Analysis of Random Mobility Models with PDE's

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ABSTRACT

In this paper we revisit two classes of mobility models which are widely used to represent users' mobility in wireless networks: Random Waypoint (RWP) and Random Direction (RD). For both models we obtain systems of partial differential equations which describe the evolution of the users' distribution. For the RD model, we show how the equations can be solved analytically both in the stationary and transient regime adopting standard mathematical techniques. Our main contributions are i) simple expressions which relate the transient duration to the model parameters; ii) the definition of a generalized random direction model whose stationary distribution of mobiles in the physical space corresponds to an assigned distribution.

Categories and Subject Descriptors

I.6.0 [**Simulation and Modeling**]: General; C.2.1 [**Computer Communication Networks**]: Network Architecture and Design—*Wireless communication*; G.1.8 [**Numerical Analysis**]: Partial Differential Equations

General Terms

Theory, Design, Performance

Keywords

Mobility models, Partial differential equations

1. INTRODUCTION

Mobility models play a fundamental role in the analysis and design of wireless systems [1, 2]. In the past several years, researchers have proposed a number of mobility models for the purpose of simulating the movement of users in a wireless network. Two widely used models are the Random Waypoint model (RWP) and the Random Direction model (RD). In both models, users independently follow a sequence of linear segments and traverse each segment at constant speed. The two models differ in how a user chooses the next segment to traverse: under the RWP model, a user selects a random destination point within the space; instead, under the RD model a user chooses a direction to travel in and a duration for the travel. In both cases, the speed on a segment is taken from some given distribution. Moreover, before starting to travel on the new segment users can stop for a random time, thus alternating phases in which they move with phases in which they keep still.

Despite their wide use in simulation studies, properties of the above mobility models have only recently been established and fully understood. In [5] the authors have used Palm calculus to study the stationary regime of a large class of mobility models (including RWP and RD), explaining a number of previously observed phenomena such as speed decay [9] and non-uniform distribution of nodes [7]. Their analysis generalizes findings in [3, 6, 8] about existence and uniqueness of a stationary regime, and provides the correct methodology to start a simulation in steady state so as to avoid transient effects (perfect simulation). Moreover, the proposed perfect sampling technique applies to quite general area shapes (e.g., the Swiss Cross), without requiring the computation of complex geometric integrals (like for example in [4]).

It turns out that the RWP mobility pattern, which is appealing because of its natural physical interpretation, is more difficult to analyze and control in terms of the stationary distributions of location and speed of mobiles, and does not usually lead to a uniform density of nodes in the space. On the contrary, the RD mobility pattern has the nice property that users always tend to be uniformly distributed in the space, irrespective of the boundary conditions imposed (wrap around or reflection). Moreover, the distribution of location and speed at a random time instant are the same as at a transition instant [5], which greatly simplifies the analysis.

Two important issues in the analysis of mobility models still need to be solved. The first is the study of the convergence rate to the stationary regime from arbitrary initial conditions. That is, how long does it take to approach the stationary distribution if the simulation starts away from the equilibrium? So far, in the literature, the transient behavior of mobility models has mostly been considered a nuisance, and many efforts have been devoted precisely to eliminate transient effects from simulations. However, capacity planning, network resilience and reliability, etc., usually require to test applications and protocols in time-varying, critical conditions, not in the steady state. Take, for example, the case of a large number of mobile nodes forming an ad-hoc network initially confined in a small area (such as a conference room, a football stadium, or the like), who at some point start dispersing away. One would like to simulate such a scenario to see how the network behaves while nodes get more and more far apart till connectivity is lost. This

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paper will show that theoretical mobility models permit to do this in a controlled and predictable fashion, i.e., it is possible to choose parameters of the mobility model to obtain a desired nodes' dispersion rate and duration of the transient. Our dynamical viewpoint thus brings what might be regarded as ideal, unrealistic mobility models much closer to practical applications.

The second issue is the reverse of the problem considered so far in analytical studies appeared in the literature: is it possible to devise a mobility pattern that achieves a desired stationary distribution of nodes in space? So far, theoretical studies have just predicted the stationary distribution generated by a given mobility model. However, the ability to design a mobility model that produces an assigned distribution of nodes in the area would be of much greater interest in real problems, where node densities are almost always non-uniform. As an example, one could be interested in simulating scenarios in which nodes are more densely concentrated in some portions of the area, like in a urban context.

In this paper we propose an analysis of the RWP and RD models that allows us to address both issues above, filling the existing gap in the analysis of mobility models. We use partial differential equations (PDE's) to describe how the mobiles' state distribution evolves over time. Our novel formulation provides the analytical basis for solving both transient and non-uniform cases. In particular, it permits to study the transient dynamics of a system starting from arbitrary initial condition, for both RWP and RD models. For the RD model, we show how the partial differential equations can be solved *analytically* in the transient regime adopting standard mathematical techniques. Moreover we re-derive known results about the stationary distribution of the RD model in a more straightforward manner than previous approaches based on Palm Calculus. Our methodology allows, for the first time to the best of our knowledge: i) to derive simple expressions relating the transient duration to the model parameters; ii) to generalize the RD model so as to obtain a desired stationary distribution of nodes in the space.

The remainder of paper is organized as follows. In Sections 2 and 3 we present the equations describing the behavior of a single user moving according to the RD and RWP model, respectively. In Section 4 we statistically re-interpret the previously obtained equation for a single users, showing that they can be used as well to describe the dynamics of a large population of mobile users. The steady state analysis of the standard RD model is provided in Section 5, whereas the extension of this model to achieve arbitrary distributions of nodes in the area is described in Section 6. The transient analysis of the RD model is presented in Section 7. In Section 8 we validate our analysis by simulation on a few examples and present possible applications of our methodology. Finally Section 9 concludes the paper.

2. EQUATIONS OF THE RD MODEL

We start considering a single user moving according to the random direction model over a unidimensional domain, further assuming that move and pause times are exponentially distributed. Then we generalize our approach to the case in which move and pause times have a general distribution. Finally we extend our equations to the multidimensional case.

2.1 Unidimensional case with exponential phases

We assume that the domain in which the mobile can move is the interval $[x_l, x_u]$, Move and pause times are taken from an exponential distribution of parameter μ and λ, respectively. When the mobile starts travelling on a new segment it selects a speed from the generic distribution $f_V(v)$. We further assume that the absolute speed value is upper bounded by a constant $V_{\max}$, i.e., support of $f_V(v)$ is in the interval $[-V_{\max}, V_{\max}]$. This is a reasonable assumption for all cases of practical interest.

The dynamics of the mobile can be described in terms of a Markov Process over a general space state [10], in which the instantaneous mobile state $K(t)$ is characterized by: i) the phase $P(t) \in \mathcal{P} = \{\textit{move}, \textit{pause}\}$; ii) the instantaneous position $X(t) \in [x_l, x_u]$; iii) the current speed $V(t)$ (in case $P(t) = \textit{move}$).

Let $N(x, v, t)$ be the cumulative probability that at time t the mobile is in the *move* phase at a position $X(t) \in [x_l, x]$ with a speed $V(t) \in [-V_{\max}, v]$:

$$N(x,v,t) \triangleq \Pr\{P(t) = \textit{move}, X(t) \in [x_l, x], V(t) \in [-V_{max}, v]\}$$

Let $S(x, t)$ be the cumulative probability that at time t the mobile is in the *pause* phase at a position $x \in [x_l, x]$:

$$S(x,t) \triangleq \Pr\{P(t) = \textit{pause}, X(t) \in [x_l, x]\}$$

Consider a small interval $T = [t, t + \Delta t)$. Conditionally over the fact that no phase transition occurs in T, according to the RD model at time $t+\Delta t$ state $K(t) = (\textit{move}, x, v)$ is deterministically transformed into state $K(t + \Delta t) = (\textit{move}, x + v\Delta t, v)$, whereas state $K(t) = (\textit{pause}, x)$ is deterministically transformed into state $K(t + \Delta t) = (\textit{pause}, x)$; thus, conditionally over the fact that no phase transition occurs in T, we have:

$$\begin{aligned} N(x + v\Delta t, v, t + \Delta t) &= N(x, v, t) \\ S(x, t + \Delta t) &= S(x, t) \end{aligned}$$

If a transition occurs at $t+\tau \in T$, state $K(t) = (\textit{pause}, x-v(\Delta t-\tau))$ is deterministically transformed into state $K(t + \Delta t) = (\textit{move}, x, v)$, whereas state $K(t) = (\textit{move}, x - v\tau, v)$ is transformed into state $K(t + \Delta t) = (\textit{pause}, x)$. Thus, conditionally over the fact that a phase transition occurs in T, it results:

$$\begin{aligned} N(x, v, t + \Delta t) &= S(x, t) \int_{-V_{\max}}^{v} f_V(v)\, \mathrm{d}v + O(\Delta t) \\ S(x, t + \Delta t) &= \int_{-V_{\max}}^{V_{\max}} N(x, v, t)\, \mathrm{d}v + O(\Delta t) \end{aligned}$$

Due to the exponential distribution of the move and pause times, the probability that a phase transition occurs in T from *pause* to *move* is $\lambda\Delta t + o(\Delta t)$; the probability that a phase transition occurs during T from *move* to *pause* is $\mu\Delta t + o(\Delta t)$; the probability that more than one phase transition occurs in T is, instead, $o(\Delta t)$. Therefore, we can write:

$$N(x + v\Delta t, v, t + \Delta t) = N(x, v, t)(1 - \mu\Delta t) + \lambda\Delta t S(x + v\Delta t, t) + o(\Delta t)$$

$$S(x, t + \Delta t) = (1 - \lambda\Delta t)S(x, t) + \mu\Delta t \int_{-V_{\max}}^{V_{\max}} N(x, v, t)\, \mathrm{d}v + o(\Delta t)$$

