Stochastic Petri nets with low variation matrix exponentially distributed firing time

P. Buchholz[*], A. Horváth[†], M. Telek[‡]

February 28, 2011

Abstract

The set of order n matrix exponential (ME) distributions is strictly larger than the set of phase type (PH) distributions for $n > 2$ and contains elements with squared coefficient of variation (scv) significantly lower than $1/n$. ME distributions with very low scv are such that the density function becomes zero at some points in $(0, \infty)$. For such distributions there is no equivalent finite dimensional PH representation, which inhibits the application of existing methodologies for the numerical analysis of stochastic Petri nets (SPNs) with this kind of ME distributed firing time.

To overcome the limitations of existing methodologies we apply the flow interpretation of ME distributions introduced by Bladt and Neuts and study the transient and the stationary behaviour of stochastic Petri nets with low variation matrix exponentially distributed firing times via extended differential and linear equations, respectively. Since the proof we present in the paper is general, it applies to all kind of ME distributions and therefore shows that ME distributions can be used like PH distributions in stochastic Petri nets and a numerical computation of transient or stationary measures like token populations at places or transition throughputs is possible with methods similar to the methods used for Markov models.

Keywords: stochastic Petri net, phase type distribution, matrix exponential distribution, extended Markov chain.

1 Introduction

The method of extended Markov chain (EMC) is a widely used analysis technique for stochastic Petri nets with phase type (PH) distributed firing times. This technique is based on the generation of a Markov chain that describes the behaviour of the marking process and additionally the phase processes of the involved PH distributions. The resulting Markov chain can then be analyzed with established numerical techniques for transient or stationary analysis.

Following the general results in [10] it was likely that in a stochastic model ME distributions can be used in place of PH distributions and several results will carry over. There are some results to this direction, but it is not easy to prove results in the general setting because probabilistic arguments associated with PH distributions do no longer hold. In [4] it has been shown that matrix geometric methods can be applied for quasi birth death processes (QBDs) with rational arrival processes (RAPs) [2], which can be viewed as an extension of ME distributions to arrival processes. To prove that the matrix geometric relations hold, the authors of [4] use an interpretation of RAPs that has been proposed in [2]. However, the resulting proofs are limited to QBDs.

The closest related result considers a *subclass* of SPNs with ME distributed firing times [6]. That paper proves the applicability of an extended system of differential equations for the transient analysis and an extended system of linear equations for stationary analysis in case of ME distributions with strictly positive density in $(0, \infty)$. Due to the similarity to the EMC based solution we refer to this solution method as EMC-like solution. [6] proofs that the EMC-like solution is applicable for SPNs with ME distributed firing times whose density is *strictly positive* in $(0, \infty)$.

[*]Informatik IV, TU Dortmund,D-44221 Dortmund, Germany
[†]Dipartimento di Informatica, Universita di Torino, I-10149 Torino, Italy
[‡]Department of Telecommunications, Technical University of Budapest, H-1521 Budapest, Hungary

In this paper we extend the result of [6] for the case of ME distributed firing times whose density might be zero in $(0, \infty)$. The importance of this extension comes from the fact that the density of important and practically convenient ME distributions is zero in some points in $(0, \infty)$. Some of the most important examples of these ME distributions are the ME distributions with very low coefficient of variation as it is detailed below.

The methodology applied in this paper is different from the one used in [6]. The proof of [6] is based on the fact that any ME distribution with strictly positive density on $(0, \infty)$ can be represented as a PH distribution with a potentially larger vector-matrix pair. This approach is not applicable for ME distributed firing time whose density might be zero on $(0, \infty)$, since these ME distributions cannot be represented as a PH distribution with finite dimension [3]. Instead, we provide a proof of the applicability of EMC-like solution based on the interpretation of ME distributions provided by Bladt and Neuts in [5]. In [5] the authors provide an interpretation of the stochastic process defining a ME distribution through some deterministic flow among infinite containers.

