

A Lambda Model Characterizing Computational Behaviours of Terms

joint paper with Silvia Ghilezan

RPC'01, Sendai, October 26, 2001

Plan of the talk

- normalization properties
- inverse limit model
- Stone dualities
- type assignment system
- reducibility method
- replaceability

Normalization properties

$M \in \mathcal{N}$ iff $M \rightarrow_{\beta}$ a normal form

$M \in \mathcal{HN}$ iff $M \rightarrow_{\beta} \lambda \vec{x}.y\vec{N}$

$M \in \mathcal{WN}$ iff $M \rightarrow_{\beta} \lambda x.N$ or $M \rightarrow_{\beta} x\vec{N}$

Persistent normalization properties

$M \in \mathcal{PN}$ iff $M\vec{N} \in \mathcal{N}$ for all terms \vec{N} in \mathcal{N}

$M \in \mathcal{PHN}$ iff $M\vec{N} \in \mathcal{HN}$ for all terms \vec{N}

$M \in \mathcal{PWN}$ iff $M\vec{N} \in \mathcal{WN}$ for all terms \vec{N}

$\mathbf{I} \equiv \lambda x.x$, $\Delta \equiv \lambda x.xx$, $\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, $\mathbf{K} \equiv \lambda xy.x$

$\lambda x.x\Delta\Delta \in \mathcal{N}$, but $\notin \mathcal{PW}\mathcal{N}$, since $(\lambda x.x\Delta\Delta)\mathbf{I} \twoheadrightarrow_{\beta} \Delta\Delta \notin \mathcal{WN}$

$\lambda x.y(\Delta\Delta) \in \mathcal{PH}\mathcal{N}$, but $\notin \mathcal{N}$

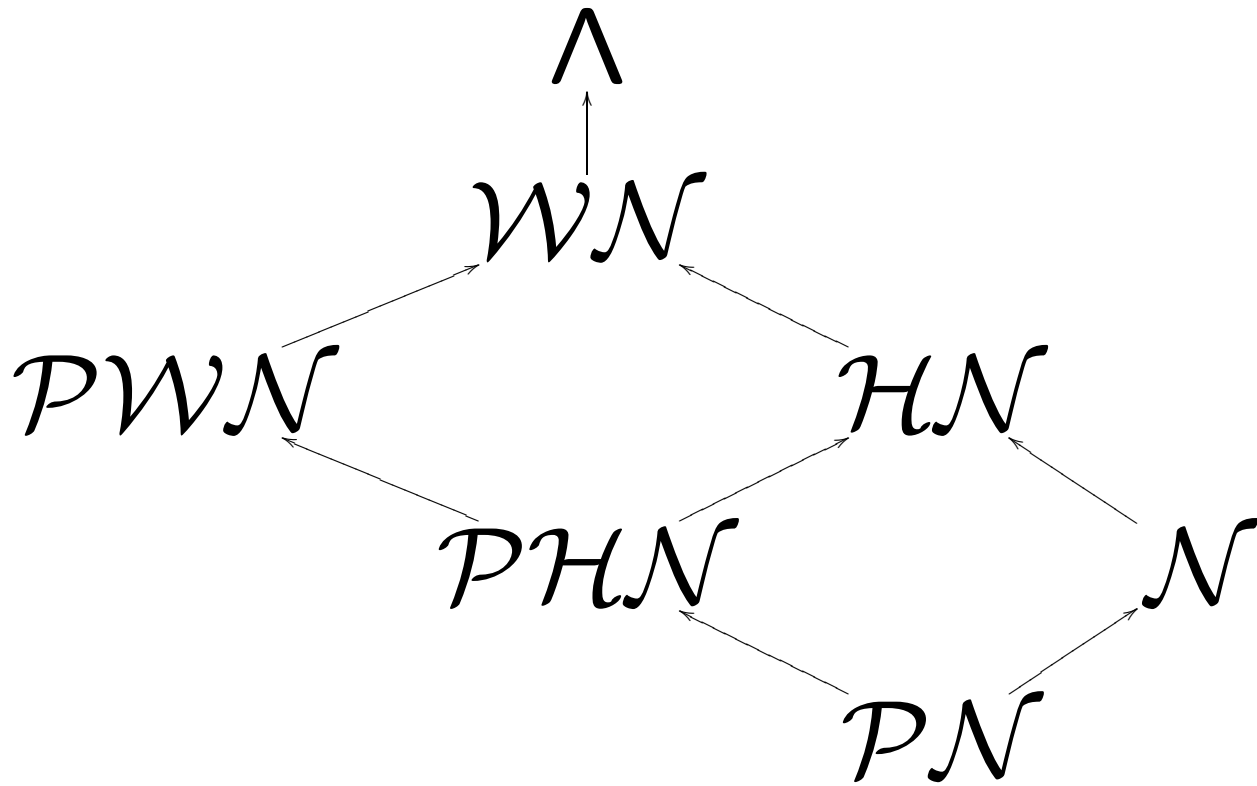
$\lambda x.x(\Delta\Delta) \in \mathcal{HN}$, but $\notin \mathcal{N}$ and $\notin \mathcal{PW}\mathcal{N}$, since

$(\lambda x.x(\Delta\Delta))\Delta \rightarrow_{\beta} \Delta(\Delta\Delta) \notin \mathcal{WN}$

$\mathbf{YK} \in \mathcal{PW}\mathcal{N}$, but $\notin \mathcal{HN}$

$\lambda x.\Delta\Delta \in \mathcal{WN}$, but $\notin \mathcal{HN}$ and $\notin \mathcal{PW}\mathcal{N}$, since

$(\lambda x.\Delta\Delta)M \twoheadrightarrow_{\beta} \Delta\Delta \notin \mathcal{WN}$