Letting $\Delta t \to 0$, assuming $f_V(v)$ to be a continuous and derivable function, and defining

$$n(x, v, t) = \frac{\partial^2 N(x, v, t)}{\partial x\, \partial v} \quad ; \quad s(x, t) = \frac{\partial S(x, t)}{\partial x}$$

we obtain the following coupled differential equations:

$$\frac{\partial n(x, v, t)}{\partial t} = -v \frac{\partial n(x, v, t)}{\partial x} + \lambda f_V(v) s(x) - \mu\, n(x, v, t) \quad (1)$$

$$\frac{\partial s(x, t)}{\partial t} = -\lambda s(x, t) + \mu \int_v n(x, v, t)\, \mathrm{d}v \quad (2)$$

Boundary Conditions

A problem that arises in the RD model is what to do when the mobile hits a boundary. Several strategies have been proposed; among them the most popular are *wrap around* and *reflection*. In the *wrap around* model, the mobile hitting a boundary with speed v instantaneously reappears at the opposite side maintaining the same speed. Thus the boundary conditions of the *wrap around* model for (1) and (2) are:

$$\begin{aligned} n(x_l, v, t) &= n(x_u, v, t) & \forall v, t \\ \lim_{x \to x_l^+} \frac{\partial n(x, v, t)}{\partial x} &= \lim_{x \to x_u^-} \frac{\partial n(x, v, t)}{\partial x} & \forall v, t \\ s(x_l, t) &= s(x_u, t) & \forall t \\ \lim_{x \to x_l^+} \frac{\partial s(x, t)}{\partial x} &= \lim_{x \to x_u^-} \frac{\partial s(x, t)}{\partial x} & \forall t \end{aligned}$$

In the *reflection* model, instead, the mobile is bounced back reversing its speed. The boundary conditions of the *reflection* model for (1) are:

$$\begin{aligned} n(x_l, v, t) &= n(x_l, -v, t) & \forall v, t \\ n(x_u, v, t) &= n(x_u, -v, t) & \forall v, t \end{aligned}$$

In both cases the initial condition is assumed to be given:

$$\begin{aligned} n(x, v, 0) &= n_o(x, v) \\ s(x, 0) &= s_o(x) \end{aligned}$$

We remark that the initial condition must satisfy the constraints related to its physical interpretation as pdf of the mobile position, speed and phase at time $t = 0$. In particular, $n_o(x, v) \geq 0$, $s_o(x) \geq 0$ and

$$\iint n_o(x, v) \, \mathrm{d}x \, \mathrm{d}v + \int s_o(x) \, \mathrm{d}x = 1$$

Uniqueness of solution

In Appendix A we prove that the mathematical problem defined by equations (1) and (2) subject to the boundary and initial conditions defined above either for the *wrap-around* or *reflection* models, admits no more that one solution. We emphasize that this is a fundamental step of our analysis; indeed, under uniqueness assumptions, if we find a solution of the equations with assigned initial and boundary conditions we can conclude that it corresponds to the actual system trajectory.

2.2 Extension to general phase times distributions

Let $g(y)$ be the pdf of *move* time, and $\mu(y)$ the associated hazard function $\mu(y) = g(y)/(1 - G(y))$, being $G(y)$ the cdf of *move* time. Similarly, let $h(z)$ be the pdf of *pause* times, and $\lambda(z)$ the associated hazard function $\lambda(z) = h(z)/(1 - H(z))$, being $H(z)$ the cdf of *pause* time. The systems dynamics can still be described as a Markov process over a general space state. However, in this case the state space becomes more complex since phase durations are not memoryless: the mobile state $K(t)$ is now characterized by the time $W(t)$ elapsed since the last phase transition, in addition to its current phase $P(t)$, instantaneous position $X(t)$ and current speed $V(t)$ (in case $P(t) =$ *move*).

The mobile dynamics satisfies the following system of differential equations:

$$\frac{\partial n(x, v, y, t)}{\partial t} = -v \frac{\partial n(x, v, y, t)}{\partial x} - \frac{\partial n(x, v, y, t)}{\partial y} + \\ + \delta(y) \int_z \lambda(z) f_V(v) s(x, z, t) \, \mathrm{d}z - \mu(y) n(x, v, y, t) \quad (3)$$

$$\frac{\partial s(x, z, t)}{\partial t} = -\frac{\partial s(x, z, t)}{\partial z} - \lambda(z) s(x, z, t) \\ + \delta(z) \iint \mu(y) n(x, v, y, t) \, \mathrm{d}v \, \mathrm{d}y \quad (4)$$

2.3 The multidimensional case

The extension to a k-dimensional domain $\in \mathbb{R}^k$ is rather straightforward. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ be the position of the mobile and $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ the current speed vector (each component represents the mobile's position/speed along the corresponding dimension). In case of exponential move and pause times the Chapman-Kolmogorov equations of the system are:

$$\begin{aligned} \frac{\partial n(\mathbf{x}, \mathbf{v}, t)}{\partial t} &= -\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t) \\ &\quad + \lambda f_V(\mathbf{v}) s(\mathbf{x}) - \mu \, n(\mathbf{x}, \mathbf{v}, t) & (5) \\ \frac{\partial s(\mathbf{x}, t)}{\partial t} &= -\lambda s(\mathbf{x}, t) + \mu \int n(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}\mathbf{v} & (6) \end{aligned}$$

being $\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t)$ the inner product between $\mathbf{v}$ and $\nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, t)$.

In the case of general distributions of phase durations, we have

$$\begin{aligned} &\frac{\partial n(\mathbf{x}, \mathbf{v}, y, t)}{\partial t} = -\mathbf{v}.\nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, y, t) - \frac{\partial n(x, \mathbf{v}, y, t)}{\partial y} \\ &+ \delta(y) \int_z \lambda(z) f_V(\mathbf{v}) s(\mathbf{x}, z) \, \mathrm{d}z - \mu(y) n(\mathbf{x}, \mathbf{v}, y, t) \quad (7) \\ &\frac{\partial s(\mathbf{x}, z, t)}{\partial t} = -\frac{\partial s(\mathbf{x}, z, t)}{\partial z} - \lambda(z) s(\mathbf{x}, z, t) \\ &\quad + \delta(z) \iint \mu(y) n(\mathbf{x}, \mathbf{v}, y, t) \, \mathrm{d}v \, \mathrm{d}y \quad (8) \end{aligned}$$

3. EQUATIONS OF THE RWP MODEL

Similarly to what we have done for the RD model, we start considering a mobile moving along a unidimensional domain, assuming that pause times are exponentially distributed. Notice that in the RWP model users do not choose a duration for the move phase, which instead depends on the selected destination point and speed. Next we generalize our approach to the case in which pause times are generally distributed, and finally to the multidimensional case.

3.1 The unidimensional case with exponential pauses

Let $[x_l, x_u]$ be the domain in which the mobile can move, and λ the parameter of the exponentially distributed pause time. The mobile in x, when choosing the next segment to travel in, first selects a destination point d according to the distribution $r(d)$, then selects a speed according to the distribution $f_V(v|d, x)$. We notice that if $d > x$ it must be $f_V(v|d, x) = 0$ for $v < 0$, while if $d < x$ it must be $f_V(v|d, x) = 0$ for $v > 0$. We again assume that the absolute speed value is upper bounded by a constant V_{max}; i.e., support of $f_V(v|d, x)$ falls in the interval $[-V_{max}, V_{max}], \forall d, x$.

The dynamics of the mobile can be described in terms of a Markov Process over an general space state in which the instantaneous state $K(t)$ is characterized by: i) the phase $P(t) \in \mathcal{P} = \{$*move*, *pause*$\}$; ii) the instantaneous position $X(t) \in [x_l, x_u]$; iii) the current destination $D(t) \in [x_l, x_u]$; iv) the current speed $V(t) \in [-V_{max}, V_{max}]$ (in case $P(t) = move$).

Let $N(x, v, d, t)$ be the cumulative probability that at time t the mobile is in the *move* phase at a position $X(t) \in [x_l, x]$, with a destination $D(t) \in [x_l, d]$, and a speed $V(t) \in [-V_{\max}, v]$:

$$N(x, v, d, t) \triangleq \Pr\{P(t) = \textit{move}, X(t) \in [x_l, x], \\ , D(t) \in [x_l, d], V(t) \in [-V_{max}, v]\}$$

Let $S(x, t)$ be the cumulative probability that at time t the mobile is in the *pause* phase at a position $X(t) \in [x_l, x]$:

$$S(x, t) \triangleq \Pr\{P(t) = \textit{pause}, X(t) \in [x_l, x]\}$$

Introducing the derivatives

$$n(x, v, d, t) = \frac{\partial^3 N(x, v, d, t)}{\partial x\, \partial v\, \partial d} \quad ; \quad s(x, t) = \frac{\partial S(x, t)}{\partial x}$$

we obtain the following pair of equations, in a way similar to what has been done for the RD model:

$$\frac{\partial n(x, v, d, t)}{\partial t} = -v \frac{\partial n(x, v, d, t)}{\partial x} + \lambda f_V(v \mid d) r(d) s(x, t) \quad (9)$$

$$\frac{\partial s(x, t)}{\partial t} = -\lambda s(x, t) + \int v\, n(x, v, x, t)\, \mathrm{d}v \quad (10)$$

where (9) is defined for $d \geq x$ and $v > 0$, or $d \leq x$ and $v < 0$.