Other methodologies might also be used to prove the main theorem. For example, one might start from the Laplace domain description of non-Markovian SPN [7] using the fact that ME distributions are rational functions in transform domain, or from the supplementary variable description [9] using the matrix exponential description of the firing time distribution. We found that the proof based on the flow interpretation of ME distributions is the simplest.

The rest of the paper is organised as follows. Section 2 presents the basic definitions of ME distributions and some important results about their representations. Examples of ME distributions with low coefficient of variation are reported in Section 3. Section 4 defines SPNs with ME distributed firing time and gives the necessary elements for their analysis. In Section 5 we provide examples to show possible applications of the approach. Finally, Section 6 concludes the paper.

2 Matrix Exponential Distributions

2.1 Basic Definitions and Notations

We quickly recall the basic definition of ME [5] and PH [12] distributions for completeness.

Definition 1. *Let X be a random variable with cumulative distribution function (cdf)*

$$F_X(x) = Pr(X < x) = 1 - \alpha e^{\mathbf{A}x} \mathbb{1},$$

where α is an initial row vector of size n, $\mathbf{A}$ is a square matrix of size $n \times n$ and $\mathbb{1}$ is the column vector of ones of size n. In this case, we say that X is matrix exponentially distributed with representation $\alpha, \mathbf{A}$, or shortly, ME($\alpha, \mathbf{A}$) distributed.

Definition 2. *If X is an ME($\alpha, \mathbf{A}$) distributed random variable, where α and $\mathbf{A}$ have the following properties:*

- $\alpha_i \geq 0$, $\alpha \mathbb{1} = 1$ *(there is no probability mass at $t = 0$),*
- $A_{ii} < 0$, $A_{ij} \geq 0$ *for $i \neq j$,* $\mathbf{A}\mathbb{1} \leq 0$,
- $\mathbf{A}$ *is non-singular,*

then we say that X is phase type distributed with representation $\alpha, \mathbf{A}$, or shortly, PH($\alpha, \mathbf{A}$) distributed.

The probability density function (pdf), the Laplace transform and the moments of X are

$$f_X(x) = \alpha e^{\mathbf{A}x} a, \tag{1}$$

$$f_X^*(s) = E(e^{-sX}) = \alpha (s\mathbf{I} - \mathbf{A})^{-1} a, \tag{2}$$

$$\mu_n = E(X^n) = n! \alpha (-\mathbf{A})^{-n} \mathbb{1}, \tag{3}$$

where $a = -\mathbf{A}\mathbb{1}$.

Order	min_scv	1/min_scv	min_scv	1/min_scv
	real poles		real and complex poles	
3	0.276583	3.61556	0.200902	4.97756
4	0.19333	5.17251	0.149808	6.6752
5	0.138453	7.22266	0.0812643	12.3055
6	0.108623	9.20619		
7	0.0861277	11.6107	0.04288	23.3209
8	0.0717026	13.9465		
9	0.0600486	16.6532	0.0261569	38.2309
10	0.0518365	19.2914		
11	0.0449173	22.2632	0.017494	57.1625
12	0.0397335	25.1677		
13	0.0352403	28.3766	0.0124696	80.1951
14	0.031726	31.5199		
15	0.0286172	34.944	0.00931281	107.379

Table 1: Minimal squared coefficient of variation (min_scv) of ME distributions with quadratic polynomials

2.2 A convenient subclass of ME distributions

One of the main problems of working with ME distributions is that the monotone increasing property of $F_X(x)$ (or the non-negativity of $f_X(x)$) is hard to check. Although there are special subclasses of ME distributions whose construction ensures that the associated PDF is non-negative.