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Functional Behaviours

step function $a \Rightarrow b$ $\lambda d. \text{ if } a \sqsubseteq d \text{ then } b \text{ else } \perp$

$$\mathcal{WN} \quad \perp \Rightarrow \perp$$

$$D = [D \rightarrow D]_{\perp}$$

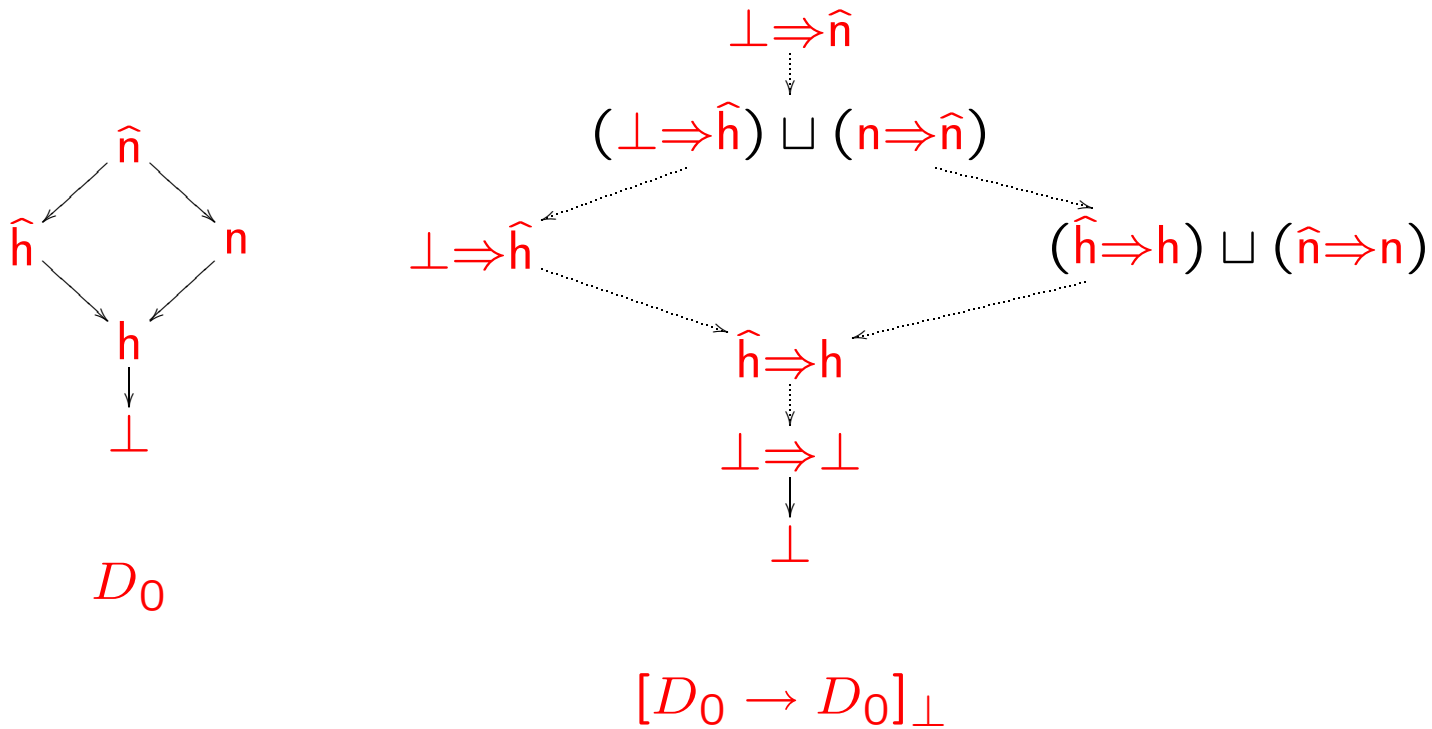
$$\mathcal{PWN} \quad \sqcup_{n \in \mathbb{N}} (\underbrace{\perp \Rightarrow \dots \Rightarrow \perp}_{\hat{n}} \Rightarrow \perp)$$

$$\mathcal{HN} \quad h \quad \hat{h} \Rightarrow h$$

$$\mathcal{PHN} \quad \hat{h} \quad \perp \Rightarrow \hat{h}$$

$$\mathcal{N} \quad n \quad (\hat{h} \Rightarrow h) \sqcup (\hat{n} \Rightarrow n)$$

$$\mathcal{PN} \quad \hat{n} \quad (\perp \Rightarrow \hat{h}) \sqcup (n \Rightarrow \hat{n})$$



$$i_0(\hat{n}) = (\perp \Rightarrow \hat{h}) \sqcup (n \Rightarrow \hat{n}) \quad i_0(n) = (\hat{h} \Rightarrow h) \sqcup (\hat{n} \Rightarrow n)$$

$$i_0(\hat{h}) = \perp \Rightarrow \hat{h} \quad i_0(h) = \hat{h} \Rightarrow h \quad i_0(\perp) = \perp$$

Main Theorem, Version I

$$M \in \mathcal{PN} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq \hat{n}$$

$$M \in \mathcal{N} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq n$$

$$M \in \mathcal{PHN} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq \hat{h}$$

$$M \in \mathcal{HN} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq h$$

$$M \in \mathcal{PWN} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq \bigsqcup_{n \in \mathbb{N}} (\underbrace{\perp \Rightarrow \dots \Rightarrow \perp}_n \Rightarrow \perp)$$

$$M \in \mathcal{WN} \text{ iff } \llbracket M \rrbracket_{\rho_{\hat{n}}}^{\mathcal{D}_{\infty}} \sqsupseteq \perp \Rightarrow \perp$$

$$\rho_{\hat{n}}(x) = \hat{n}$$

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Stone dualities

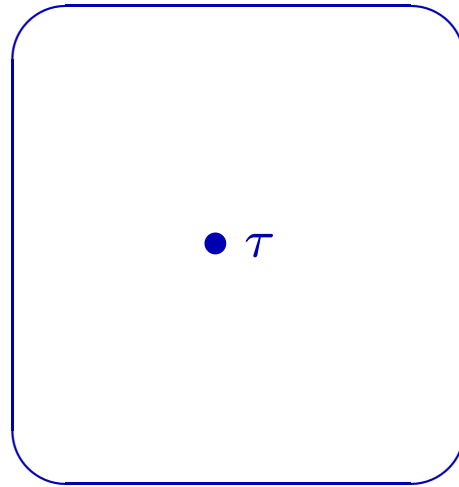
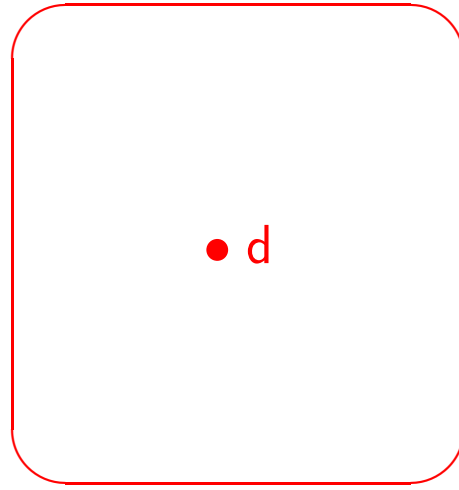
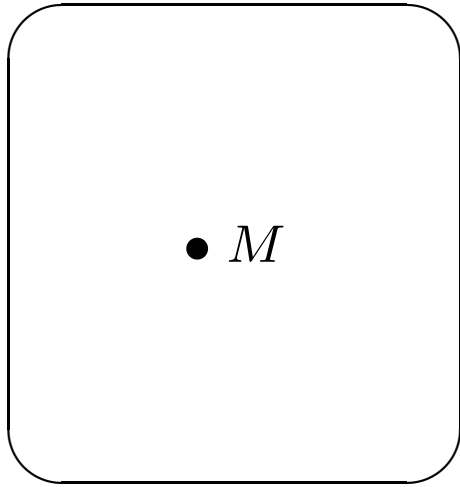
topological spaces as partial orders