Boundary conditions

In the RWP model, the boundary conditions express the fact that the probability for the mobile to hit the boundaries is null:

$$\begin{aligned} n(x_l, v, d, t) &= 0 \qquad \forall v, d, t \\ n(x_u, v, d, t) &= 0 \qquad \forall v, d, t \\ s(x_l, t) &= 0 \qquad \forall t \\ s(x_u, t) &= 0 \qquad \forall t \end{aligned}$$

In addition we impose the initial conditions:

$$\begin{aligned} n(x, v, d, 0) &= n_o(x, v, d) \\ s(x, 0) &= s_o(x) \end{aligned}$$

which must be a proper pdf for the mobile's initial position, speed, and destination.

3.2 Extension to general pause time distribution

Let $h(z)$ be the pdf of pause time, and $\lambda(z)$ the associated hazard function. The system dynamics can still be described by a Markov Process over a general state space; we only need to add to the state associated to the pause phase the time z elapsed since the mobile entered the pause phase. The model equations become:

$$\frac{\partial n(x, v, d, t)}{\partial t} = -v \frac{\partial n(x, v, d, t)}{\partial x} + \\ f_V(v|x, d) r(d) \int \lambda(z) s(x, z, t)\, \mathrm{d}z \quad (11)$$

defined for $d \geq x$ and $v > 0$ or $d \leq x$ and $v < 0$, and

$$\frac{\partial s(x, z, t)}{\partial t} = -\frac{\partial s(x, z, t)}{\partial z} - \\ -\lambda(z) s(x, z, t) + \delta(z) \int v\, n(x, v, x, t)\, \mathrm{d}v \quad (12)$$

3.3 Multidimensional case

Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ be the position of the mobile, $\mathbf{d} = (d_1, d_2, \ldots, d_k)$ the current destination, and v the current speed.

Considering the case in which the pause time is generally distributed, with hazard function $\lambda(z)$, we obtain

$$\frac{\partial n(\mathbf{x}, \mathbf{v}, \mathbf{d}, t)}{\partial t} = -\mathbf{v} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{v}, \mathbf{d}, t) \\ + f_V(\mathbf{v}|\mathbf{d}, \mathbf{x}) r(\mathbf{d}) \int_z \lambda(z) s(\mathbf{x}, z, t)\, \mathrm{d}z$$

$$\frac{\partial s(\mathbf{x}, z, t)}{\partial t} = -\frac{\partial s(\mathbf{x}, z, t)}{\partial z} - \\ -\lambda(z) s(\mathbf{x}, z, t) + \delta(z) \int ||\mathbf{v}|| n(\mathbf{x}, \mathbf{v}, \mathbf{x}, t)\, \mathrm{d}\mathbf{v}$$

4. STATISTICAL INTERPRETATION OF PREVIOUS EQUATIONS

In this section we provide a statistical interpretation of the equations derived in Sections 2 and 3, valid when the population of mobile users becomes large. We restrict ourselves to the unidimensional random direction model under general phases distributions, however the same interpretation holds in all other cases.

Consider a population of N mobiles, moving independently of each other. The complete state for mobile i at time t is denoted by $K_i(t) = (P(t), X(t), V(t), W(t))$. Let $\mathcal{M}$ be the set of all states in which the mobile is in the *move* phase, and $\mathcal{S}$ the set of all states in which the mobile is in the *pause* phase. Let A be any (Lebesgue measurable) set of states, and define $A_{\mathcal{M}} = A \cap \mathcal{M}$ and $A_{\mathcal{S}} = A \cap \mathcal{S}$. Let $\mathbb{1}_{K_i(t) \in A}$ be an indicator function which returns 1 if mobile i at time t is in a state belonging to A, i.e. $K_i(t) \in A$, and 0 otherwise.

By the strong law of large numbers, it results:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{K_i(t) \in A} = E[\mathbb{1}_{K_1(t) \in A}] = \Pr\{K_1(t) \in A\} = \\ = \int_{A_{\mathcal{M}}} n(x, v, y, t)\, \mathrm{d}A_{\mathcal{M}} + \int_{A_{\mathcal{S}}} s(x, z, t)\, \mathrm{d}A_{\mathcal{S}}$$

Now we observe that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{K_i(t) \in A}$$

has an immediate physical interpretation as the fraction of mobiles whose instantaneous state at time t belongs to A; as a consequence (3) and (4) describe the statistical density evolution of a large population of users moving according to the considered mobility model.

5. STEADY STATE ANALYSIS

In this Section we compute the steady-state solutions (i.e. solutions which are invariant with respect to time) of the RD model. We start considering the unidimensional case with exponential phase times. Next we generalize our solution to the case in which phases are generally distributed, and finally to the multidimensional case.

5.1 The exponential case

The system dynamics are described by a Markov process over an uncountable compact [1] space state, whose properties have recently been studied proving that the steady state distribution exists unique

[1] Any closed bounded subset of $\mathbb{R}^n$, $\forall n$, is compact.

[5, 6]. [2] Moreover, regardless of the initial condition, $n(x, v, t)$ and $s(x, t)$ tend to the steady state distribution for $t \to \infty$.

By setting the derivative with respect to time equal to zero in both (1) and (2), we obtain that steady state solutions $n(x, v)$, $s(x)$ must satisfy the following equations:

$$v\frac{\partial n(x,v)}{\partial x} = \lambda f_V(v)s(x) - \mu\, n(x,v) \qquad (13)$$

$$\lambda s(x) = \mu \int n(x,v)\,\mathrm{d}v \qquad (14)$$

with the boundary conditions defined in Section 2.

Considering product-form candidate solutions for $n(x, v)$, i.e. $n(x, v) = \alpha(x)\beta(v)$, we obtain the following solution of steady-state equations:

$$n(x,v) = \frac{\lambda f_V(v)}{(\lambda+\mu)|x_u - x_l|} \quad ; \quad s(x) = \frac{\mu}{(\lambda+\mu)|x_u - x_l|}$$

which satisfies the boundary conditions for both *wrap around* and *reflection*, in the latter case under the mild assumption that the speed distribution is symmetric, i.e., $f_V(v) = f_V(-v)$. Notice that we have basically reobtained the known result that the steady state distribution of nodes is uniform in space, while the speed distribution is the same as that used to select a new speed at the transition points.

5.2 General phase times distributions

When phase times have a general distribution, the steady state distribution still exists unique, under the only condition that average phase durations are finite [5, 6]. In addition, regardless of the initial condition, $n(x, v, t)$ and $s(x, t)$ tend to the steady state distribution for $t \to \infty$.

Now we show how the steady state analysis of RD models with generally distributed phases can be reconducted to the analysis of RD models with exponential phases. Setting the derivative with respect to time equal to zero in both (3) and (4), we have:

$$-v\frac{\partial n(x,v,y)}{\partial x} = -\frac{\partial n(x,v,y)}{\partial y} + $$
$$+\delta(y)\int_z \lambda(z) f_V(v) s(x,z)\,\mathrm{d}z - \mu(y)n(x,v,y) \qquad (15)$$

$$\frac{\partial s(x,z)}{\partial z} = -\lambda(z)s(x,z) + \delta(z)\iint \mu(y)n(x,v,y)\,\mathrm{d}v\,\mathrm{d}y \qquad (16)$$

Considering product-form candidate solutions of the type $n(x, v, y) = m(x, v)k(y)$ and $s(x, z) = p(x)h(z)$ with $\int_{0-}^{\infty} h(z)dz = \int_{0-}^{\infty} k(y)dy = 1$, and defining:

$$\lambda_{\text{eff}} = \int_{0-}^{\infty} \lambda(z)h(z)\,\mathrm{d}z = \frac{1}{E[T\mathit{pause}]}$$

$$\mu_{\text{eff}} = \int_{0-}^{\infty} \mu(y)k(y)\,\mathrm{d}y = \frac{1}{E[T\mathit{move}]}$$

it results that $m(x, v)$, $p(x)$, $k(y)$, and $h(z)$ must satisfy:

$$v\frac{\partial m(x,v)}{\partial x} = \lambda_{\text{eff}}\, f_V(v)p(x) - \mu_{\text{eff}}\, m(x,v) \qquad (17)$$

$$\lambda_{\text{eff}}\, p(x) = \mu_{\text{eff}} \int m(x,v)\,\mathrm{d}v \qquad (18)$$

[2] In [5, 6] the properties of RD models have been analyzed, by considering the embedded discrete time Markov process which is obtained by sampling the system dynamics at instants in which the mobile changes phase. An exhaustive analysis of Markov processes over uncountable space states can be found in [10] for the discrete time case.

$$\frac{\partial k(y)}{\partial y} p_{\text{move}} = \lambda_{\text{eff}}\,\delta(y)p_{\text{pause}} - \mu(y)k(y)p_{\text{move}} \qquad (19)$$

$$\frac{\partial h(z)}{\partial y} p_{\text{pause}} = \mu_{\text{eff}}\,\delta(z)p_{\text{move}} - \lambda(z)h(z)p_{\text{pause}} \qquad (20)$$

where p_{pause} and p_{move} are, respectively, the probability for the mobile of being in pause and move phase at steady state:

$$p_{\text{pause}} = \frac{E[T\mathit{pause}]}{E[T\mathit{move}] + E[T\mathit{pause}]}$$

$$p_{\text{move}} = \frac{E[T\mathit{move}]}{E[T\mathit{move}] + E[T\mathit{pause}]}$$

We observe that equations (17) and (18) are structurally identical to equations (13) and (14), thus they admit the same solution (with proper parameter substitutions). Instead, equations (19) and (20) admit the following solutions:

$$k(y) = \frac{e^{-\int_0^y \mu(\alpha)\,\mathrm{d}\alpha}}{\int_{0-}^{\infty} e^{-\int_0^y \mu(\alpha)\,\mathrm{d}\alpha}\,\mathrm{d}y} = \frac{1 - G(y)}{E[T\mathit{move}]}$$

$$h(z) = \frac{e^{-\int_0^z \lambda(\alpha)\,\mathrm{d}\alpha}}{\int_{0-}^{\infty} e^{-\int_0^z \lambda(\alpha)d\alpha}\,\mathrm{d}z} = \frac{1 - H(z)}{E[T\mathit{pause}]}$$

which correspond, as expected, to the residual time spent in the move or in the pause phase when sampling the system at a random point in time. Also in this case we have found the unique steady state solution for both *wrap around* and *reflection* (in the latter case under the assumption that $f_V(v) = f_V(-v)$).

