(B, w) = \mathbf{1})$. A *entails* B *in* **K** iff A entails B on all valuations v in **K**. Finally, A *entails* B (*positively*) iff A entails B in all $+ms$ **K**.

Semantic completeness for **ITD** will amount to the claim that $A \leq B$ is a theorem iff A *entails* B. Before proving it we enter some important definitions. First, where $U, V \subseteq Form$, define the fusion operation $\circ$ by

$$D\circ. \quad U \circ V \ =_{df} \ \{B : \ \exists A \in Form(A \rightarrow B \in U \text{ and } A \in V)\}$$

A *theory* U is any non-empty subset of $Form$ which is closed under $\leq$ and $\wedge$. [11] I.e., U must satisfy

$$
\begin{array}{ll}
\leq\text{E}. & A \leq B \text{ in } \mathbf{ITD} \Rightarrow (A \in U \Rightarrow B \in U) \\
\wedge\text{I}. & A \in U \text{ and } B \in U \Rightarrow A \wedge B \in U
\end{array}
$$

The *empty theory*, to which we have sometimes appealed in the past, is ruled out here. So *every theory* must therefore contain the constant $\mathbf{T}$, in view of $\leq$E and Top above.

The *calculus of theories* $\mathbf{CT} =< CT, \circ >$ is the structure such that
(i) CT is the collection of all theories and
(ii) $\circ$ is the fusion operation defined by $D\circ$.
It is easy to verify that if U and V are theories so also is $U \circ V$. To each $A \in Form$ there corresponds its *principal theory* $A\!\uparrow = \{B : \ A \leq B \text{ in } \mathbf{ITD}\}$. The *canonical valuation* c in **CT** is the valuation such that, for all $p \in Var$ and $U \in CT$, $c(p, U) = \mathbf{1}$ iff $p \in U$. It is elementary to observe that the extension of c to a canonical interpretation $\mathcal{C}$ on the rubric above extends the property to $\mathcal{C}(A, U) = \mathbf{1}$ iff $A \in U$, for all formulas A and theories U, invoking $T \rightarrow$, etc.

Canonical lemma. For all $A, B \in Form$, $A \leq B$ in **ITD** iff A entails B on c in **CT**.
Proof. ($\Rightarrow$) Assume $A \leq B$ and $\mathcal{C}(A, U) = \mathbf{1}$. Then $A \in U$; so $B \in U$ by $\leq$E, whence $\mathcal{C}(B, U) = \mathbf{1}$.
($\Leftarrow$) Assume A entails B on the canonical valuation c. Then in particular $\mathcal{C}(A, A\!\uparrow) = \mathbf{1} \Rightarrow \mathcal{C}(B, A\!\uparrow) = \mathbf{1}$. But $\mathcal{C}(A, A\!\uparrow) = \mathbf{1}$. Whence $A \leq B$ in **ITD** by definition of $A\!\uparrow$, ending the proof of the canonical lemma.

---

[8] Fine develops (mainly independently) an Urquhart-Routley style *operational* relevant semantics.

[9] We usually indicate composition under $\circ$ by juxtaposition, writing (e.g.) 'wx' instead of '$w \circ x$'.

[10] I agrees with v on variables by Tp, and it is extended to all formulas by truth-conditions $T\mathbf{T}$, $T\wedge$, $T\rightarrow$.

[11] This is *Logicese*. In *Algebraese* it is called a *filter*, as in Dunn's [11] and in BCD's [6].

We get immediately an appropriate
Completeness theorem for ITD. ITD $\models A \leq B$ **iff** A **(positively) entails** B.
Proof. ($\Rightarrow$) By deductive induction.
($\Leftarrow$) Suppose A entails B. Then in particular A entails B on the canonical valuation c, whence by the canonical lemma $A \leq B$ in **ITD**.

2. The calculus of theories CT is a model for λ and CL

In Algebraese these are already principal results of [6] and [10] respectively. But here we are speaking Logicese, whence we say *theory* where the cited papers say *filter*. By λ we mean the type-free λK β-calculus, invented by Church in the birth year of one of us, and exhaustively studied by Barendregt in [5]. By **CL** we mean Curry's (weak) combinatory logic, as summarised in [15]. As **CL** is already definable in λ in well-known ways [12], it will suffice here to recount the [6] proof that $\mathbf{CT} =< CT, \circ >$ is a model of λ. First, we define an equivalence $\equiv$ in **ITD** on $Form$ by

$$D\equiv . \quad A \equiv B \ =_{df} \ A \leq B \text{ and } B \leq A$$

[6] (which uses '$\sim$' where we here use '$\equiv$') rightly suggests that **ITD** may be considered modulo $\equiv$, in which case $\leq$ becomes a partial order. They also prove an important lemma, which goes into our notation as

Bubbling lemma (i) $A \to B \equiv \mathbf{T}$ iff $B \equiv \mathbf{T}$. (ii) Assume it is NOT the case that $D \equiv \mathbf{T}$. Assume moreover that we have $\bigwedge_{i\in I}(A_i \to B_i) \leq C \to D$ for the finite non-empty index set I. Then there is a finite non-empty subset J of I such that

$$C \leq \bigwedge_{i\in J} A_i \text{ and } \bigwedge_{i\in J} B_i \leq D.$$

The Bubbling lemma (ii) is exceedingly important in [6]. Indeed, that it *fails* in the richer environment of all of $\mathbf{B}_+$ greatly complicates the story that we are telling here.

But let us dwell first on (more or less) easy success, which is preferable where available. A λ-valuation v in **CT** shall be a function which assigns theories to λ-variables x, y, etc. If U is a theory, by $v[U/x]$ we mean the λ-valuation [13] defined by:

$$v[U/x](y) = \begin{cases} U & \text{if } x = y \\ v(y) & \text{otherwise.} \end{cases}$$

Each λ-valuation v is extended to the corresponding λ-interpretation $\mathcal{V}$ on the following rubric:

$$\begin{array}{ll} \mathcal{V}x. & \mathcal{V}(x) = v(x) \\ \mathcal{V}\circ. & \mathcal{V}(MN) = \mathcal{V}(M)\circ\mathcal{V}(N) \\ \mathcal{V}\lambda. & \mathcal{V}(\lambda x.M) = \{\bigwedge_{i\in I}(A_i \to B_i) :\ B_i \in \mathcal{V}[A_i\!\uparrow /x](M)\} \end{array}$$

[12]Translating the combinators **K** by $\lambda xy.x$ and **S** by $\lambda xyz.xz(yz)$, etc.