Stone spaces as Boolean algebras
(Stone, 36)

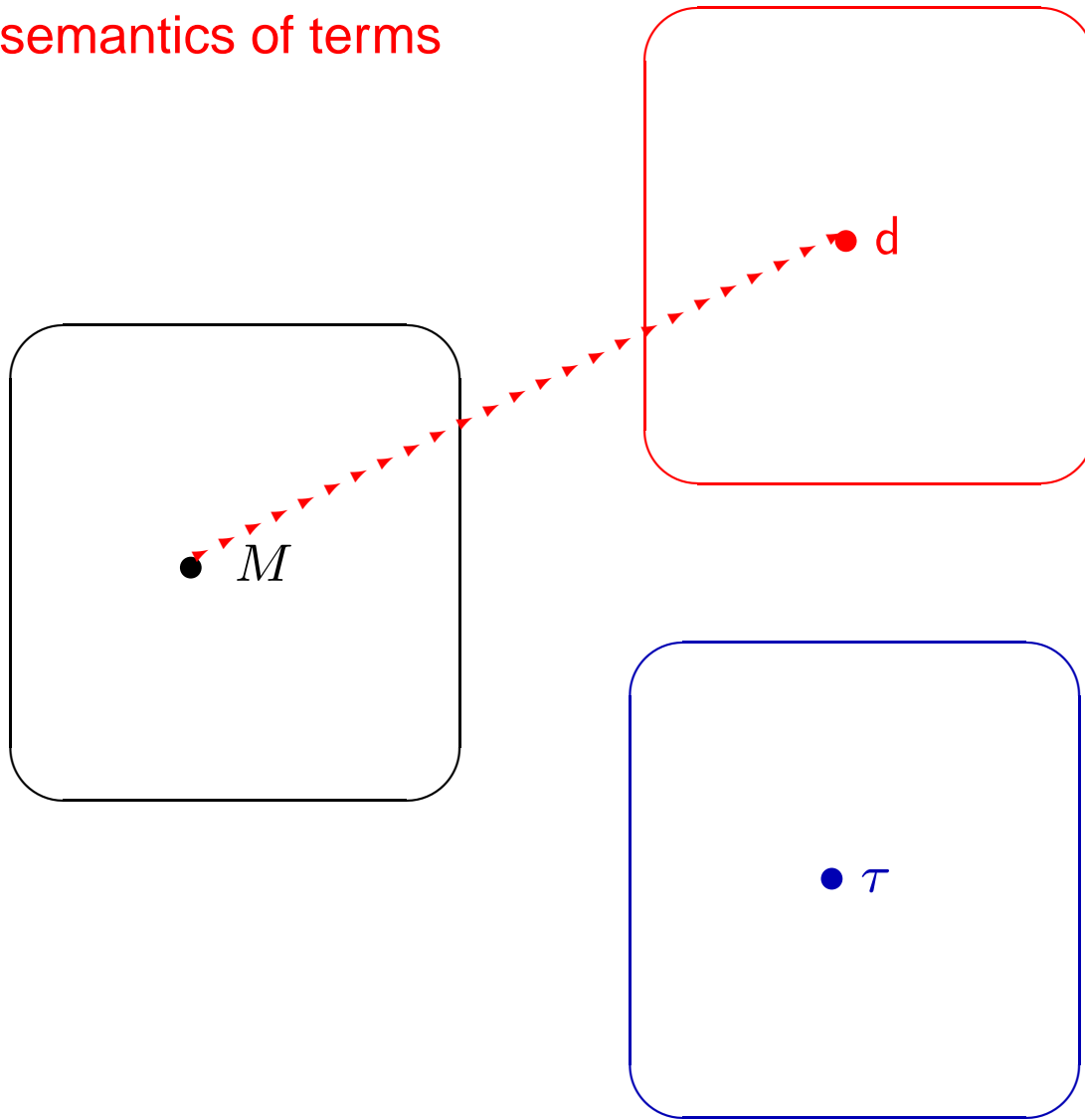
Scott domains as information systems
(Scott, 82)

ω -algebraic complete lattices as intersection type theories
(Coppo et al., 84)

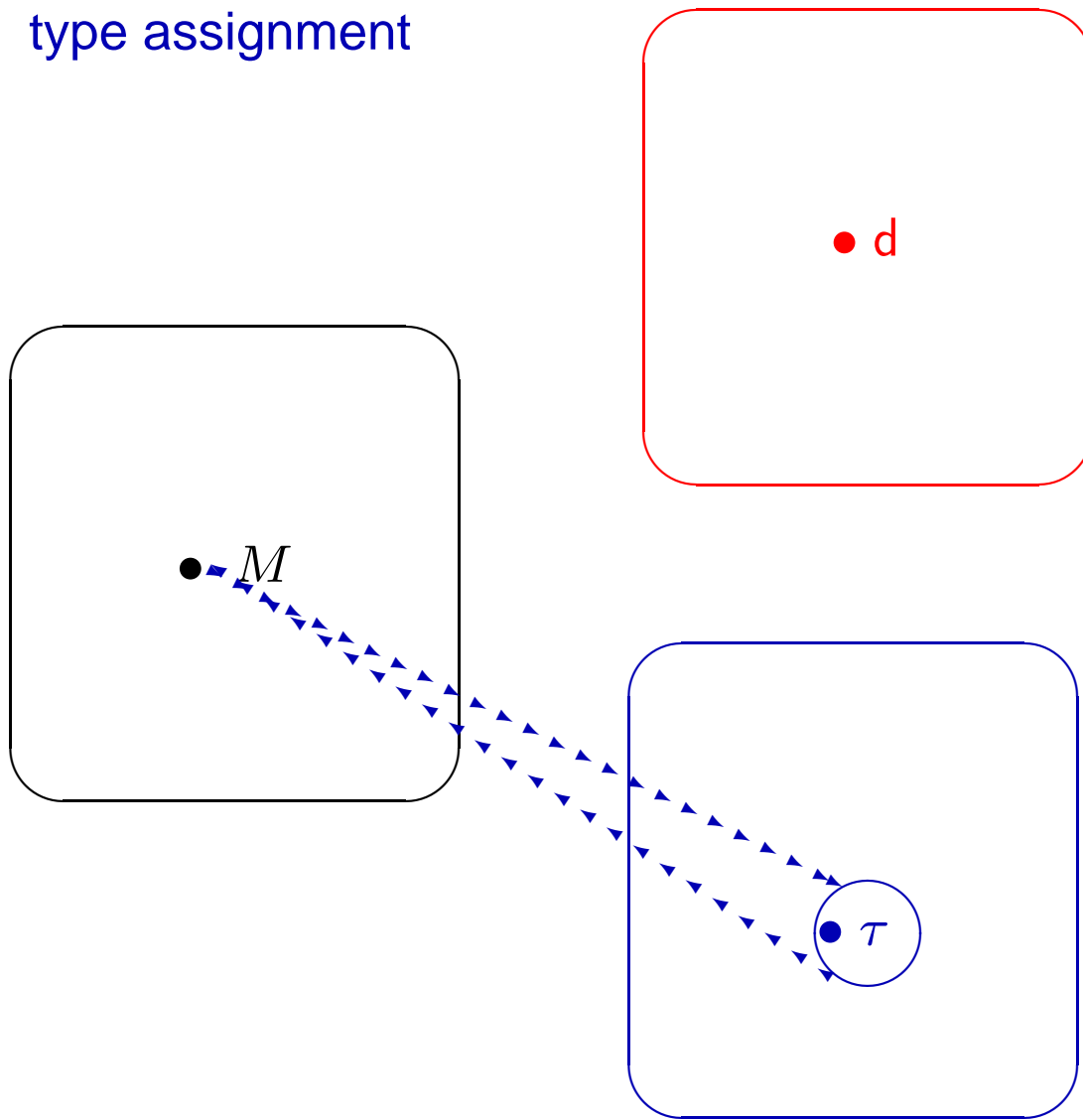
SFP domains as pre-locales
(Abramsky, 91)