5.3 The multidimensional case

Previous results can be immediately generalized to a multidimensional domain, since in this case steady-state equations admit product form solutions

$$n(\mathbf{x}, \mathbf{v}, y) = n_1(x_1, v_1)n_2(x_1, v_1)\cdots n_k(x_k, v_k)\,k(y)$$
$$s(\mathbf{x}, z) = s_1(x_1)s_2(x_1)\cdots s_k(x_k)\,h(z)$$

and thus can be decoupled into unidimensional equations which are structurally identical to those presented in the previous Section.

5.4 Discussion

As a final remark of our steady-state analysis, we emphasize the our approach based on differential equations allows to obtain the steady-state distribution of RD models with *wrap-around* or *reflection* (in the latter case under the condition that $f_V(v) = f_V(-v)$) in a straightforward manner, providing an alternative to approaches based on Palm Calculus [6, 5].

6. GENERALIZED RD MODEL WITH NON UNIFORM STATIONARY SOLUTION

The standard random direction model brings to a steady state in which nodes are uniformly distributed in space. However, in many practical cases one would like to have an anisotropous node density in the area. For this reason we now generalize the RD model in such a way that the stationary distributions of nodes in the move and/or pause phases are not necessarily uniform in space, but follow a desired (assigned) distribution. In particular, we consider a random direction model in which: i) the pause time may depend on the position x where the mobile stops; ii) the speed of mobiles during the move phase can vary with the instantaneous position x. To simplify the presentation, we consider only the unidimentional case with exponential phase times distribution, however the same results apply to the case of general phase distributions and to the multidimensional case.

When a mobile starts travelling on a new segment, we assume it chooses a "base speed" ζ from a generic distribution $f_V(\zeta)$. The actual speed v is a deterministic function of the position x and the base speed ζ. For simplicity we assume that the actual speed is simply proportional to the base speed ζ through a factor $\psi(x)$ that depends only on the position, i.e., $v(x,\zeta) = \psi(x)\zeta$.

The equations of the generalized RD model are:

$$\frac{\partial n(x,\zeta,t)}{\partial t} = -\frac{\partial[v(x,\zeta)n(x,\zeta,t)]}{\partial x} + \\ +\lambda(x) f_V(\zeta) s(x) - \mu\, n(x,\zeta,t) \quad (21)$$

$$\frac{\partial s(x,t)}{\partial t} = -\lambda(x)s(x,t) + \mu \int n(x,\zeta,t)\,\mathrm{d}\zeta \quad (22)$$

from which the steady-state equations are:

$$\frac{\partial[v(x,\zeta)n(x,\zeta)]}{\partial x} = \lambda(x) f_V(\zeta) s(x) - \mu\, n(x,\zeta) \quad (23)$$

$$\lambda(x)s(x) = \mu \int n(x,\zeta)\,\mathrm{d}\zeta \quad (24)$$

Substituting the expression of $s(x)$ obtained from (24) into (23), we obtain:

$$\frac{\partial[v(x,\zeta)n(x,\zeta)]}{\partial x} = f_V(\zeta)\mu \int n(x,\zeta)\,\mathrm{d}\zeta - \mu\, n(x,\zeta) \quad (25)$$

Now, considering product-form candidate solutions, i.e., solutions of the form $n(x,\zeta) = m(x)\beta(\zeta)$, with $\int \beta(\zeta) d\zeta = 1$, it results:

$$\zeta\beta(\zeta)\frac{\partial[\psi(x)m(x)]}{\partial x} = \mu\, m(x)[f_V(\zeta) - \beta(\zeta)] \quad (26)$$

and we can decouple the previous equation into two ordinary differential equations:

$$\frac{\mathrm{d}[\psi(x)m(x)]}{\mathrm{d}x} = C\mu\, m(x)$$

$$C\zeta = \frac{f_V(\zeta)}{\beta(\zeta)} - 1$$

from which:

$$\beta(\zeta) = \frac{f_V(\zeta)}{[C\zeta + 1]}$$

Since $\beta(\zeta) \geq 0$ and $\int \beta(\zeta)\,\mathrm{d}\zeta = 1$, it results $C = 0$;
hence, $\beta(\zeta) = f_V(\zeta)$ and $m(x) = \frac{a}{\psi(x)}$ for some a such that $\int m(x)\,\mathrm{d}x = 1$.

In conclusion, we can obtain any assigned profiles $\tilde{n}(x)$ and $\tilde{s}(x)$ of the mobiles' density in the move and pause phases, respectively, by setting

$$\psi(x) = \frac{1}{\tilde{n}(x)}$$

$$\lambda(x) = \frac{\mu\,\tilde{n}(x)}{\tilde{s}(x)}$$

7. TRANSIENT ANALYSIS

In this section we present an analytical solution for the transient regime of the RD model. We start considering the case of *wrap around* boundary conditions. As we will see at the end of this section, the transient analysis of the RD model with *reflection* comes for free once we know how to solve the RD model with *wrap around*. As usual, we first consider the unidimensional case with exponential phase times. Then we extend the analysis to the case of general phase times, and finally to the multidimensional case.

7.1 Unidimensional case with exponential phase times

We apply the methodology of separation of variables to find solutions for the system of equations (1) and (2), in case of *wrap around* boundary conditions. Consider product-form candidate solutions: $n(x,v,t) = \tau(t)m(x,v)$ and $s(x,t) = \tau(t)r(x)$; substituting into (1) and (2), we obtain:

$$\frac{\mathrm{d}\tau(t)}{\mathrm{d}t} m(x,v) = -v\tau(t)\frac{\partial m(x,v)}{\partial x} + \\ +\lambda f_V(v) r(x)\tau(t) - \mu m(x,v)\tau(t)$$

$$\frac{\mathrm{d}\tau(t)}{\mathrm{d}t} r(x) = -\lambda r(x)\tau(t) + \mu\tau(t)\int m(x,v)\,\mathrm{d}v$$

From which we can separate the dependency on time from the dependency on space and speed, yielding:

$$\frac{\mathrm{d}\tau(t)}{\mathrm{d}t} = \gamma\tau(t) \quad (27)$$

$$v\frac{\partial m(x,v)}{\partial x} = \lambda f_V(v) r(x) - (\mu+\gamma)m(x,v) \quad (28)$$

$$r(x) = \frac{\mu}{\lambda+\gamma}\int m(x,v)\,\mathrm{d}v \quad (29)$$

Now substituting the expression of $r(x)$ provided by (29) into (28) we have:

$$v\frac{\partial m(x,v)}{\partial x} = f_V(v)\frac{\lambda\mu}{\lambda+\gamma}\int m(x,v)\,\mathrm{d}v - (\mu+\gamma)m(x,v)$$

For $m(x,v)$, we consider again product-form candidate solutions $m(x,v) = \alpha(x)\beta(v)$, obtaining:

$$v\beta(y)\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} = \\ f_V(v)\frac{\lambda\mu}{\lambda+\gamma}\alpha(x)\int\beta(v)\,\mathrm{d}v - (\mu+\gamma)\alpha(x)\beta(v)$$

in which we can separate the functions which depend on x from the functions which depend on v:

$$\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} = \eta\alpha(x) \quad (30)$$

$$\frac{\beta(v)}{\int\beta(w)dw} = f_V(v)\frac{\lambda\mu}{(\lambda+\gamma)(\mu+\gamma+\eta v)} \quad (31)$$

Functions $\alpha(x) = e^{\eta x}$, being η any complex number, are solutions of (30). Instead from (31), since $\int_v \frac{\beta(v)}{\int\beta(w)\,\mathrm{d}w}\,\mathrm{d}v = 1$, we obtain a fundamental relation between γ and η:

$$\frac{\lambda\mu}{\lambda+\gamma}\int_v \frac{f_V(v)}{\mu+\gamma+\eta v}\,\mathrm{d}v = 1 \quad (32)$$

Wrap around boundary conditions require that $\alpha(x_l) = \alpha(x_u)$, $\lim_{x\to x_l^+}\frac{\partial\alpha(x)}{\partial x} = \lim_{x\to x_u^-}\frac{\partial\alpha(x)}{\partial x}$. This constraint is satisfied when $\alpha(x)$ is periodic with period $1/f_x = x_u - x_l$. It follows that wrap around boundary conditions are satisfied when $\eta = j2\pi f_x k$, with $k \in \mathbb{Z}$. Notice that solutions $\alpha_k(x) = e^{j2\pi f_x kx}$ correspond to the standard Fourier basis for the interval $[x_u, x_l]$, which is dense in $C^0([x_u, x_l])$, the class of continuous functions defined over $[x_u, x_l]$.