[13]Note that $v[U/x]$ is what Leblanc [17] calls an x-variant. I.e. it agrees with v everywhere, except possibly at x.

where I is a finite non empty set of indices. [14]
For the correctness of our definition, we need all the λ-interpretations to be theories. Proof is by induction on the construction of the λ-interpretation $\mathcal{V}$ defined above.
The crucial case which requires the Bubbling lemma (ii) is clause $\mathcal{V}\lambda$.
A preliminary observation is that our λ-interpretations are monotone, i.e. if $v(x) \subseteq v'(x)$ for all variables x which occur free in M, then $\mathcal{V}(M) \subseteq \mathcal{V}'(M)$. This can be easily checked by induction on the construction of λ-interpretations.
For $\{\bigwedge_{i\in I}(A_i \to B_i) : B_i \in \mathcal{V}[A_i\uparrow /x](M)\}$ to be a theory, we need that $D \in \mathcal{V}[C\uparrow /x](M)$ whenever $\bigwedge_{i\in I}(A_i \to B_i) \le C \to D$ and $B_i \in \mathcal{V}[A_i\uparrow /x](M)$ for all $i \in I$. By the Bubbling lemma (ii) we get $C \le A_i$ for all $i \in J$ and $\bigwedge_{i\in J} B_i \le D$ for some $J \subseteq I$. This implies $C\uparrow \supseteq A_i\uparrow$ for all $i \in J$, which together with $B_i \in \mathcal{V}[A_i\uparrow /x](M)$ for all $i \in J$, gives us $B_i \in \mathcal{V}[C\uparrow /x](M)$ for all $i \in J$ by the monotonicity of λ-interpretations. So we get $D \in \mathcal{V}[C\uparrow /x](M)$, since $B_i \in \mathcal{V}[C\uparrow /x](M)$ for all $i \in J$ and $\mathcal{V}[C\uparrow /x](M)$ is a theory by induction.

It is easy to verify (and already done in [6]), that for all v the λ-interpretation $\mathcal{V}$ is a *syntactic λ-model* according to [14], i.e. that :

$Ix.$ $\mathcal{V}(x) = v(x)$
$I\circ.$ $\mathcal{V}(MN) = \mathcal{V}(M)\circ\mathcal{V}(N)$
$I\lambda\circ.$ $\mathcal{V}(\lambda x.M)\circ U = \mathcal{V}[U/x](M)$
$Iv.$ if $v(x) = v'(x)$ for all variables x which occur free in M, then $\mathcal{V}(M) = \mathcal{V}'(M)$
$I\alpha.$ $\mathcal{V}(\lambda x.M) = \mathcal{V}(\lambda y.M[y/x])$ if y does not occur free in M
$I\xi.$ $\mathcal{V}[U/x](M) = \mathcal{V}[U/x](N)$ for all $U \Rightarrow \mathcal{V}(\lambda x.M) = \mathcal{V}(\lambda x.N)$

A crucial observation to prove clause $I\lambda\circ$ is that λ-interpretations are compositional, i.e. $\mathcal{V}[U/x](M) = \bigcup_{A\in U} \mathcal{V}[A\uparrow /x](M)$. Also this property can be easily shown by induction on the construction of λ-interpretations.
By definition of $\circ$ and clause $\mathcal{V}\lambda$ we get $\mathcal{V}(\lambda x.M)\circ U = \{B : \exists A \in Form(A \to B \in \mathcal{V}(\lambda x.M)$ and $A \in U)\} = \{B : \exists A \in Form(B \in \mathcal{V}[U/x](M)$ and $A \in U)\} = \bigcup_{A\in U} \mathcal{V}[A\uparrow /x](M)$, so we can conclude using the compositionality of λ-interpretations.

3. The calculus of theories on $\mathbf{CT}\vee$ is not a model for λ and CL

We can enrich **ITD** by adding the following axiom schemes and rules:

Idem$\vee$.	$A \vee A \le A$
$\vee$I.	$A \le A \vee B, B \le A \vee B$
$\to\vee$E.	$(A \to C) \wedge (B \to C) \le A \vee B \to C$
Dist$\wedge\vee$.	$A \wedge (B \vee C) \le A \wedge B \vee A \wedge C$
Mon$\vee$.	$A \le A', B \le B' \Rightarrow A \vee B \le A' \vee B'$

We call this extension $\mathbf{ITD}\vee$. Now we can transform $\mathbf{B}_+$ into $\mathbf{ITD}\vee$ with only two moves. It suffices to replace $\to$ when it is the *principal* connective of a formula with $\le$. And to add $\mathbf{T}$ with the axioms Top and Top$\to$. The difference with the translation * described in section 1 is that we don't drop $\vee$. We call ** this new translation. So the old translation * maps **B**$\wedge$**T** into **ITD**; the new translation ** generalises the old one, since it maps $\mathbf{B}_+$ into $\mathbf{ITD}\vee$. As expected, the coincidence theorem also holds for the translation **; i.e. we have:

[14] We extend our convention by making $\mathcal{V}[U/x](y)$ the interpretation $\mathcal{V}$ determines by the x-variant $v[U/x](y)$.

Extended coincidence theorem. $\mathbf{ITD}\vee$ = $\mathbf{B}_+$ on the ** translation.
The proof can be given using the same metavaluation argument that we introduced for proving the coincidence theorem. It suffices to add to the definition of the class MTR the clause:

$$v\vee. \quad A^{**}\vee B^{**}\in \texttt{MTR} \text{ iff either } A^{**}\in\texttt{MTR} \text{ or } B^{**}\in\texttt{MTR}$$

In fact it is easy to verify that the coherence lemma still holds, i.e. that $A\in\mathbf{B}_+ \Rightarrow A^{**}\in$ MTR.