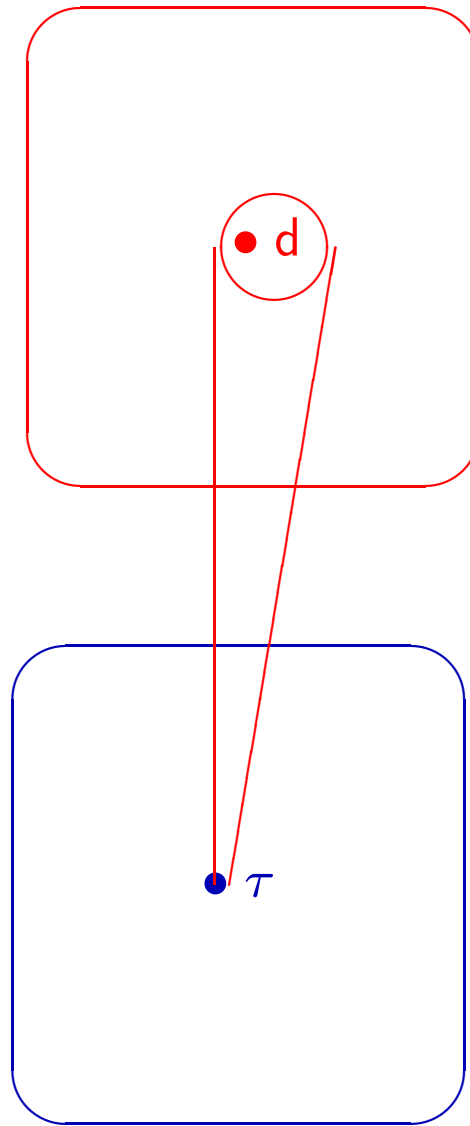
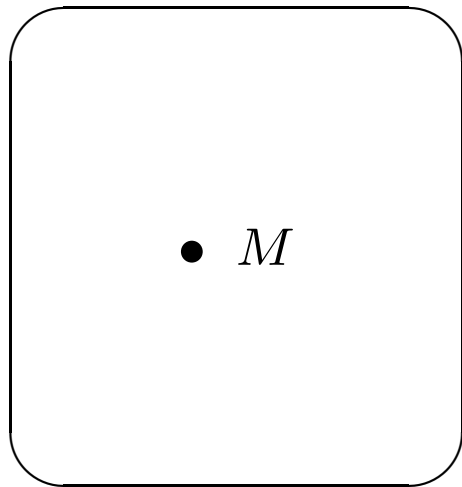
semantics of terms