For any given $k \in \mathbb{Z}$, (32) provides an implicit equation that defines exponent $\gamma(k)$:

$$\frac{\lambda\mu}{\lambda+\gamma}\int_v \frac{f_V(v)}{\mu+\gamma+j2\pi k f_x v}\,\mathrm{d}v = 1 \quad (33)$$

In particular, when $k = 0$, (33) admits the solution $\gamma_1 = 0$, corresponding to the steady-state distribution of the system already found in Section 5. There is also the solution $\gamma_2 = -(\lambda+\mu)$, which has a different physical interpretation: it is the rate at which the system converges to the steady state distribution from the condition in which the probability of being in the move or pause phases are uniform over space but not in equilibrium.

For $k \neq 0$, the existence of real solutions for γ can be guaranteed when the probability density function of nodes' speed is symmetric, i.e. $f_V(v) = f_V(-v)$. In this case, the imaginary component of $\int_v \frac{f_V(v)}{(\mu+\gamma+j2\pi kv)}\,\mathrm{d}v$ is null.

For example, in case $f_V(v)$ is uniform in the interval $[-V, V]$ equation (33) reduces to:

$$\frac{\lambda\mu}{(\lambda+\gamma)2\pi k f_x V}\arctan\left(\frac{2\pi k f_x V}{\mu+\gamma}\right) = 1$$

from which γ can be easily obtained numerically. In general, it can be shown that γ has two negative solutions γ_1 and γ_2 for every k ($\gamma_2 < \gamma_1 < 0$).

As a result of previous calculations, the class of elementary functions:

$$n_k(x,v,t) = \frac{\lambda\mu f_V(v) e^{j2\pi f_x kx} e^{\gamma t}}{(\lambda+\gamma)(\mu+\gamma+j2\pi k f_x v)} \quad (34)$$

$$s_k(x,t) = \frac{\mu}{\lambda+\gamma} e^{j2\pi f_x kx} e^{\gamma t} \quad (35)$$

are solutions of (1) and (2), with *wrap around* boundary conditions. Moreover, letting $\hat{n}_k(x,t) = \int n_k(x,v,t)\,\mathrm{d}v$, we obtain the elementary solution vector:

$$\begin{pmatrix} \hat{n}_k(x,t) \\ s_k(x,t) \end{pmatrix} = \begin{pmatrix} \lambda+\gamma \\ \mu \end{pmatrix} e^{j2\pi f_x kx} e^{\gamma t} \quad (36)$$

Recalling that for every value of k there exist two solutions of γ, it turns out that any solution of (1) and (2) in which the initial distribution profiles in the move and pause phases are continuous with respect to the space coordinate (i.e.,$\in C^0([x_u, x_l])$) can be expanded in series of the above elementary vectors [3].

The procedure to compute the system state at an arbitrary time instant t can be summarized into the following steps:

1. Compute the values of $\gamma_1(k)$ and $\gamma_2(k)$ associated to every elementary vector.
2. Compute the Fourier series expansion of the initial distribution of mobiles' in terms of elementary vectors evaluated at time $t = 0$.
3. Multiply each coefficient of the (possibly truncated) series expansion by the exponential decay factor of the corresponding solution vector (either $e^{-\gamma_1(k)t}$ or $e^{-\gamma_2(k)t}$).
4. Reconstruct the distribution of mobiles' using the new values of coefficients at time t.

Of course, steps 1 and 2 has to be performed only once, not for any t. As expected, as time tends to infinity all 'propagation modes' $\alpha_k(x)$, with $k \neq 0$, tend to vanish exponentially, leaving only the uniform distribution associated to $k = 0$ ($\gamma_1 = 0$). Moveover, we observe that the duration of the transient is essentially determined by the periodic component with the minimum absolute value of γ_1.

[3] This is due to the fact that Fourier system represents a complete orthogonal system in the class of square summable functions which comprise functions in $C^0([x_u, x_l])$.

7.2 General phase times distributions

We have not tried to solve exactly the transient analysis of the system in case the move and/or pause times have general (non-exponential) distributions. However, an approximate analysis can be performed using a stage decomposition approach. This means that a separate differential equation has to be written for each stage of the decomposition. For example, consider the case in which the move time is described by an hyper-exponential distribution of the second order:

$$H_2(t) = p_1\mu_1 e^{-\mu_1 t} + p_2\mu_2 e^{-\mu_2 t}$$

This simple 2-stage approximation allows to match the first two moments an any distribution having a coefficient of variation larger than one. Let $n_1(x,v,t)$ and $n_2(x,v,t)$ be the pdf over (x,v) of the mobile in move stages 1 and 2, respectively, at time t. The transient behavior is then described by the following system of differential equations

$$\begin{cases} \dfrac{\partial n_1}{\partial t} = -v\dfrac{\partial n_1}{\partial t} + \lambda p_1 f_V s - \mu_1 n_1 \\ \dfrac{\partial n_2}{\partial t} = -v\dfrac{\partial n_2}{\partial t} + \lambda p_2 f_V s - \mu_2 n_2 \\ \dfrac{\partial s}{\partial t} = -\lambda s + \mu_1 \int n_1\,\mathrm{d}v + \mu_2 \int n_2\,\mathrm{d}v \end{cases} \quad (37)$$

which can be solved in a way analogous to the exponential case. In this case the equation relating γ to η is

$$p_1\mu_1 \int \frac{f_V(v)}{\mu_1+\gamma+\eta v}\,\mathrm{d}v + p_2\,\mu_2 \int \frac{f_V(v)}{\mu_2+\gamma+\eta v}\,\mathrm{d}v = \frac{\lambda+\gamma}{\lambda}$$

Similarly, we can analyze the case in which the pause time is described by an hyper-exponential distribution of the second order, obtaining the set of equations

$$\begin{cases} \dfrac{\partial n}{\partial t} = -v\dfrac{\partial n}{\partial t} + \lambda_1 f_V s_1 + \lambda_2 f_V s_2 - \mu\, n \\ \dfrac{\partial s_1}{\partial t} = -\lambda_1 s_1 + p_1\mu \int n\,\mathrm{d}v \\ \dfrac{\partial s_2}{\partial t} = -\lambda_2 s_2 + p_2\mu \int n\,\mathrm{d}v \end{cases} \quad (38)$$

In this case the equation relating γ to η becomes

$$\left(\frac{p_1\lambda_1}{\lambda_1+\gamma} + \frac{p_2\lambda_2}{\lambda_2+\gamma}\right)\mu \int_v \frac{f_V(v)}{\mu+\gamma+\eta v} dv = 1$$

Instead, in the case in which move time is described by an Erlang distribution of the second order (the sum of two exponential distributions with rate μ_1 and μ_2), we obtain:

$$\begin{cases} \dfrac{\partial n_1}{\partial t} = -v\dfrac{\partial n_1}{\partial t} + \lambda p_1 f_V s - \mu_1 n_1 \\ \dfrac{\partial n_2}{\partial t} = -v\dfrac{\partial n_2}{\partial t} + \mu n_1 - \mu_2 n_2 \\ \dfrac{\partial s}{\partial t} = -\lambda s + \mu_2 \int n_2\,\mathrm{d}v \end{cases} \quad (39)$$

In this case the equation relating γ to η is

$$\mu_1\mu_2 \int \frac{f_V(v)}{(\mu_1+\gamma+\eta v)(\mu_2+\gamma+\eta v)}\,\mathrm{d}v = \frac{\lambda+\gamma}{\lambda}$$

From the examples above it is clear that one can write a set of equations for a generic stage decomposition of the move and/or pause time, and obtain an implicit equation of γ for each feasible value of η. We remark that to analyze the transient it is not necessary to solve the entire set of differential equations, just to derive (numerically) the proper value of γ for each value of η, and apply the same procedure described at the end of Section 7.1. Thus, using a stage decomposition approach the analysis of the transient behavior of the system in the general case has the same computationally complexity as the exponential case.

7.3 The multidimensional case

The transient analysis can be extended also to the multidimensional case. Since the most interesting applications of mobility models arise on the bi-dimensional space, here we describe more in detail the transient analysis in the 2D case, restricting ourselves to the case of exponential sojourn times in the move and pause states. We assume that mobiles are free to move in the rectangular area $[x_u, x_l] \times [y_u, y_l]$, and independently choose the speed components along x and y from arbitrary distributions f_{V_x} and f_{V_y} (possibly different). Similarly to the mono-dimensional case, we apply the methodology of separation of variables, looking for solutions of equations (5) and (6) in the form

$$\begin{aligned} n(x, y, v_x, v_y, t) &= \tau(t)\, \alpha_X(x)\, \beta_X(v_x)\, \alpha_Y(y)\, \beta_Y(v_y) \\ s(x, y, t) &= \tau(t)\, r_X(x)\, r_Y(y) \end{aligned}$$

Plugging these expressions into (5) and (6) we have $\frac{\mathrm{d}\tau(t)}{\mathrm{d}t} = \gamma\tau(t)$, while boundary conditions of the wrap around model imply that the only feasible solutions for $\alpha_X(x)$ and $\alpha_Y(y)$ belong to the discrete sets $\alpha_{k_X}(x) = e^{j2\pi f_x k_x x}$ and $\alpha_{k_Y}(y) = e^{j2\pi f_y k_x y}$, respectively. Now we have that exponents γ associated to each pair $(k_x, k_x) \in \mathbb{Z}^2$, have to satisfy the equation

$$\frac{\lambda\mu}{\lambda + \gamma} \iint \frac{f_{V_x}(v_x) f_{V_y}(v_y)}{\mu + \gamma + j2\pi(k_x f_x v_x + k_y f_y v_y)}\, \mathrm{d}v_x\, \mathrm{d}v_y = 1 \quad (40)$$

The procedure to evaluate the system state at a generic time instant t follows the same steps described in Section 7.1, except that now we need to perform a bi-dimensional (Fourier) expansion of the initial distribution of mobiles in the move and pause phases over the rectangular region.