We can continue as in section 1. Let $\mathbf{K}$ be a $+ms$ and $Form\vee$ be the set of all formulas in $\mathbf{ITD}\vee$. We can define an interpretation $\mathcal{I}$ from $Form\vee\times K$ to **2** by adding the following clause:

$$T\vee. \quad \mathcal{I}(A\vee B,w)=max[\mathcal{I}(A,w),\mathcal{I}(B,w)]$$

to the clauses $Tp,\ T\wedge,\ T\rightarrow,\ T\mathbf{T}$.
We can borrow from section 1 the definitions of entailment, fusion and theory, obviously considering formulas in $Form\vee$ instead of formulas in $Form$. In this way we get a calculus of theories $\mathbf{CT}\vee$. We do have the following:
Soundness theorem for $\mathbf{ITD}\vee$. If $\mathbf{ITD}\vee\ \models A\leq B$ then A (positively) entails B.

This is *halfway* to where we arrived happily at the end of section 1. We would like to supply the other (completeness) half and then to continue as in section 2. Note however that the canonical lemma above does NOT extend smoothly to $\mathbf{CT}\vee$. Extending the canonical valuation we obtain an interpretation which does not satisfy clause $T\vee$. The obvious example is $(A\vee B)\uparrow$: $A\vee B\in(A\vee B)\uparrow$ but $A,B\notin(A\vee B)\uparrow$.

We can generalise $\equiv$ to $\mathbf{ITD}\vee$ in the obvious way. But we do not know how to go on. The first problem is that the Bubbling lemma (ii) no longer holds. The counter-example is under the eyes of everybody: it is just the axiom $\rightarrow\vee$E. The unpleasant consequence of this is that $\{\bigwedge_{i\in I}(A_i\rightarrow B_i)\ :\ \ B_i\in\mathcal{V}[A_i\uparrow\ /x](M)\}$ is no longer a theory for all λ-terms M and all λ-valuations v. The counter-example is again related to axiom $\rightarrow\vee$E. Take $M_0\equiv\lambda x.yxx$ and $v_0(y)=((A\rightarrow A\rightarrow C)\wedge(B\rightarrow B\rightarrow C))\uparrow$. We get $(A\rightarrow C)\wedge(B\rightarrow C)\in\mathcal{V}_0(M_0)$, but $A\vee B\rightarrow C\notin\mathcal{V}_0(M_0)$. To see why, observe that $A\vee B\rightarrow C$ is an element of $\mathcal{V}_0(M_0)$ only if C is an element of $\mathcal{V}_0[A\vee B\uparrow\ /x](yxx)$. And C is an element of $\mathcal{V}_0[(A\vee B)\uparrow\ /x](yxx)$ only if we can find D such that $D\rightarrow C\in\mathcal{V}_0[(A\vee B)\uparrow\ /x](yx)$ and $D\in\mathcal{V}_0[(A\vee B)\uparrow\ /x](x)$. But such a D does not exist, since it is easy to verify that $\mathcal{V}_0[(A\vee B)\uparrow\ /x](yx)=((A\rightarrow C)\vee(B\rightarrow C))\uparrow$, and therefore also $\mathcal{V}_0[(A\vee B)\uparrow\ /x](yxx)=\mathbf{T}\uparrow$.

An obvious recipe to remedy this drawback is to force the interpretation of an abstraction to be a theory, by defining

$$\mathcal{V}\lambda\vee. \quad \mathcal{V}(\lambda x.M)=\{A\rightarrow B:\ B\in\mathcal{V}[A\uparrow\ /x](M)\}\uparrow$$ [15]

where by $U\uparrow$ we mean the minimal theory containing the set of formulas U, i.e. the closure of U under $\wedge$ and $\leq$. But the problem we pushed out of the door will come back through the window. For this new definition of λ-interpretation loses the key property characterising models of λ and **CL** – i.e., the property $I\lambda\circ$. The previously introduced λ-term M_0 and

[15]The closure ($\uparrow$) allows us to avoid intersections of arrow formulas (cf. clause $\mathcal{V}\lambda$).

the λ-valuation v_0 are again good choices to point out our failure. In fact now we oblige $A \vee B \to C$ to be an element of $\mathcal{V}_0(M_0)$; therefore we have

$$C \in \mathcal{V}_0(M_0) \circ (A \vee B)\uparrow$$

But the other clauses of λ-interpretation are unchanged, so we have as before $\mathcal{V}_0[(A \vee B)\uparrow /x](yxx) = \mathbf{T}\uparrow$. We must conclude that $I\lambda\circ$ fails! The underlying point of this counter example is that the set of $\vee$-prime theories is NOT closed under fusion. As usual, a theory is $\vee$-*prime* iff it contains either A or B whenever it contains $A \vee B$. So, $\vee$-prime theories are exactly the theories which satisfy clause $T\vee$. We can easily show that $\vee$-prime theories are NOT closed under fusion, as follows: let p, q, r be (type) variables,

$$\begin{aligned} X &= (p \to (q \vee r))\uparrow \text{ is } \vee\text{-prime, at the level of } \mathbf{B}_+, \\ Y &= p\uparrow \text{ is also } \vee\text{-prime.} \end{aligned}$$

Set $Z = X \circ Y$. Then $q \vee r \in Z$. But $q \notin Z$ and $r \notin Z$.

4. The calculus of Harrop theories HCT is a model for λ and CL

The crucial idea to which we appeal in this paper to overcome the failure of the previous section is in Harrop's paper [13]. To take advantage of it, we define the set $HForm \subseteq Form\vee$ of *Harrop formulas* as follows:

$p \in HForm$ for all $p \in Var$
$\mathbf{T} \in HForm$
if $A, B \in HForm$ then $A \wedge B \in HForm$
if $A \in Form\vee$ and $B \in HForm$ then $A \to B \in HForm$

Using this definition we can easily verify that:

Claim. If $C \in HForm$ then there are two finite sets I and K of indices, variables $p_k \in Var$ for all $k \in K$ and formulas $A_i \in Form\vee$, $B_i \in HForm$ for all $i \in I$ such that $I \cup K$ is non-empty and $C \equiv (\bigwedge_{i \in I}(A_i \to B_i)) \wedge (\bigwedge_{k \in K} p_k)$.

In fact, if C is $\mathbf{T}$, then $C \equiv \mathbf{T} \to \mathbf{T}$. If C is $A \wedge B$ with $A, B \in HForm$ the claim follows by induction, and lastly if C is a variable or of the form $A \to B$ the claim is immediate.

The main feature of Harrop formulas is that they allow us to recover a (restricted) version of the Bubbling lemma.