type assignment



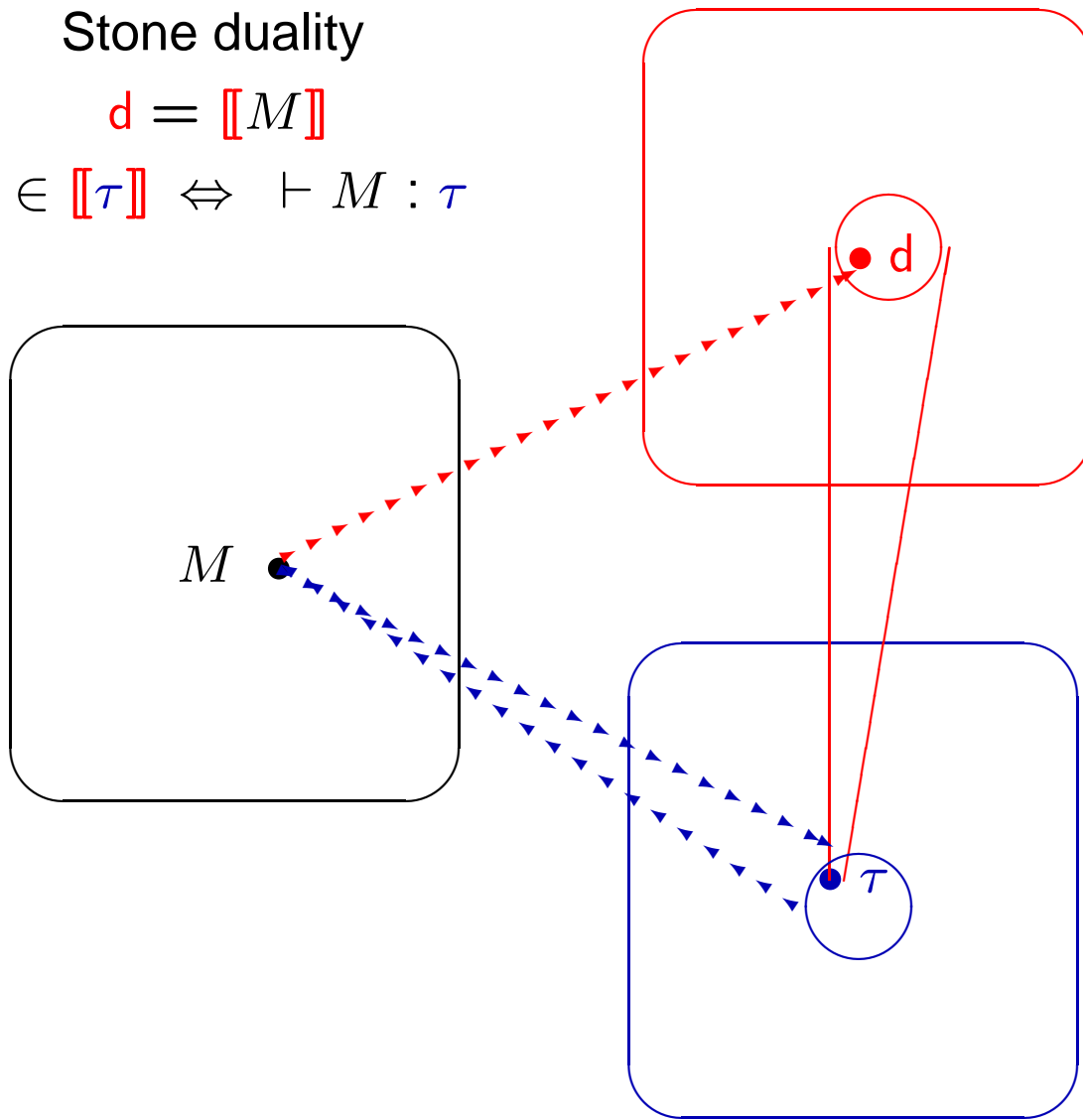
semantics of types



Stone duality

$$d = \llbracket M \rrbracket$$

$$d \in \llbracket \tau \rrbracket \Leftrightarrow \vdash M : \tau$$



We can describe an inverse limit model \mathcal{D}_∞ by taking:

- the **types** freely generated by closing (a set of atomic types corresponding to) the **elements** of \mathcal{D}_0 under the **function type constructor** \rightarrow and the **intersection type constructor** \cap ;
- the **preorder** between types induced by reversing the **order** in \mathcal{D}_0 and by encoding the **initial projection**, according to the correspondence:

function type constructor	\mapsto	step function
intersection type constructor	\mapsto	join

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The set \mathcal{T} of types

$$\mathcal{T} = \nu \mid \hat{\nu} \mid \mu \mid \hat{\mu} \mid \Omega \mid \mathcal{T} \rightarrow \mathcal{T} \mid \mathcal{T} \cap \mathcal{T}$$

mapping $m : \mathcal{T} \rightarrow \mathcal{D}_\infty$

$$\begin{aligned} m(\nu) &= \mathbf{n} & m(\hat{\nu}) &= \hat{\mathbf{n}} & m(\mu) &= \mathbf{h} & m(\hat{\mu}) &= \hat{\mathbf{h}} & m(\Omega) &= \perp \\ m(\sigma \rightarrow \tau) &= m(\sigma) \Rightarrow m(\tau) & & & & & m(\sigma \cap \tau) &= m(\sigma) \sqcup m(\tau). \end{aligned}$$

Preorder on \mathcal{T}

$$\hat{\nu} \leq \nu$$

$$\nu \leq \mu$$

$$\sigma \leq \Omega$$

$$\hat{\nu} \sim (\Omega \rightarrow \hat{\mu}) \cap (\nu \rightarrow \hat{\nu})$$

$$\hat{\mu} \sim \Omega \rightarrow \hat{\mu}$$

$$\sigma \leq \sigma$$

$$\sigma \leq \sigma \cap \sigma$$

$$\sigma \cap \tau \leq \sigma, \sigma \cap \tau \leq \tau$$

$$(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \zeta) \leq \sigma \rightarrow \tau \cap \zeta$$

$$\hat{\nu} \leq \hat{\mu}$$

$$\hat{\mu} \leq \mu$$

$$\sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$$

$$\nu \sim (\hat{\mu} \rightarrow \mu) \cap (\hat{\nu} \rightarrow \nu)$$

$$\mu \sim \hat{\mu} \rightarrow \mu$$

$$\sigma \leq \sigma', \tau \leq \tau' \Rightarrow \sigma \cap \sigma' \leq \tau \cap \tau'$$

$$\sigma \leq \tau, \tau \leq \zeta \Rightarrow \sigma \leq \zeta$$

$$\sigma \leq \sigma', \tau \leq \tau' \Rightarrow \sigma' \rightarrow \tau \leq \sigma \rightarrow \tau'$$

$\sigma \sim \tau$ is short for $\sigma \leq \tau$ and $\tau \leq \sigma$.

The set \mathcal{F} of filters

A filter is a set $X \subseteq \mathcal{T}$ such that:

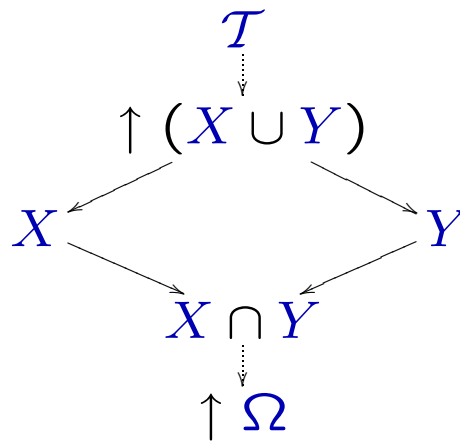
- $\Omega \in X$
- if $\sigma \leq \tau$ and $\sigma \in X$, then $\tau \in X$
- if $\sigma, \tau \in X$, then $\sigma \cap \tau \in X$

\mathcal{F} is the set of filters over \mathcal{T}

$\uparrow X$ is the filter generated by X

$\uparrow \sigma$ is $\uparrow \{\sigma\}$

$\langle \mathcal{F}, \subseteq \rangle$ is an ω -algebraic complete lattice



\mathcal{F} and \mathcal{D}_∞ are isomorphic as ω -algebraic complete lattices

$$m^* : \mathcal{F} \rightarrow \mathcal{D}_\infty$$

$$m^*(X) = \sqcup_{\sigma \in X} m(\sigma)$$

$$m^*(\uparrow \sigma) = m(\sigma)$$

The type assignment system

$$(ax) \quad \Gamma, x : \sigma \vdash x : \sigma \quad (\Omega) \quad \Gamma \vdash M : \Omega$$

$$(\rightarrow E) \quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$(\rightarrow I) \quad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau}$$

$$(\cap I) \quad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau} \quad (\leq) \quad \frac{\Gamma \vdash M : \sigma, \sigma \leq \tau}{\Gamma \vdash M : \tau}$$

Finitary logical description

For any lambda term M and environment $\rho : \text{var} \rightarrow \mathcal{D}_\infty$

$$\llbracket M \rrbracket_\rho^{\mathcal{D}_\infty} = \mathfrak{m}^*(\{\tau \in \mathcal{T} \mid \exists \Gamma \models \rho. \Gamma \vdash M : \tau\})$$

where $\Gamma \models \rho$ if and only if $(x : \sigma) \in \Gamma$ implies $\mathfrak{m}(\sigma) \sqsubseteq \rho(x)$.