7.4 Extension to reflection boundary conditions

The transient analysis of the RD model with *reflection* can be easily reconducted to the analysis of the RD model with *wrap around*. Here we provide an intuitive explanation of how this can be done.

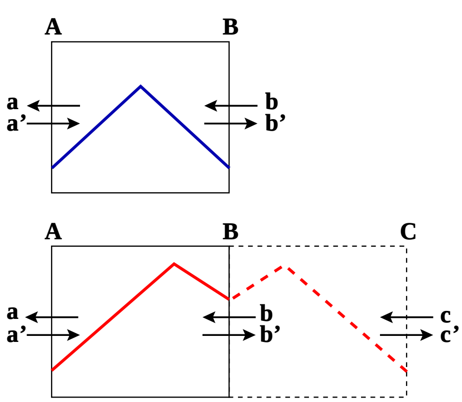

Figure 1: Reduction of RD model with *reflection* to RD model with *wrap around*

A formal proof is reported in Appendix B. Consider first the simple case in which the initial distribution of mobiles in the *move* and *pause* phases are symmetric (top of Figure 1). In this case, the solution of the RD model with *reflection* is exactly the same as that of the RD model with *wrap around*. Indeed, the flows of mobiles hitting the boundaries are the same, $\mathbf{a} = \mathbf{b}'$, and since in the *wrap around* $\mathbf{b}' = \mathbf{a}'$ we have $\mathbf{a} = \mathbf{a}'$ (similarly, $\mathbf{b} = \mathbf{b}'$), which means that the dynamics are the same as in the *reflection* model. If the initial conditions are not symmetric, we double the area adding a specular 'image' of the initial domain to the right (or to the left), as shown in the bottom part of Figure 1. Doing so, we obtain a scenario in which the initial conditions are symmetric, thus $\mathbf{a} = \mathbf{c}' = \mathbf{a}'$. Moreover, by construction we have $\mathbf{b} = \mathbf{b}'$. Therefore the dynamics of the *wrap around* model in the extended area 'contain' those of the *reflection* model in the restricted area.

8. VALIDATION AND APPLICATIONS

In this Section we validate our analysis of the RD model comparing analytical prediction with simulation results obtained from an event-driven simulator. At the same time we offer examples of possible applications of our methodology.

8.1 Generalized mobility model.

Suppose that we want to achieve a given non-uniform stationary distribution of mobiles' on the 2D plane. In particular, consider a metropolitan area divided into 3 concentric rings R_1, R_2, R_3 in a square area of edge 20 kilometers, as depicted in Figure 2.

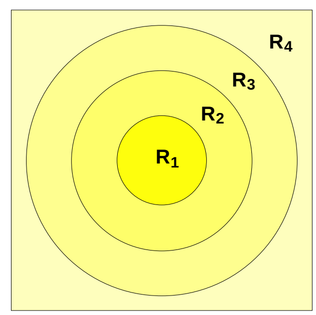

Figure 2: Regions of the metropolitan area

The outer region is denoted by R_4. Let's assume the population density is maximum in R_1, equal to ρ_1, whereas the density in region R_i, $(i > 1)$, is $\rho_i = \rho_1 / i$. We can design a generalized RD model whose stationary distribution of mobiles' location follows exactly this distribution. Assuming an equal fraction of users in the move and pause states ($\lambda = \mu$), we have to set the scaling factor of speed velocity $\psi(x, y) = i\,, \forall (x, y) \in R_i$.

Figure 3 contains the results of a simulation in which 8 million mobiles move according to the generalized mobility model specified above. Irrespective of their initial position, distributions of move/pause times, distribution of speed, they tend to the desired non-uniform density. In particular, the plot in Figure 3 reports the total number of users measured in simulation in each square of edge 100m after 5 hours of simulated time, assuming a base speed uniformly distributed in [-10 km/h, 10 km/h] in each direction.

8.2 Transient analysis in 2D.

We now present an example of transient analysis on the 2D plane. We assume that mobiles are initially uniformly distributed within a circle of radius 2 in the middle of square area of edge 20. At time $t = 0$ they start moving with a speed uniformly distributed in [-1,1] in each direction. Move and pause times are exponentially distributed with mean 1. This scenario could represent how the center of a city empties at the end of a working day.

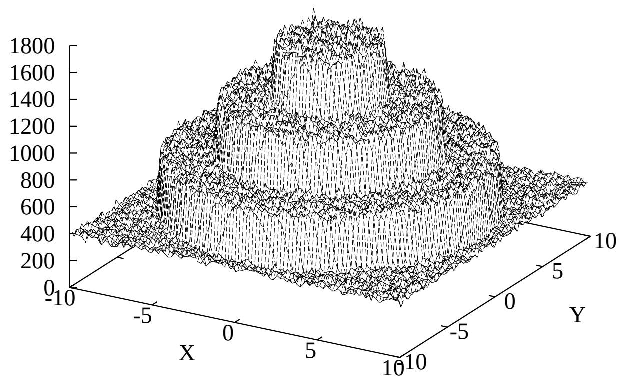

Figure 3: Distribution of mobiles measured on simulation after 5 hours, resulting from the generalized RD model

We follow the steps to perform the transient analysis of the system as described in Section 7. First, we compute parameters $\gamma_1(k_x, k_y)$ and $\gamma_2(k_x, k_y)$ associated to each elementary vector (see Section 7.3). Figure 4 depicts parameter γ_1 for all combinations of the first 30 positive values of k_x and k_y as a continuous surface (for a better representation). Note that

$$\gamma_1(k_x, k_y) = \gamma_1(-k_x, k_y) = \gamma_1(k_x, -k_y) = \gamma_1(-k_x, -k_y)$$

thus positive values of k_x and k_y provide all information. We observe that γ_1, which is a negative number, decreases rapidly for increasing k_x or k_y. In practice, the duration of the transient is determined by the smallest absolute values of γ_1. This suggests that there is no need to keep too many terms of the Fourier series expansion of the initial distributions: solutions corresponding to large k_x or k_y decay very fast over time and therefore do not provide a significant contribution to the overall solution except in the very beginning of the transient.

Next we compute the elementary vector expansion of the initial distribution of mobiles' location (a bi-dimensional Fourier series expansion), truncating the series to 128×128 coefficients k_x and k_y. This is enough to produce a satisfactory representation of the initial distribution since the very beginning (i.e. $t = 0$), as illustrated in Figure 5.

Now, suppose that we want to compute the distribution of mobiles at an arbitrary time instant $t > 0$. It is sufficient to reconstruct the distribution from the elementary vectors series expansion, having multiplied each term of the series by the corresponding factor (either $e^{-\gamma_1(k_x,k_y)t}$ or $e^{-\gamma_2(k_x,k_y)t}$). For example, Figures 6 and 7 report the distribution of mobiles at time $t = 10$, according to analysis and simulation, respectively.

The mobiles' distribution has been obtained in simulation considering 10 million nodes, and counting how many of them are present at $t = 10$ in each square of a 100×100 grid. Note that we have used such a large number of nodes to obtain a clean distribution on the chosen grid after a single simulation run. We could have considered a smaller number of nodes (or even a single node), but in this case it would have been necessary to average the results

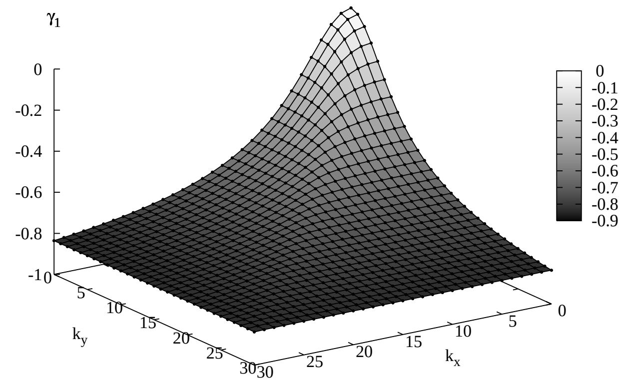

Figure 4: Parameter γ_1 for all combinations of (k_x, k_y), $1 \leq k_x, k_y \leq 30$.

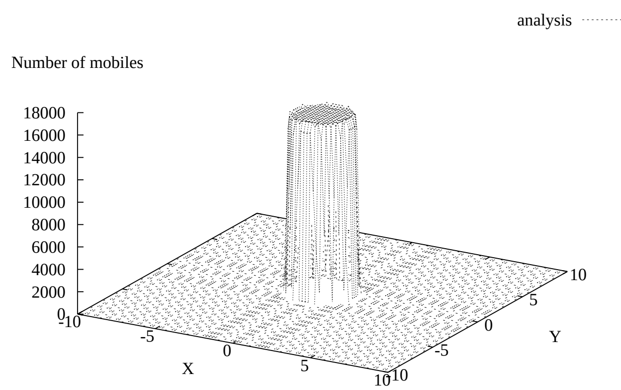

Figure 5: Representation of the initial distribution of mobiles' location (a circle of radius 2) limiting the Fourier series expansion to 128×128 terms.

of many independent simulation runs.