Bubbling lemma for $Form\vee$. (i) $A \to B \equiv \mathbf{T}$ iff $B \equiv \mathbf{T}$. (ii) Assume $C \in HForm$ and it is NOT the case that $D \equiv \mathbf{T}$. Assume moreover that we have $(\bigwedge_{i \in I}(A_i \to B_i)) \wedge (\bigwedge_{k \in K} p_k) \leq C \to D$ for the finite index sets I, K. Then I is non-empty and there is a finite non-empty subset J of I such that

$$C \leq \bigwedge_{i \in J} A_i \text{ and } \bigwedge_{i \in J} B_i \leq D.$$

The proof if point (i) by induction on the construction of $\equiv$ is standard. The proof of point (ii) involves a stratification of formulas and we give it in the Appendix.

We want to consider only theories which are essentially based on Harrop formulas. For this reason we say that a theory $U \subseteq Form\vee$ is an *Harrop theory* if and only if for all $A \in U$ there is $A' \in U$ such that $A' \in HForm$ and $A' \leq A$. In the remaining of this section we will deal only with the set $\mathbf{HCT}$ of Harrop theories.

We show the soundness of the calculus of theories $\mathbf{HCT} =< HCT, \circ >$, i.e., that Harrop theories are closed under the fusion operation $\circ$.
By definition $U \circ V = \{B : \exists A \in Form\vee(A \to B \in U$ and $A \in V)\}$. We will prove that for all $B \in U \circ V$ there is $B' \in U \circ V$ such that $B' \in HForm$ and $B' \leq B$. The case $B \equiv \mathbf{T}$ is trivial, so in the following we assume $B \not\equiv \mathbf{T}$. Now $A \in V$, where V is an Harrop theory, implies that there is $A' \in V$ such that $A' \in HForm$ and $A' \leq A$. From $A \to B \in U$ we get $A' \to B \in U$, since $A' \leq A$ implies $A \to B \leq A' \to B$ and U being a theory is closed under $\leq$. Since also U is an Harrop theory, there is $C \in U$ such that $C \in HForm$ and $C \leq A' \to B$. By the claim we have $C \equiv (\bigwedge_{i \in I}(A_i \to B_i)) \wedge (\bigwedge_{k \in K} p_k)$ for some sets I, K of indices, variables $p_k \in Var$, and formulas $A_i \in Form\vee$, $B_i \in HForm$ for all $i \in I$. Now $(\bigwedge_{i \in I}(A_i \to B_i)) \wedge (\bigwedge_{k \in K} p_k) \leq A' \to B$, $B \not\equiv \mathbf{T}$, and $A' \in HForm$ imply that there is a finite non-empty subset J of I such that $A' \leq \bigwedge_{i \in J} A_i$ and $\bigwedge_{i \in J} B_i \leq B$ by the Bubbling lemma (ii) for $Form\vee$. We will show now that $\bigwedge_{i \in J} B_i$ is a correct choice for B'. First notice that each $B_i \in HForm$, whence $\bigwedge_{i \in J} B_i \in HForm$ by definition. Since $A' \leq \bigwedge_{i \in J} A_i$ we get $\bigwedge_{i \in J} A_i \in V$. Moreover $\bigwedge_{i \in J} A_i \to \bigwedge_{i \in J} B_i \in U$, since $C \leq \bigwedge_{i \in I}(A_i \to B_i) \leq \bigwedge_{i \in I}(\bigwedge_{i \in J} A_i \to B_i) \leq \bigwedge_{i \in J} A_i \to \bigwedge_{i \in J} B_i$. Therefore we get $\bigwedge_{i \in J} B_i \in U \circ V$, and this concludes our proof.

Point (ii) of the Bubbling lemma for $Form\vee$ suggests that the Harrop formulas are good guys. To make the most of this property in the construction of our model we build the interpretation of λ-abstraction starting only from formulas of this shape. More precisely we extend a *λ-H-valuation* v (assigning Harrop theories to λ-variables) to the corresponding *λ-H-interpretation* $\mathcal{V}^H$ as follows:

$\mathcal{V}^H x.$ $\quad \mathcal{V}^H(x) = v(x)$
$\mathcal{V}^H \circ.$ $\quad \mathcal{V}^H(MN) = \mathcal{V}^H(M) \circ \mathcal{V}^H(N)$
$\mathcal{V}^H \lambda.$ $\quad \mathcal{V}^H(\lambda x.M) = \{A \to B : A \in HForm$ and $B \in \mathcal{V}^H[A\uparrow /x](M)\}\uparrow$

The soundness of this definition requires that all λ-H-interpretations are Harrop theories. This can be proved by induction on the construction of λ-H-interpretations itself. The only non-trivial case is clause $\mathcal{V}^H\lambda$. Now $\mathcal{V}^H(\lambda x.M)$ is a theory by construction. To show that it is an Harrop theory, we need to build $C' \in \mathcal{V}^H(\lambda x.M)$ such that $C' \in HForm$ and $C' \leq C$ given an arbitrary $C \in \mathcal{V}^H(\lambda x.M)$. Now $C \in \mathcal{V}^H(\lambda x.M)$ implies $\bigwedge_{i \in I}(A_i \to B_i) \leq C$, for some set of indices I and formulas $A_i \in HForm$, $B_i \in Form\vee$ such that $B_i \in \mathcal{V}^H[A_i\uparrow /x](M)$ for all $i \in I$. By induction each $\mathcal{V}^H[A_i\uparrow /x](M)$ is an Harrop theory, and therefore we can find $B'_i \in \mathcal{V}^H[A_i\uparrow /x](M)$ such that $B'_i \in HForm$ and $B'_i \leq B_i$. We want to show that $\bigwedge_{i \in I}(A_i \to B'_i)$ is a correct choice for C'. First notice that by definition $\bigwedge_{i \in I}(A_i \to B'_i) \in \mathcal{V}^H(\lambda x.M)$ since $B'_i \in \mathcal{V}^H[A_i\uparrow /x](M)$. Moreover $B'_i \in HForm$ implies $A_i \to B'_i \in HForm$ for all $i \in I$, whence $\bigwedge_{i \in I}(A_i \to B'_i) \in HForm$. Lastly $\bigwedge_{i \in I}(A_i \to B'_i) \leq C$, since $B'_i \leq B_i$ for all $i \in I$ (whence $A_i \to B'_i \leq A_i \to B_i$ and $\bigwedge_{i \in I}(A_i \to B'_i) \leq \bigwedge_{i \in I}(A_i \to B_i)$) and $\bigwedge_{i \in I}(A_i \to B_i) \leq C$.