If $\Gamma \vdash M : \tau$ and $M =_\beta N$, then $\Gamma \vdash N : \tau$

Main Theorem, Version II

$$M \in \mathcal{PN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \hat{\nu}$$

$$M \in \mathcal{N} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \nu$$

$$M \in \mathcal{PHN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \hat{\mu}$$

$$M \in \mathcal{HN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \mu$$

$$M \in \mathcal{PWN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \Omega^n \rightarrow \Omega \text{ for all } n \in \mathbf{N}$$

$$M \in \mathcal{WN} \text{ iff } \Gamma_{\hat{\nu}} \vdash M : \Omega \rightarrow \Omega$$

$$\Gamma_{\hat{\nu}} = \{x : \hat{\nu} \mid \forall x \in \text{var}\}$$

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Proof of \Leftarrow

Reducibility method

- interpret types by suitable sets
- show soundness of the type assignment

we consider Λ as the applicative structure whose domain is the set of lambda terms and where the application is just the application of terms

valuation of types $\llbracket - \rrbracket : \mathcal{T} \rightarrow 2^\Lambda$

- $\llbracket \nu \rrbracket = \mathcal{N}$ $\llbracket \hat{\nu} \rrbracket = \mathcal{PN}$ $\llbracket \mu \rrbracket = \mathcal{HN}$ $\llbracket \hat{\mu} \rrbracket = \mathcal{PHN}$ $\llbracket \Omega \rrbracket = \Lambda$
- $\llbracket \sigma \cap \tau \rrbracket = \llbracket \sigma \rrbracket \cap \llbracket \tau \rrbracket$
- $\llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \rightarrow \llbracket \tau \rrbracket = \{M \in \mathcal{WN} \mid \forall N \in \llbracket \sigma \rrbracket \quad MN \in \llbracket \tau \rrbracket\}$

$$\begin{aligned} \llbracket \Omega \rightarrow \Omega \rrbracket &= \{M \in \mathcal{WN} \mid \forall N \in \llbracket \Omega \rrbracket \quad MN \in \llbracket \Omega \rrbracket\} \\ &= \{M \in \mathcal{WN} \mid \forall N \in \Lambda \quad MN \in \Lambda\} = \mathcal{WN} \end{aligned}$$

$$\begin{aligned} \bigcap_{n \in \mathbf{N}} \llbracket \Omega^n \rightarrow \Omega \rrbracket &= \bigcap_{n \in \mathbf{N}} \llbracket \Omega^n \rightarrow \Omega \rightarrow \Omega \rrbracket \\ &= \{M \in \mathcal{WN} \mid \forall \vec{N} \in \llbracket \Omega \rrbracket \quad M\vec{N} \in \llbracket \Omega \rightarrow \Omega \rrbracket\} \\ &= \{M \in \mathcal{WN} \mid \forall \vec{N} \in \Lambda \quad M\vec{N} \in \mathcal{WN}\} = \mathcal{PW} \end{aligned}$$

valuation of terms $\llbracket - \rrbracket_\rho : \Lambda \rightarrow \Lambda$

$$\llbracket M \rrbracket_\rho = M[x_1 := \rho(x_1), \dots, x_n := \rho(x_n)]$$

where

$\rho : \text{var} \rightarrow \Lambda$ is a valuation of term variables in Λ

$$\text{FV}(M) = \{x_1, \dots, x_n\}$$

semantic satisfiability relation \models

- $\rho \models M : \tau$ iff $\llbracket M \rrbracket_\rho \in \llbracket \tau \rrbracket$
- $\rho \models \Gamma$ iff $(\forall (x : \sigma) \in \Gamma) \rho \models x : \sigma$
- $\Gamma \models M : \tau$ iff $(\forall \rho \models \Gamma) \rho \models M : \tau$

soundness

$$\Gamma \vdash M : \tau \quad \Rightarrow \quad \Gamma \models M : \tau$$

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Proof of \Rightarrow

• \mathcal{WN}

- $M =_{\beta} \lambda x.N$

$$\frac{\frac{}{\Gamma_{\hat{\nu}} \vdash N : \Omega} (\Omega)}{\Gamma_{\hat{\nu}} \vdash \lambda x.N : \Omega \rightarrow \Omega} (\rightarrow I)$$

- $M =_{\beta} x\vec{N}$

$$\begin{array}{c}
 \frac{}{\Gamma_{\hat{\nu}} \vdash x : \hat{\nu}} \text{ (ax)} \\
 \frac{}{\Gamma_{\hat{\nu}} \vdash x : \vec{\Omega} \rightarrow \Omega \rightarrow \Omega} \text{ (}\leq\text{)} \quad \frac{}{\Gamma_{\hat{\nu}} \vdash \vec{N} : \vec{\Omega}} \text{ (}\Omega\text{)} \\
 \hline
 \Gamma_{\hat{\nu}} \vdash x\vec{N} : \Omega \rightarrow \Omega \text{ (}\rightarrow E\text{)}
 \end{array}$$

- \mathcal{PWN}

- for all n , there is N such that $M =_{\beta} \lambda x_1 \dots x_n. N$

$$\frac{\frac{\text{—————}}{\Gamma_{\hat{\nu}} \vdash N : \Omega} (\Omega)}{\Gamma_{\hat{\nu}} \vdash \lambda x_1 \dots x_n. N : \Omega^n \rightarrow \Omega} (\rightarrow I)$$

- $M =_{\beta} \lambda \vec{x}. y \vec{N}$

$$\begin{array}{c}
 \frac{}{\Gamma_{\hat{\nu}} \vdash y : \hat{\nu}} \text{ (ax)} \\
 \frac{}{\Gamma_{\hat{\nu}} \vdash y : \vec{\Omega} \rightarrow \vec{\Omega} \rightarrow \Omega} \text{ (}\leq\text{)} \quad \frac{}{\Gamma_{\hat{\nu}} \vdash \vec{N} : \vec{\Omega}} \text{ (}\Omega\text{)} \\
 \hline
 \Gamma_{\hat{\nu}} \vdash y \vec{N} : \vec{\Omega} \rightarrow \Omega \quad \text{ (}\rightarrow E\text{)} \\
 \hline
 \Gamma_{\hat{\nu}} \vdash \lambda \vec{x}. y \vec{N} : \vec{\Omega} \rightarrow \Omega \quad \text{ (}\rightarrow I\text{)}
 \end{array}$$