Recall that we can regard the analytical prediction as the probability of finding a single mobile at a given point of the plane after time t, starting from an initial pdf of its location at $t = 0$. Actually, this point of view opens a wide range of possible applications of our analysis. For example, one could study the persistence of a wireless connection between a mobile and a base station, and use our probabilistic analysis to design better hand-off strategies. Another interesting application is the analysis of link duration and availability in mobile ad-hoc networks [11].

8.3 Impact of parameters on the transient duration.

We now turn to the 1D case and study the impact of various parameters of the random direction model on the duration of the transient. As already observed, the time constant of the system is essentially given by the smallest absolute value of γ_1, i.e. the one associated with the fundamental mode $k = 1$. Thus, we look now at how $\gamma_1(1)$ depends on the system parameters. We fix the region

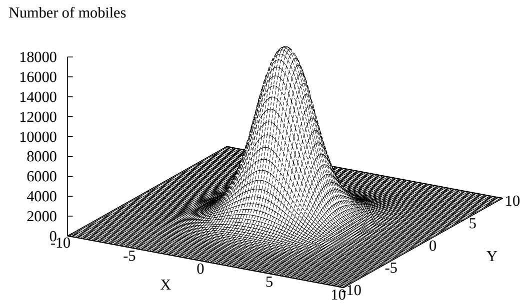

Figure 6: Distribution of mobiles according to analysis at time t = 10.

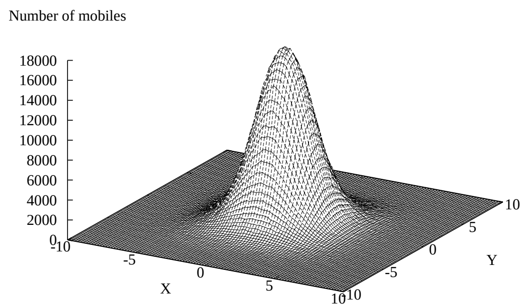

Figure 7: Distribution of mobiles according to simulation at time t = 10.

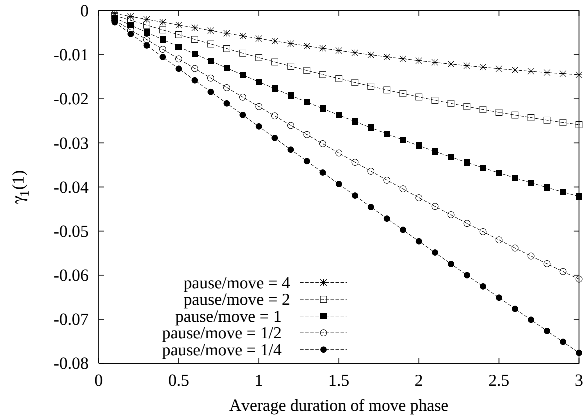

Figure 8: The dependence of $\gamma(1)$ on move/pause dynamics.

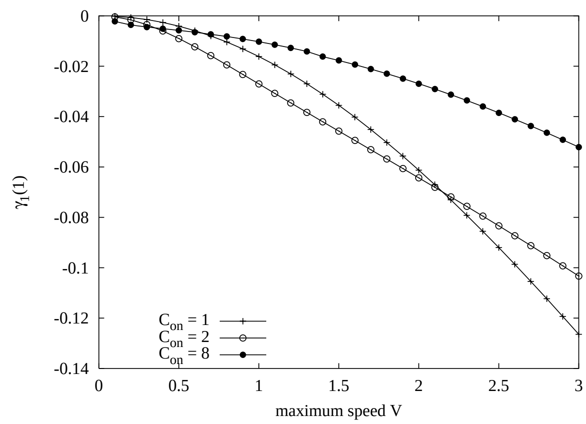

Figure 9: The dependence of $\gamma(1)$ on maximum speed V, for different variation coefficients of move time.

where mobiles move to the interval [-10,10]. First, we consider the impact of move/pause dynamics, while keeping the speed uniformly distributed in [-1,1]. Figure 8 reports the value of $\gamma_1(1)$ as a function of the average duration of move time, for different ratios between pause and move times. Both of them are assumed to be exponentially distributed.

We observe that, for a given pause/move ratio, $\gamma_1(1)$ becomes more negative (which implies a shorter transient) for increasing duration of the move time, because mobiles spread faster if they keep the same direction and speed for prolonged period of time. For a given value of average move time, the absolute value of $\gamma_1(1)$ decreases (which implies a longer transient) for increasing persistence in the pause state.

Next, we fix the average duration of the move and pause times equal to 1, and vary the maximum speed V of mobiles, which is assumed to be uniformly distributed in $[-V, V]$. Figure 9 reports the value of $\gamma_1(1)$ as a function of V for different values of the variation coefficient $C_{\rm on}$ of move time, whose distribution is assumed to be hyper-exponential of the second order. Pause times are instead exponentially distributed, i.e. $C_{\rm off} = 1$.

While the effect of speed distribution is more intuitive, i.e. $\gamma_1(1)$ becomes more negative for increasing V (shorter transient), the dependency on the variation coefficient $C_{\rm on}$ is quite intriguing, with multiple intersections among curves corresponding to $C_{\rm on} = 1, 2, 8$. Therefore we decided to check on simulation this peculiar behavior. In particular, we consider the case of $V = 1$. According to Figure 9 the fastest transient should be for $C_{\rm on} = 2$, whereas $C_{\rm on} = 8$ should produce the slower transient. We take as initial distribution of mobiles' position a gaussian distribution with a variance of 1, centered in the origin. Figures 10, 11, 12, reports, respectively, the distributions mobiles for $C_{\rm on} = 1, 2, 8$ sampled every 10 time units.

Analytical predictions match perfectly with simulation results in all cases, and confirm the impact of the variation coefficient: the curves referring to $C_{\rm on} = 2$ flatten more rapidly than the curves for $C_{\rm on} = 1$, which in turn flatten more rapidly than the curves for $C_{\rm on} = 8$.

Finally, Figure 13 reports the values of $\gamma_1(k)$ up to $k = 1000$ for the three considered values of $C_{\rm on}$. The most significant values of $\gamma_1(k)$ in determining the duration of the transient are shown in the inset of Figure 13.

9. CONCLUSIONS AND FUTURE WORK

So far in the literature, the theoretical investigation of random direction and random waypoint mobility models has mainly focused on the analysis of the steady state distributions. The approach proposed in this paper permits to extends the analysis to the transient regime. We have started from the observation that Chapman-Kolmogorov equations describing the dynamics of a single mobile can be used to describe the dynamics of large population of users. We have obtained Chapman-Kolmogorov equations

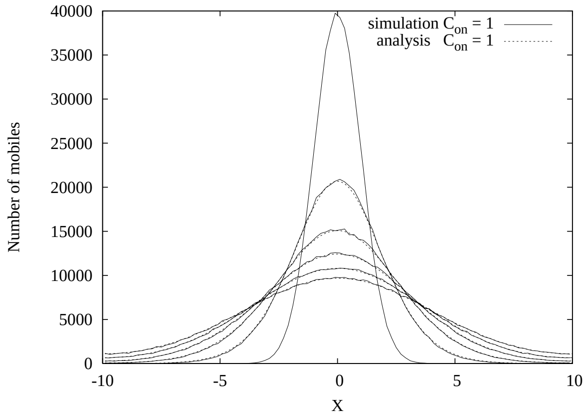

Figure 10: Distribution of mobiles for $C_{on} = 1$. Comparison between analysis and simulation at different time instants.

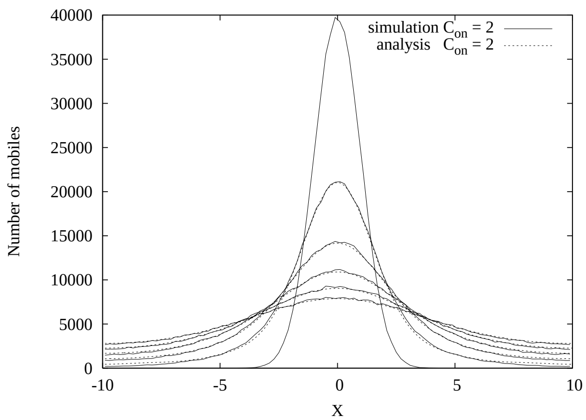

Figure 11: Distribution of mobiles for $C_{on} = 2$. Comparison between analysis and simulation at different time instants.

of a mobile moving according to either RD or RWP model. Then we have applied standard mathematical techniques to *analytically* solve the equations for RD models in both steady state and transient regime, either with *wrap around* or *reflection* boundary conditions. We have derived simple expressions relating the transient duration to the model parameters; moreover, we have proposed generalized RD models to achieve a desired stationary distribution of mobiles in the space, a problem that has received so far little attention. Our dynamical viewpoint indeed opens many new directions in the theory and practice of random mobility models.

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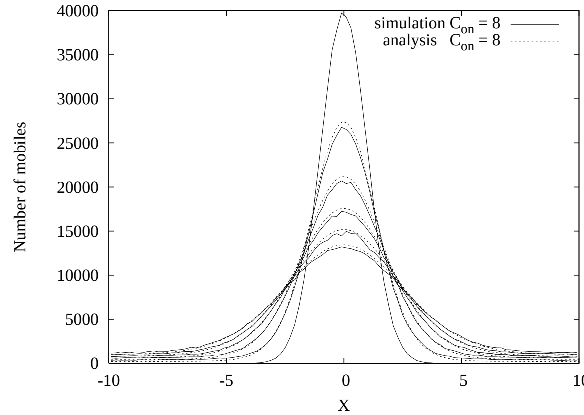

Figure 12: Distribution of mobiles for $C_{on} = 8$. Comparison between analysis and simulation at different time instants.