As in section 2, to prove that we have really obtained a λ-model it is crucial to show compositionality of interpretations. In the present case this is stronger, since we can limit our consideration to Harrop formulas.

Compositionality Lemma. For all Harrop theories U, λ-H-valuations v and λ-terms M

$$\mathcal{V}^H[U/x](M) = \bigcup_{A \in U \cap HForm} \mathcal{V}^H[A\!\uparrow /x](M).$$

Proof. By induction on the construction of λ-H-interpretations. For clause $\mathcal{V}^H x$ notice that if U is an Harrop theory, then $U = \{A : \exists A' \in HForm(A' \in U \text{ and } A' \leq A)\}$. The other clauses follow by induction.

A further useful property of λ-H-interpretations concerns abstraction.

Abstraction Lemma. If $A \to B \in \mathcal{V}^H(\lambda x.M)$ and $A \in HForm$, then $B \in \mathcal{V}^H[A\!\uparrow /x](M)$.
Proof. If $A \to B \in \mathcal{V}^H(\lambda x.M)$ then $\bigwedge_{i \in I}(A_i \to B_i) \leq A \to B$, for some set of indices I and formulas $A_i \in HForm$, $B_i \in Form\vee$ such that $B_i \in \mathcal{V}^H[A_i\!\uparrow /x](M)$ for all $i \in I$. The Bubbling lemma (ii) for $Form\vee$ implies that there is $J \subseteq I$ such that $A \leq \bigwedge_{i \in J} A_i$ and $\bigwedge_{i \in J} B_i \leq B$, where $A \in HForm$ by hypothesis. Now $A \leq \bigwedge_{i \in J} A_i$ implies $A\!\uparrow \supseteq (\bigwedge_{i \in J} A_i)\!\uparrow \supseteq A_i\!\uparrow$ for all $i \in J$. Since the λ-H-interpretations are monotone we get $B_i \in \mathcal{V}^H[A\!\uparrow /x](M)$ for all $i \in J$. Whence we conclude $B \in \mathcal{V}^H[A\!\uparrow /x](M)$ using $\bigwedge_{i \in J} B_i \leq B$.

The condition $A \in HForm$ in the previous lemma is necessary, since for example $A \vee B \to C \in \mathcal{V}_0^H(M_0)$ but $C \notin \mathcal{V}_0^H[(A \vee B)\!\uparrow /x](yxx)$, where $M_0 \equiv \lambda x.yxx$ and $v_0(y) = ((A \to A \to C) \wedge (B \to B \to C))\!\uparrow$.

To conclude our job we want to prove our main result, i.e. that $\mathbf{HCT}$ is a λ-model, showing that $\mathcal{V}^H$ is a syntactic interpretation according to the definition given at page 7.

Main Theorem. $\mathbf{HCT}$ is a λ-model.
Proof. We already know that the only interesting case is clause $I\lambda\circ$ in the definition of syntactic interpretations. We have

$$\mathcal{V}^H(\lambda x.M)\circ U = \{B : \exists A \in Form\vee(A \to B \in \mathcal{V}^H(\lambda x.M) \text{ and } A \in U)\}.$$

Since U is an Harrop theory, there is $A' \in U$ such that $A' \in HForm$ and $A' \leq A$. Now $A' \leq A$ implies $A \to B \leq A' \to B$, whence $A' \to B \in \mathcal{V}^H(\lambda x.M)$. We get

$$\mathcal{V}^H(\lambda x.M)\circ U = \{B : \exists A' \in HForm(A' \to B \in \mathcal{V}^H(\lambda x.M) \text{ and } A' \in U)\}.$$

By the abstraction lemma from $A' \in HForm$ and $A' \to B \in \mathcal{V}^H(\lambda x.M)$ we have $B \in \mathcal{V}^H[A'\!\uparrow /x](M)$. Therefore we obtain

$$\mathcal{V}^H(\lambda x.M)\circ U = \{B : \exists A' \in HForm(B \in \mathcal{V}^H[A'\!\uparrow /x](M) \text{ and } A' \in U)\},$$

so by the compositionality lemma we conclude

$$\mathcal{V}^H(\lambda x.M)\circ U = \mathcal{V}^H[U\!\uparrow /x](M).$$

Our last remark is that in the case of Harrop theories completeness fails. In fact we have that $\mathbf{ITD}\vee \;\not\models p \to q \vee r \leq (p \to q) \vee (p \to r)$. So we would like to find an Harrop theory U such that $p \to q \vee r \in U$ but $(p \to q) \vee (p \to r) \notin U$. By definition of Harrop theory $p \to q \vee r \in U$ implies there is $A \in HForm \cap U$ such that $A \leq p \to q \vee r$. Now clearly we can only choose either $p \to q$ or $p \to r$ as A.

Conclusion

The main result of the present paper is that the calculus of Harrop theories over the minimal relevant logic $\mathbf{B}_+$ is a model of λ and **CL**. We seek nonetheless a *better model* in a wider class of $\vee$-prime $\mathbf{B}_+$-theories, as a direction for future research and for the further illumination of logics and of types. Recently further progress has been made in this direction: [9] compares $\mathbf{B}_+$ with the semantics-based approach to subtyping introduced by Frisch, Castagna and Benzaken [12] in the definition of a type system with intersection and union. [9] shows that – for the functional core of the system – such notion of subtyping, which is defined in purely set-theoretical terms, coincides with the relevant entailment of the logic $\mathbf{B}_+$.

Appendix

We will use a stratification of $Form\vee$. A similar stratification was considered in [3].