• \mathcal{HN}

$$M =_{\beta} \lambda \vec{x}. y \vec{N} \quad \Gamma^* = \{z : \hat{\mu} \mid \forall z \in \vec{x}\} \cup \{z : \hat{\nu} \mid \forall z \in \text{var} - \vec{x}\}$$

$$\begin{array}{c}
\frac{}{\Gamma^* \vdash y : \hat{\mu}} \text{ (ax)} \\
\frac{}{\Gamma^* \vdash \vec{N} : \vec{\Omega}} \text{ (\Omega)} \\
\frac{}{\Gamma^* \vdash y : \vec{\Omega} \rightarrow \mu} \text{ (\leq)} \quad \frac{}{\Gamma^* \vdash \vec{N} : \vec{\Omega}} \text{ (\Omega)} \\
\frac{}{\Gamma^* \vdash y \vec{N} : \mu} \text{ (\rightarrow E)} \\
\frac{}{\Gamma^* \vdash \lambda \vec{x}. y \vec{N} : \vec{\mu} \rightarrow \mu} \text{ (\rightarrow I)} \\
\frac{}{\Gamma_{\hat{\nu}} \vdash \lambda \vec{x}. y \vec{N} : \mu} \text{ (\leq)}
\end{array}$$

- \mathcal{PHN}

$M =_{\beta} \lambda \vec{x}. y \vec{N}$ y is free

$\Gamma^* = \{z : \Omega \mid \forall z \in \vec{x}\} \cup \{z : \hat{\nu} \mid \forall z \in \text{var} - \vec{x}\}$

$$\begin{array}{c}
 \frac{}{\Gamma^* \vdash y : \hat{\nu}} \text{ (ax)} \\
 \frac{}{\Gamma^* \vdash y : \vec{\Omega} \rightarrow \hat{\mu}} (\leq) \quad \frac{}{\Gamma^* \vdash \vec{N} : \vec{\Omega}} (\Omega) \\
 \frac{}{\Gamma^* \vdash y \vec{N} : \hat{\mu}} (\rightarrow E) \\
 \frac{}{\Gamma^* \vdash \lambda \vec{x}. y \vec{N} : \vec{\Omega} \rightarrow \hat{\mu}} (\rightarrow I) \\
 \frac{}{\Gamma_{\hat{\nu}} \vdash \lambda \vec{x}. y \vec{N} : \hat{\mu}} (\leq)
 \end{array}$$

• \mathcal{N}

- $M =_{\beta} x\vec{N}$

$$\frac{\frac{\frac{}{\Gamma_{\hat{\nu}} \vdash x : \hat{\nu}} (ax)}{\Gamma_{\hat{\nu}} \vdash x : \vec{\nu} \rightarrow \nu} (\leq)}{\Gamma_{\hat{\nu}} \vdash x\vec{N} : \nu} (\rightarrow E)}{\Gamma_{\hat{\nu}} \vdash \vec{N} : \vec{\nu}} (HI)$$