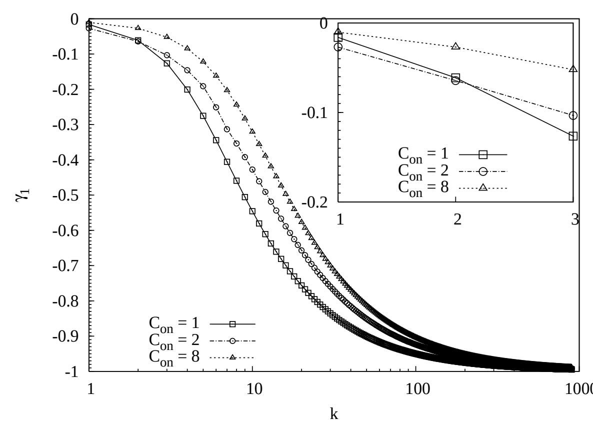

Figure 13: Values of $\gamma_1(k)$ for different variation coefficients.

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APPENDIX

A. UNIQUENESS OF SOLUTION

Here we prove that the problem defined by equations (1), (2) with boundary and initial conditions specified in section 2.1 admits at most one solution in both the *wrap-around* and *reflection* cases. First we introduce the following two lemmas which will be used to prove our main result.

LEMMA 1. *Suppose for simplicity that* $f_V(v)$ *is uniformly distributed between* $[-1/2, 1/2]$. *Then the functional*

$$L(t) = \frac{\mu}{2} \iint n^2(x,v,t)\,\mathrm{d}x\,\mathrm{d}v + \frac{\lambda}{2} \int s(x,t)^2\,\mathrm{d}x$$

is a non-increasing function of time, for any pair of functions $n(x,v,t)$, $s(x,t)$ *which are solutions of (1) and (2), respectively.*

Proof: We first define:

$$\begin{aligned} ||s(x,t)||^2 &= \int s^2(x,t)\,\mathrm{d}x = \iint s^2(x,t)\mathbb{1}(v)\,\mathrm{d}v\,\mathrm{d}x \\ ||f_V(v)||^2 &= \int f^2(v)\,\mathrm{d}v = 1 \\ ||s(x,t)f_V(v)||^2 &= \iint s^2(x,t)f^2(v)\,\mathrm{d}v\,\mathrm{d}x = \\ & ||s(x,t)||^2||f_V(x)||^2 = ||s(x,t)||^2 \end{aligned}$$

Now let us evaluate:

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu}{2}\iint n(x,v,t)^2\,\mathrm{d}x\,\mathrm{d}v + \frac{\lambda}{2}\int s(x,t)^2\,\mathrm{d}x\right] =$$

$$= \mu \iint n(x,v,t)\frac{\partial n(x,v,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}v + \lambda \int s(x,t)\frac{\partial s(x,t)}{\partial t}\,\mathrm{d}x =$$

$$= \iint n(x,v,t)\left[-v\frac{\partial n(x,v,t)}{\partial x} + \lambda f_V(v)s(x) - \mu n(x,v,t)\right]\mathrm{d}x\mathrm{d}v +$$

$$+ \lambda \iint s(x,t)\left[-\lambda s(x,t) + \mu \int n(x,v,t)\,\mathrm{d}v\,\mathrm{d}x\right]\mathrm{d}x =$$

$$= -\mu \iint v\frac{\partial n^2(x,v,t)}{\partial x}\,\mathrm{d}x\,\mathrm{d}v + \lambda\mu \iint f_V(v)s(x)n(x,v,t)\,\mathrm{d}v\mathrm{d}x -$$

$$-\mu^2 \iint n^2(x,v,t)\,\mathrm{d}v + \mu\lambda \iint s(x)n(x,v,t)\,\mathrm{d}v\,\mathrm{d}x -$$

$$-\lambda^2 \int s^2(x,t)\,\mathrm{d}x \leq$$

$$\leq -\mu^2||n^2(x,v,t)|| + 2\lambda\mu||s(x)||\,||n(x,v,t)|| - \lambda^2||s(x)||^2 \leq 0$$

In case in which $f_V(v)$ is not uniform the previous result can be extended redefining the function $L(t)$. ∎

LEMMA 2. *Let* $f_V(v)$ *be a regular pdf whose associated cdf is* $F_V(v)$. *Having defined* $m(x,v,t) = n(x,v,t)/f_V(v)$ *for any* $\epsilon > 0$ *the functional*

$$L(t) = \frac{\mu}{2} \iint_{v:f_V(v)>\epsilon} m(x,v,t)^2\,\mathrm{d}x\,\mathrm{d}F_V(v) + \frac{\lambda}{2}\int s(x,t)^2\,\mathrm{d}x$$

is not increasing.

Proof: this statement can be proved repeating the passages of previous proof. ∎

From the monotonicity of functional $L(t)$ we can easily show that solutions of equations (1) and (2) are unique.

Proof: By contradiction, suppose that two different pairs of functions $n_1(x,v,t)$, $s_1(x,t)$ and $n_2(x,v,t)$, $s_2(x,t)$ are solutions of the equations (1) and (2) with the same initial and boundary conditions; then by linearity of the equations (1) and (2), $n_1(x,v,t) - n_2(x,v,t)$ and $s_1(x,t) - s_2(x,t)$ are solutions of the equations (1) and (2) with null initial conditions; i.e., $n_1(x,v,0) - n_2(x,v,0) = 0$ and $s_1(x,0) - s_2(x,0) = 0$. As a consequence, since $L(t) \geq 0$ (by definition), $L(0) = 0$, and $L(t)$ is not increasing, it results $L(t) = 0\,, \forall t$. Therefore it must be $n_1(x,v,t) - n_2(x,v,t) = 0$ and $s_1(x,t) - s_2(x,t) = 0$; i.e., $n_1(x,v,t) = n_2(x,v,t)$ and $s_1(x,t) = s_2(x,t)$. ∎

B. TRANSIENT ANALYSIS OF RD MODEL WITH REFLECTION

Here we prove that the transient analysis of the RD model with *reflection* can be reduced to the analysis of the RD model with *wrap around*, as explained in Section 7.4. We assume that the the speed distribution is symmetric, i.e., $f_V(v) = f_V(-v)$. The proof is articulated in four steps.

Step 1 Consider the unidimensional RD model with *wrap around*. Without loss of generality, let the domain be the interval $[x_l = -1, x_u = 1]$. If $n(x,v,t)$ and $s(x,t)$ are the solution of (1) and (2) corresponding to the initial conditions $n_o(x,v)$ and $s_o(x)$, then $n(-x,-v,t)$ and $s(-x,t)$ are the solution of (1) and (2) corresponding to the initial conditions $n_o(-x,-v)$ and $s_o(-x)$. This property can be easily checked directly on equations (1) and (2) through the change of variables $(x,v) \rightarrow (-x,-v)$.

Step 2 As a consequence of previous step the following property follows: if the initial condition is symmetrical, i.e., $n_o(x,v) = n_o(-x,-v)$, $s_o(x) = s_o(-x)$, then the solution $n(x,v,t)$, $s(x,t)$ is symmetrical for all t, i.e., $n(x,v,t) = n(-x,-v,t)$ and $s(x,t) = s(-x,t)$.

step 3 For any symmetrical initial condition $n_o(x,v) = n_o(-x,-v)$ and $s_o(x) = s_o(-x)$, the RD models with *wrap around* and *reflection* admit the same solution. Indeed, the *wrap around* solution must satisfy the boundary conditions $n(1,v,t) = n(-1,v,t)$. This, combined with the invariance under the transformation $(x,v) \rightarrow (-x,-v)$, implies that $n(1,v,t) = n(1,-v,t)$, and similarly $n(-1,v,t) = n(-1,-v,t)$, thus the *wrap around* solution satisfies also the *reflection* boundary conditions and therefore provides a solution for the reflection model. Finally, from the uniqueness of the solution of the RD model, no other solution for the *reflection* model exists.

Step 4 Now, without loss of generality, consider a *reflection* model over the domain $[0,1]$, under an arbitrary initial condition $n_o(x,v)$ and $s_o(x,t)$. We compare the solution of this model with that of a *wrap around* model over the extended domain $[-1,1]$, under the initial condition $n_o(x,v) + n_o(-x,-v)$ and $s_o(x) + s_o(-x)$: we claim that the restriction of the latter model (with wrap around) over the domain $[0,1]$ provides the solution of the *reflection* model over the same domain. Indeed consider $n(x,v,t)$ and $s(x,t)$, the solution of the *wrap around* model over $[-1,1]$. Observe that, by construction, the initial conditions of this *wrap around* model are symmetric. Being $n(x,v,t)$ and $s(x,t)$ invariant under the transformation $(x,v) \rightarrow (-x,-v)$, we have $n(1,v,t) = n(-1,v,t) = n(1,-v,t)$, therefore the solution satisfies the reflection condition at boundary $x = 1$. Moreover, by construction $n(0,v,t) = n(0,-v,t)$, thus the reflection boundary conditions are satisfied also at $x = 0$. Since $n(x,v,0) = n_o(x,v)$ and $s(x,0) = s_o(x)$ over domain $[0,1]$, the restriction of $n(x,v,t)$, $s(x,t)$ over $[0,1]$ provide the unique solution of the *reflection* model over the same domain, under the initial condition $n_o(x,v)$ and $s_o(x)$.