Definition 1 (Stratification of $Form\vee$)**.** $T_\to, T_\vee, T_\wedge, T_{\wedge\vee}, T_{\vee\wedge} \subseteq Form\vee$ *are recursively defined by:*

$(T_\to)$ $\mathbf{T} \in T_\to$
$p \in T_\to$ *for all type variables* p
$A \in T_\wedge, B \in T_\vee \Rightarrow A \to B \in T_\to$
$(T_\vee)$ $A \in T_\to \Rightarrow A \in T_\vee$
$A, B \in T_\vee \Rightarrow A \vee B \in T_\vee$
$(T_\wedge)$ $A \in T_\to \Rightarrow A \in T_\wedge$
$A, B \in T_\wedge \Rightarrow A \wedge B \in T_\wedge$
$(T_{\wedge\vee})$ $A \in T_\vee \Rightarrow A \in T_{\wedge\vee}$
$A, B \in T_{\wedge\vee} \Rightarrow A \wedge B \in T_{\wedge\vee}$
$(T_{\vee\wedge})$ $A \in T_\wedge \Rightarrow A \in T_{\vee\wedge}$
$A, B \in T_{\vee\wedge} \Rightarrow A \vee B \in T_{\vee\wedge}$.

Specialisation of $\leq$ to the sets T_i are now introduced, whose definition exploits the syntactical form of the types in T_i.

Definition 2. $\leq_i \subseteq T_i \times T_i$ $(i = \to, \vee, \wedge, \wedge\vee, \vee\wedge)$ *are the least preorders such that*

$(\leq_\to)$ $A \leq_\to B \Leftrightarrow$ *either* $B = \mathbf{T}$ *or* $A = B$
or $A = A_1 \to A_2, B = B_1 \to B_2$ *and* $B_1 \leq_\wedge A_1, A_2 \leq_\vee B_2$

$(\leq_\vee)$ $\bigvee_{i\in I} A_i \leq_\vee \bigvee_{j\in J} B_j$ (*where* $A_i, B_j \in T_\to$) $\Leftrightarrow \forall i \in I \exists j \in J, A_i \leq_\to B_j$

$(\leq_\wedge)$ $\bigwedge_{i\in I} A_i \leq_\wedge \bigwedge_{j\in J} B_j$ (*where* $A_i, B_j \in T_\to$) $\Leftrightarrow \forall j \in J \exists i \in I, A_i \leq_\to B_j$

$(\leq_{\wedge\vee})$ $\bigwedge_{i\in I} A_i \leq_{\wedge\vee} \bigwedge_{j\in J} B_j$ (*where* $A_i, B_j \in T_\vee$) $\Leftrightarrow \forall j \in J \exists i \in I, A_i \leq_\vee B_j$

$(\leq_{\vee\wedge})$ $\bigvee_{i\in I} A_i \leq_{\vee\wedge} \bigvee_{j\in J} B_j$ (*where* $A_i, B_j \in T_\wedge$) $\Leftrightarrow \forall i \in I \exists j \in J, A_i \leq_\wedge B_j$.

Lemma 3. $\leq_i$ $(i = \to, \vee, \wedge, \wedge\vee, \vee\wedge)$ *are reflexive and transitive.*

Proof. By induction the construction of $\leq_i$. □

We will now introduce maps from arbitrary formulas belonging to $Form\vee$ into their conjunctive/disjunctive normal forms in $T_{\wedge\vee}$ and $T_{\vee\wedge}$, respectively.

Definition 4. *The maps $\mathbf{m}_{\wedge\vee} : Form\vee \to T_{\wedge\vee}$ and $\mathbf{m}_{\vee\wedge} : Form\vee \to T_{\vee\wedge}$ are defined by simultaneous induction the structure of formulae:*

(i) $\mathbf{m}_{\wedge\vee}(A) = \mathbf{m}_{\vee\wedge}(A) = A$ *if* $A = \mathbf{T}$ *or* A *is a variable.*

(ii) *If* $\mathbf{m}_{\vee\wedge}(A) = \bigvee_{i\in I} A_i$ *and* $\mathbf{m}_{\wedge\vee}(B) = \bigwedge_{j\in J} B_j$ *then*

$$\mathbf{m}_{\wedge\vee}(A \to B) = \mathbf{m}_{\vee\wedge}(A \to B) = \bigwedge_{i\in I} \bigwedge_{j\in J} (A_i \to B_j).$$

(iii) $\mathbf{m}_{\wedge\vee}(A \wedge B) = \mathbf{m}_{\wedge\vee}(A) \wedge \mathbf{m}_{\wedge\vee}(B)$*, and, if* $\mathbf{m}_{\vee\wedge}(A) = \bigvee_{i\in I} A_i$ *and* $\mathbf{m}_{\vee\wedge}(B) = \bigvee_{j\in J} B_j$ *then*

$$\mathbf{m}_{\vee\wedge}(A \wedge B) = \bigvee_{i\in I} \bigvee_{j\in J} (A_i \wedge B_j).$$

(iv) $\mathbf{m}_{\vee\wedge}(A \vee B) = \mathbf{m}_{\vee\wedge}(A) \vee \mathbf{m}_{\vee\wedge}(B)$*, and, if* $\mathbf{m}_{\wedge\vee}(A) = \bigwedge_{i\in I} A_i$ *and* $\mathbf{m}_{\wedge\vee}(B) = \bigwedge_{j\in J} B_j$ *then*

$$\mathbf{m}_{\wedge\vee}(A \vee B) = \bigwedge_{i\in I} \bigwedge_{j\in J} (A_i \vee B_j).$$

The following proposition states that conjunctive/disjunctive normal forms are logically equivalent to their counterimages under $\mathbf{m}_{\wedge\vee}()$ and $\mathbf{m}_{\vee\wedge}()$, and that the specialised relations $\leq_i$ are actually restrictions of $\leq$ to the sets T_i respectively.

Proposition 5. *For all* $A, B \in Form\vee$ *:*

(i) $A \equiv \mathbf{m}_{\wedge\vee}(A) \equiv \mathbf{m}_{\vee\wedge}(A)$.

(ii) $A, B \in T_i, A \leq_i B \Rightarrow A \leq B$ *for* $i = \to, \vee, \wedge, \vee\wedge, \wedge\vee$.

(iii) $A \leq B \Leftrightarrow \mathbf{m}_{\wedge\vee}(A) \leq_{\wedge\vee} \mathbf{m}_{\wedge\vee}(B) \Leftrightarrow \mathbf{m}_{\vee\wedge}(A) \leq_{\vee\wedge} \mathbf{m}_{\vee\wedge}(B)$.