Definition 3. *The set of ME distributions with probability density function* $f(t) = \frac{a(t)}{\int_0^\infty a(t)dt}$ *where* $a(t) = \sum_i (r_i^2(t) + ts_i^2(t))e^{-\lambda_i t} + \sum_i (q_i^2(t) + tw_i^2(t))e^{-\mu_i t}\cos^2(\omega_i t + \phi_i)$*, and* $r_i(t), s_i(t), q_i(t), w_i(t)$ *are arbitrary finite polynomials of* t *and* $\lambda_i, \mu_i, \omega_i, \phi_i$ *are positive real numbers, is referred to as ME distributions with quadratic polynomials.*

First of all, ME distributions with quadratic polynomials have the nice property that their structure (quadratic polynomials) ensures the non-negativity of $f(t)$. Additionally, as it is discussed in [8] some important extreme ME distributions belong this class. For example numerical investigations suggests that the order n ($n \geq 1$) ME distributions with real eigenvalues and minimal coefficient of variation and the order $2k+1$ ($k \geq 1$) ME distributions with real and complex eigenvalues and minimal coefficient of variation belong to the class of ME distributions with quadratic polynomials. Based on [8] Table 1 lists the minimal scv of ME distributions with quadratic polynomials of different order. To the best of the authors knowledge the values presented in Table 1 represent the minimal squared coefficient of variation (scv) of the whole ME class (with or without complex eigenvalues) of the given order, but there is no proof or counter example (an ME with lower scv) is available up to now which verify or destroy this conjecture. To avoid invalid statements below we are going to talk about ME distribution with low scv, but we think that it can be read as ME distribution with minimal scv.

PH distributions of order n are known to have squared coefficient of variation greater or equal to $1/n$. Consequently, the 1/min_scv parameters in Table 1 indicate the minimal size of a PH distribution to approximate such a low coefficient of variation. This property of the class of ME distributions with quadratic polynomials makes their use very efficient for approximating distributions with low coefficient of variation.

2.3 Interpretation of matrix exponential distributions via flows

In [5] the authors provide a stochastic interpretation of matrix exponential distributions via flows. This interpretation is the following for ME($\alpha, \mathbf{A}$) of size n with $a = -\mathbf{A}\mathbb{1}$. Consider n doubly infinite containers of liquid whose initial contents are $\alpha_1, ..., \alpha_n$ and an additional container whose content is zero initially. Assume that liquids flow from container i to container j, with $1 \leq i, j \leq n, i \neq j$, at constant rate given by $\mathbf{A}(i, j)$. That means that if the ith container has c amount of liquid at time u then $c\mathbf{A}(i, j)du$ amount of liquid flows from the ith to jth container in the interval $[u, u + du]$. Further, from container $i, 1 \leq i \leq n$, liquid flows toward the $n + 1$th container at constant rate given by the ith entry of a.

Let us denote by $v_i(u), 1 \leq i \leq n+1$ the level of liquid in container i at time u. As shown in [5], the vector $v(u) = |v_1(u), ..., v_n(u)|$ refering to the first n containers follows the set of ordinary differential equations (ODE)

$$\frac{dv(u)}{dt} = v(u)\mathbf{A}$$

with initial condition $v(0) = \alpha$. The solution of these ODEs is $v(u) = \alpha \exp(\mathbf{A}u)$. Then it is easy to see that the following relations hold between the levels of the liquids in the containers and a random variable X distributed according to ME($\alpha, \mathbf{A}$):

$$1 - F_X(u) = Pr(X > u) = \sum_{i=1}^{n} v_i(u) = 1 - v_{n+1}(u),$$

i.e., the total amount of liquid present in containers $v_1, ..., v_n$ at time u corresponds to the probability that X is greater then u, and

$$f_X(u) = v(u)a,$$

i.e., the pdf of X can be connected to the level of the liquids through the vector a and we will refer to the quantity $v(u)a$ as the firing potential. It follows that the aging of a ME distributed random variable can be captured by the real valued vector $v(u)$.