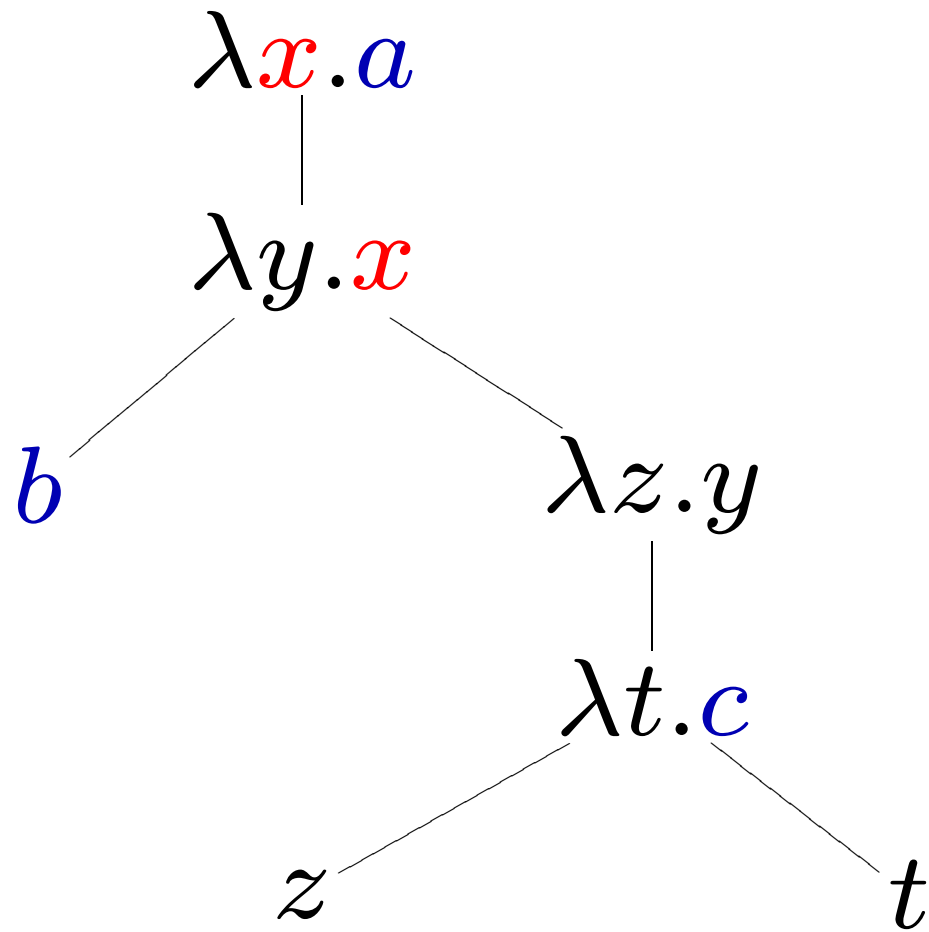
- $M =_{\beta} \lambda x.N$

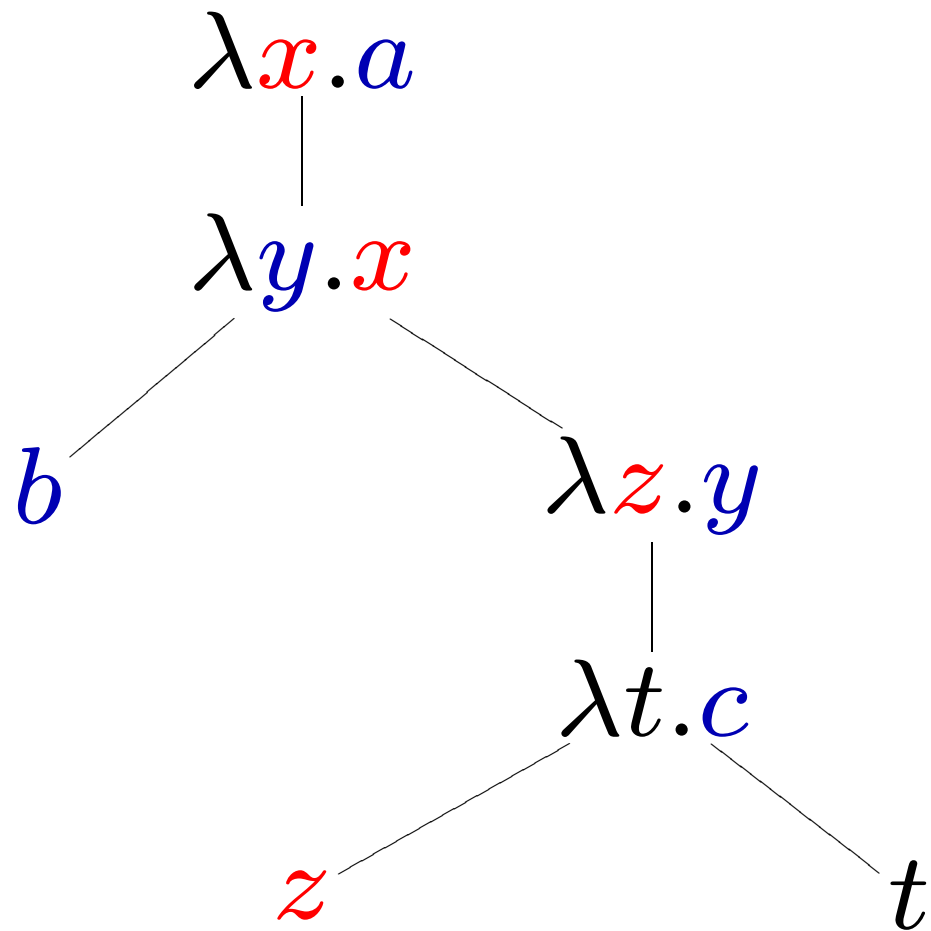
$$\begin{array}{c}
 \frac{\vdots}{\Gamma_{\hat{\nu}} \vdash N : \nu} (HI) \\
 \frac{\Gamma_{\hat{\nu}} \vdash N : \nu}{\Gamma_{\hat{\nu}} \vdash \lambda x.N : \hat{\nu} \rightarrow \nu} (\rightarrow I) \quad \frac{\vdots}{\Gamma_{\hat{\nu}} \vdash \lambda x.N : \mu} (case \mathcal{HN}) \\
 \frac{\Gamma_{\hat{\nu}} \vdash \lambda x.N : \hat{\nu} \rightarrow \nu \quad \Gamma_{\hat{\nu}} \vdash \lambda x.N : \mu}{\Gamma_{\hat{\nu}} \vdash \lambda x.N : (\hat{\nu} \rightarrow \nu) \cap \mu} (\cap I) \\
 \frac{\Gamma_{\hat{\nu}} \vdash \lambda x.N : (\hat{\nu} \rightarrow \nu) \cap \mu}{\Gamma_{\hat{\nu}} \vdash \lambda x.N : \nu} (\leq)
 \end{array}$$

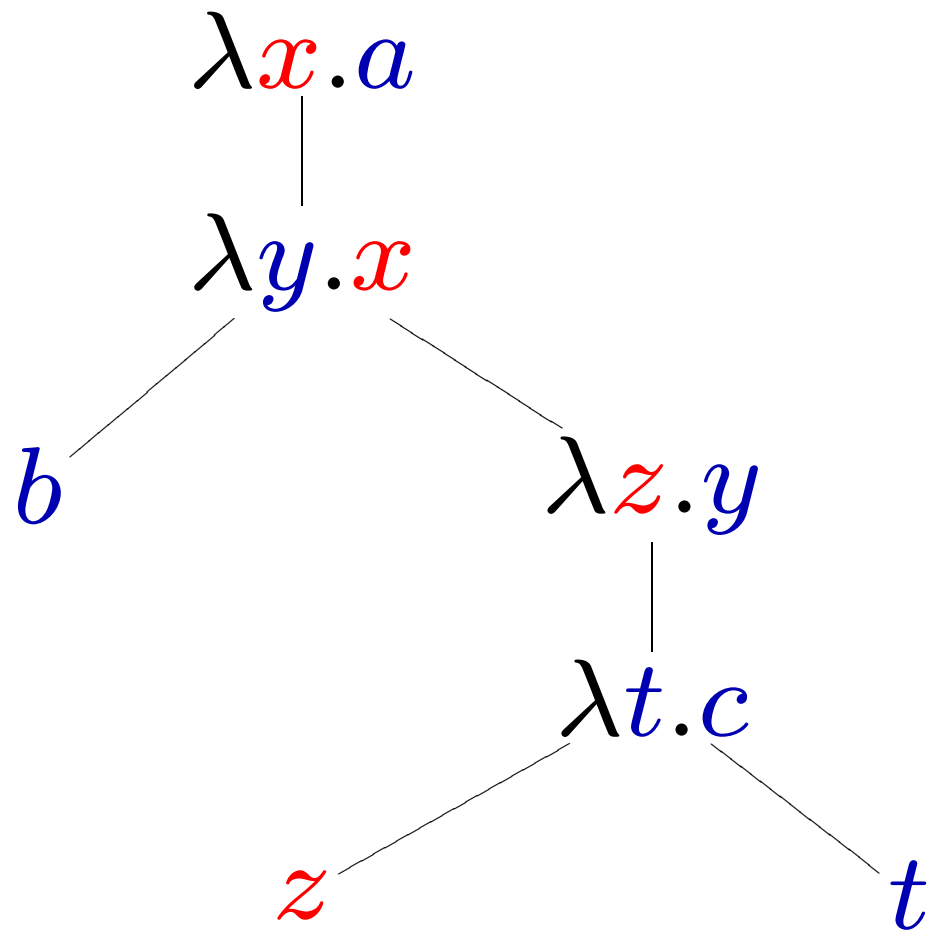
- \mathcal{PN}

replaceable and **non-replaceable** variables In a normal form M :

- all occurrences of variables bound in the initial abstractions are **replaceable**
- all occurrences of free variables are **non-replaceable**
- if $x\vec{N}(\lambda \vec{z} y.P)$ is a subterm of M and the showed occurrence of x is **replaceable** then the occurrences of y in P are **replaceable**
- if $x\vec{N}(\lambda \vec{z} y.P)$ is a subterm of M and the showed occurrence of x is **non-replaceable** then the occurrences of y in P are **non-replaceable**







a normal form $M \in \mathcal{PN}$ iff in M
all fathers of **replaceable** variables are **non-replaceable**