Proof. (i) By induction on the structure of A. E.g. if $A = B \to C$ then, by induction hypothesis, we have $B \equiv \mathbf{m}_{\vee\wedge}(B) = \bigvee_{i\in I} B_i$ and $C \equiv \mathbf{m}_{\wedge\vee}(C) = \bigwedge_{j\in J} C_j$, so that, by repeated uses of (→∧I), (→∨E) and (Mon→) we conclude that

$$B \to C \equiv \bigvee_{i\in I} B_i \to \bigwedge_{j\in J} C_j \equiv \bigwedge_{i\in I} \bigwedge_{j\in J} (B_i \to C_j) \equiv \mathbf{m}_{\wedge\vee}(B \to C) = \mathbf{m}_{\vee\wedge}(B \to C).$$

(ii) By straightforward induction on the construction of $\leq_i$.

(iii) Implications ($\Leftarrow$) are immediate consequences of (i) and (ii). To prove ($\Rightarrow$) we use induction on the construction of $\leq$. All cases are simple calculations. E.g. case (Mon∨) $A \leq B, C \leq D \Rightarrow A \vee C \leq B \vee D$: by induction hypothesis

$$\mathbf{m}_{\wedge\vee}(A) \leq_{\wedge\vee} \mathbf{m}_{\wedge\vee}(B) \Rightarrow \forall j \in J\ \exists i \in I\ \forall n \in I_i\ \exists q \in J_j, A_{i,n} \leq_{\to} B_{j,q},$$

where $\mathbf{m}_{\wedge\vee}(A) = \bigwedge_{i\in I} A_i$, $\mathbf{m}_{\vee\wedge}(A_i) = \bigvee_{n\in I_i} A_{i,n}$, and $\mathbf{m}_{\wedge\vee}(B) = \bigwedge_{j\in J} B_j$, $\mathbf{m}_{\vee\wedge}(B_j) = \bigvee_{q\in J_j} B_{j,q}$. Similarly,

$$\mathbf{m}_{\wedge\vee}(C) \leq_{\wedge\vee} \mathbf{m}_{\wedge\vee}(D) \Rightarrow \forall l \in L\ \exists k \in K\ \forall r \in K_k\ \exists s \in L_l, C_{k,r} \leq_{\to} D_{l,s},$$

where $\mathbf{m}_{\wedge\vee}(C) = \bigwedge_{k\in K} C_k$, $\mathbf{m}_{\vee\wedge}(C_k) = \bigvee_{r\in K_k} C_{k,r}$ and $\mathbf{m}_{\wedge\vee}(D) = \bigwedge_{l\in L} D_l$, $\mathbf{m}_{\vee\wedge}(D_l) = \bigvee_{s\in L_l} D_{l,s}$. Then we have

$$\begin{aligned}
& \forall j \in J, l \in L \big[\exists i \in I\ \forall n \in I_i\ \exists q \in J_j, A_{i,n} \leq_{\rightarrow} B_{j,q} \\
& \ \text{and}\ \exists k \in K\ \forall r \in K_k\ \exists s \in L_l, C_{k,r} \leq_{\rightarrow} D_{l,s}\big] \\
\Rightarrow\ & \forall j \in J, l \in L\ \exists i \in I, k \in K, \bigvee_{n\in I_i} A_{i,n} \vee \bigvee_{r\in K_r} C_{k,r} \leq_{\vee} \bigvee_{q\in J_j} B_{j,q} \vee \bigvee_{s\in L_l} D_{l,s} \\
\Rightarrow\ & \forall j \in J, l \in L\ \exists i \in I, k \in K, A_i \vee C_k \leq_{\vee} B_j \vee D_l \\
\Rightarrow\ & \bigwedge_{i\in I}\bigwedge_{k\in K}(A_i \vee C_k) \leq_{\wedge\vee} \bigwedge_{j\in J}\bigwedge_{l\in L}(B_j \vee D_l) \\
\Rightarrow\ & \mathbf{m}_{\wedge\vee}(A \vee C) \leq_{\wedge\vee} \mathbf{m}_{\wedge\vee}(B \vee D).
\end{aligned}$$

□

The converse of Proposition 5(ii) is false, an example is just axiom (→∨E).

We eventually come to the proof of the Bubbling lemma for $Form\vee$ using the notion of ∨-prime formulas.

Definition 6. *A formula A is ∨-prime iff $A \leq B \vee C \Rightarrow A \leq B$ or $A \leq C$.*

Theorem 7 (Bubbling for $\mathbf{B}_+$). (i) *Each Harrop formula is ∨-prime.*

(ii) *$(\bigwedge_{i\in I}(A_i \rightarrow B_i)) \wedge (\bigwedge_{k\in K} p_k) \leq C \rightarrow D$, $D \not\equiv \mathbf{T}$, and C is ∨-prime imply $C \leq \bigwedge_{i\in J} A_i$ and $\bigwedge_{i\in J} B_i \leq D$ for some $J \subseteq I$.*

Proof. By the claim at page 9 each Harrop formula is equivalent to $(\bigwedge_{i\in I}(A_i \rightarrow B_i)) \wedge (\bigwedge_{k\in K} p_k)$ for suitable formulas A_i, B_i and variables p_k.

(i) By proposition 5(iii) we have:

$$(\bigwedge_{i\in I}(A_i \rightarrow B_i)) \wedge (\bigwedge_{k\in K} p_k) \leq C \vee D \ \Leftrightarrow$$
$$\mathbf{m}_{\vee\wedge}((\bigwedge_{i\in I}(A_i \rightarrow B_i)) \wedge (\bigwedge_{k\in K} p_k)) \leq_{\vee\wedge} \mathbf{m}_{\vee\wedge}(C \vee D).$$

Now $\mathbf{m}_{\vee\wedge}(\bigwedge_{i\in I}(A_i \rightarrow B_i) \wedge (\bigwedge_{k\in K} p_k))$ is a conjunction of arrows and variables, namely a formula with no disjunction at the top level; on the other hand $\mathbf{m}_{\vee\wedge}(C \vee D)$ has the form $\bigvee_{j\in J} C_j \vee \bigvee_{l\in L} D_l$ where $\mathbf{m}_{\vee\wedge}(C) = \bigvee_{j\in J} C_j$ and $\mathbf{m}_{\vee\wedge}(D) = \bigvee_{l\in L} D_l$. By definition of $\leq_{\vee\wedge}$ we immediately have that

$$\mathbf{m}_{\vee\wedge}(\bigwedge_{i\in I}(A_i \rightarrow B_i) \wedge (\bigwedge_{k\in K} p_k)) \quad \leq_{\wedge} \quad C_j$$
or
$$\mathbf{m}_{\vee\wedge}(\bigwedge_{i\in I}(A_i \rightarrow B_i) \wedge (\bigwedge_{k\in K} p_k)) \quad \leq_{\wedge} \quad D_l,$$

for some j, l; therefore the thesis follows by Proposition 5(i) and (ii).