3 Examples of ME distributions with low coefficient of variation

3.1 Order 3 ME distributions with complex eigenvalues

First we consider the order 3 ME structure with minimal scv reported in [8]. It is

$$\begin{aligned} f(t) &= ue^{-at} 2\cos^2\left(\tfrac{\omega t+\phi}{2}\right) = ue^{-at}(1+\cos(\omega t+\phi)) = \\ &= e^{-at}\Big(u + u\cos(\phi)\cos(\omega t) - u\sin(\phi)\sin(\omega t)\Big) \end{aligned}$$

where from $\int f(t)dt = 1$

$$u = \frac{a(a^2+\omega^2)}{a^2+\omega^2+a^2\cos(\phi) - a\omega\sin(\phi)}.$$

On the other hand we have the following real matrix representation

$$\begin{aligned} f'(t) &= \alpha e^{\mathbf{A}x}(-\mathbf{A})\mathbb{I} = (g, c+d, c-d) \ \ \exp\left[\begin{pmatrix} -a & 0 & 0 \\ 0 & -a & -\omega \\ 0 & \omega & -a \end{pmatrix} t\right] \begin{pmatrix} a \\ a+\omega \\ a-\omega \end{pmatrix} = \\ &= (g, c-id, c+id) \ \ \exp\left[\begin{pmatrix} -a & 0 & 0 \\ 0 & -a-i\omega & 0 \\ 0 & 0 & -a+i\omega \end{pmatrix} t\right] \begin{pmatrix} a \\ a+i\omega \\ a-i\omega \end{pmatrix} = \\ &= age^{-at} + (c-id)(a+i\omega)e^{-at}e^{-i\omega t} + (c+id)(a-i\omega)e^{-at}e^{i\omega t} = \\ &= e^{-at}\Big(ag + 2(ac+\omega d)\cos(\omega t) - 2(ad-\omega c)\sin(\omega t)\Big). \end{aligned}$$

From the first matrix representation to the second one a similarity transformation is applied with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+i)/2 & (1-i)/2 \\ 0 & (1-i)/2 & (1+i)/2 \end{pmatrix}.$$

Having a, ω and ϕ fixed, from $f(t) \equiv f'(t)$ we have

$$ag = u, \quad 2(ac+\omega d) = u\cos(\phi), \quad 2(ad-\omega c) = u\sin(\phi),$$

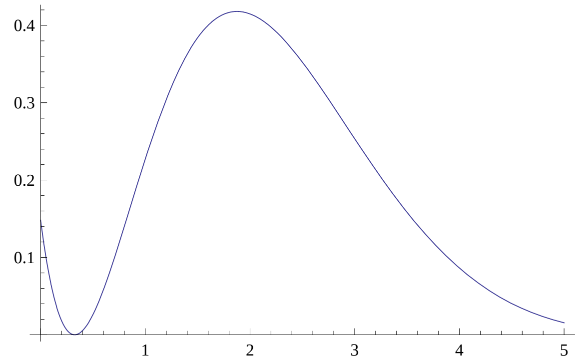

Figure 1: Probability density function of ME3 with $a = 1$, $\phi = -3.47863$, $\omega = 1.03593$

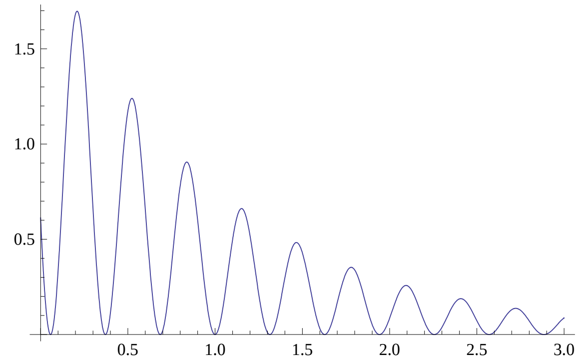

Figure 2: Probability density function of ME3 with $a = 1$, $\phi = 2$, $\omega = 20$

from which

$$g = \frac{u}{a}, \quad c = \frac{a^2\cos(\phi) - a\omega\sin(\phi)}{2(a^2+\omega^2+a^2\cos(\phi)-a\omega\sin(\phi))}, \quad d = \frac{a\omega\cos(\phi) + a^2\sin(\phi)}{2(a^2+\omega^2+a^2\cos(\phi)-a\omega\sin(\phi))}.$$