(ii) Let first compute:

$$\mathbf{m}_{\vee\wedge}(\bigwedge_{i\in I}(A_i \rightarrow B_i)) = \bigwedge_{i\in I}\left[\bigwedge_{h\in H_i}\bigwedge_{l\in L_i}(A_{i,h} \rightarrow B_{i,l})\right],$$

where $\mathbf{m}_{\vee\wedge}(A_i) = \bigvee_{h\in H_i} A_{i,h}$, and $\mathbf{m}_{\wedge\vee}(B_i) = \bigwedge_{l\in L_i} B_{i,l}$. On the other hand suppose that $\mathbf{m}_{\vee\wedge}(C \rightarrow D) = \bigwedge_{k\in K}\bigwedge_{q\in Q}(C_k \rightarrow D_q)$, where $\mathbf{m}_{\vee\wedge}(C) =$

$\bigvee_{k\in K} C_k$, and $\mathbf{m}_{\wedge\vee}(D) = \bigwedge_{q\in Q} D_q$. By Proposition 5(iii) and the definition of $\leq_{\wedge\vee}$ we have

$$\forall k \in K, q \in Q\ \exists i \in I, h \in H_i, l \in L_i.\ C_k \leq_\wedge A_{i,h}\ \&\ B_{i,l} \leq_\vee D_q.$$

By Proposition 5(i), $C \equiv \bigvee_{k\in K} C_k$: hence, since C is $\vee$-prime, there exists $k_0 \in K$ such that $C \leq C_{k_0}$. Choose one such k_0 and, for any $q \in Q$, define

$$J_q = \{i \in I \mid \exists h \in H_i, l \in L_i.\ C_{k_0} \leq_\wedge A_{i,h}\ \&\ B_{i,l} \leq_\vee D_q\},$$

which is non-empty by the above statement. Finally, we take $J = \bigcup_{q\in Q} J_q$. Now, for all $i \in J$, there exists $h \in H_i$ such that $C_{k_0} \leq A_{i,h} \leq A_i$: therefore $C \leq C_{k_0} \leq \bigwedge_{i\in J} A_i$.

To conclude, for all $q \in Q$ there is $i \in J_q$ and $l \in L_i$ such that $B_i \leq B_{i,l} \leq D_q$: then $\bigwedge_{i\in J} B_i \leq D_q$ for all q, and, therefore, $\bigwedge_{i\in J} B_i \leq \bigwedge_{q\in Q} D_q \equiv D$.

□

The condition C is $\vee$-prime in point (ii) of the above theorem is necessary. A counter-example is axiom ($\rightarrow\vee$E).

References

[1] A. R. Anderson and N. D. Belnap Jr. *Entailment*, volume I. Princeton, 1975.

[2] A. R. Anderson, N. D. Belnap Jr, and J. M. Dunn. *Entailment*, volume II. Princeton, 1992.

[3] S. van Bakel, M. Dezani-Ciancaglini, U. de'Liguoro, and Y. Motohama. The minimal relevant logic and the call-by-value lambda calculus. Technical Report TR-ARP-05-2000, Australian National University, 2000.

[4] F. Barbanera, M. Dezani-Ciancaglini, and U. de'Liguoro. Intersection and union types: syntax and semantics. *Information and Computation*, 119:202–230, 1995.

[5] H. P. Barendregt. *The lambda calculus: its syntax and semantics*. North-Holland, Amsterdam, 2nd edition, 1984.

[6] H. P. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *Journal of Symbolic Logic*, 48, 1983.

[7] H. B. Curry. *Foundations of mathematical logics*. McGraw-Hill, New York, 1963.

[8] H. B. Curry and R. Feys. *Combinatory logic*, volume 1. North-Holland, Amsterdam, 1958.

[9] M. Dezani-Ciancaglini, A. Frisch, E. Giovannetti, and Y. Motohama. The relevance of semantic sybtyping. In *ITRS 2002*, volume 70 (1) of *Electric Notes in Theoretical Computer Science*, 2002. URL: `http://www.elsevier.nl/locate/entcs/volume70.html`.

[10] M. Dezani-Ciancaglini and J. R. Hindley. Intersection types for combinatory logic. *Theoretical Computer Science*, 100:303–324, 1992.

[11] J. M. Dunn. *The algebra of intensional logics*. PhD thesis, University of Pittsburgh, 1966.

[12] A. Frisch, G. Castagna, and V. Benzaken. Semantic subtyping. In *17th IEEE Symposium on Logic in Computer Science*, pages 137–146. IEEE Computer Society Press, 2002.

[13] R. Harrop. Concerning formulas of the types $a \rightarrow b \vee c, a \rightarrow (\exists x)b(x)$ in intuitionistic formal systems. *Journal of Symbolic Logic*, 25:27–32, 1960.

[14] J. R. Hindley and G. Longo. Lambda calculus models and extensionality. *Z. Math. Logik Grundlagen Math.*, 1980.

[15] J. R. Hindley and J. P. Seldin. *Introduction to combinators and λ-calculus*. Cambridge university press, Cambridge, 1986.

[16] S. Kripke. Semantical analysis of modal logic. I. normal modal propositional calculus. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[17] H. Leblanc. On dispensing with things and worlds. In M. K. Munitz, editor, *Logic and ontology*. New York, 1973. reprinted in H. Leblanc, *Existence, truth and provability*, New York, 1982.

[18] R. K. Meyer and R. Routley. Algebraic analysis of entailment I. *Logique et Analyse*, 15:407–428, 1972.

[19] R. Routley and R. K. Meyer. The semantics of entailment III. *Journal of Philosophical Logic*, 1:192–208, 1972.

[20] R. Routley, with V. Plumwood, R. K. Meyer, and R. T. Brady. *Relevant logics and their rivals*. Ridgeview, Atascadero, California, 1982.

[21] B. Venneri. Intersection types as logical formulae. *Journal of Logic and Computation*, 4(2):109–124, 1994.