With $a = 1$, $\phi = -3.47863$, and $\omega = 1.03593$ the minimal scv of this structure is obtained and it is $0.200902 \sim 1/5$ [8]. The pdf of this distribution is depicted in Figure 1, and to indicate the flexibility of this class of distributions Figure 2 depicts the pdf obtained at $a = 1$, $\phi = 2$, and $\omega = 20$.

3.2 Order 3 ME distributions with real eigenvalues

The order 3 ME structure with real eigenvalues and with minimal scv is [8]

$$f(t) \;=\; e^{-at}\Big((w_1t+w_0)^2+v_0^2t\Big) = e^{-at}\Big(w_1^2t^2+(w_1w_0+v_0^2)t+w_0^2\Big)$$

where $w_1 = (-aw_0 - \sqrt{2a^3-2av_0^2-a^2w_0^2})/2$ ensures $\int f(t)dt = 1$.

Starting from an Erlang type matrix representation we have

$$\begin{aligned} f'(t) &= \alpha e^{\mathbf{A}x}(-\mathbf{A})\mathbb{1} = (x_1,x_2,x_3) \;\; \exp\left[\begin{pmatrix} -a & a & 0 \\ 0 & -a & a \\ 0 & 0 & -a \end{pmatrix} t\right]\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \\ &= e^{-at}\left(\frac{a^3t^2x_1}{2!}+\frac{a^2tx_2}{1!}+\frac{a^1x_3}{0!}\right). \end{aligned}$$

From the identity of the coefficients of t^i we have

$$x_1 = \frac{2w_1^2}{a^3}, \quad x_2 = \frac{w_1w_0+v_0^2}{a^2}, \quad x_3 = \frac{w_0^2}{a}.$$

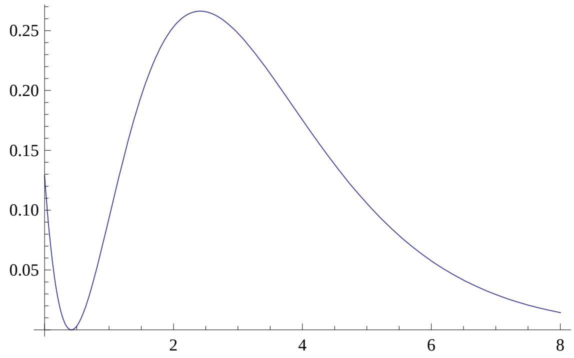

Figure 3: Probability density function of ME3 with real poles and $w_0 = 0.358998$

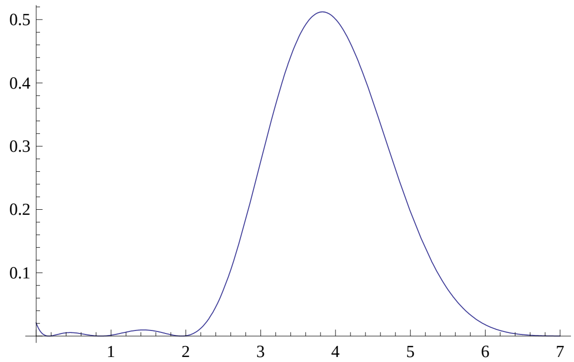

Figure 4: Probability density function of the order 7 ME with scv=0.04288

The minimal scv with real poles is obtained at $w_0 = 0.358998$ and it is 0.276583. The pdf of this distribution is depicted in Figure 3.

3.3 Higher order ME distributions with complex eigenvalues

3.3.1 $n = 7$

Let the density of a ME distribution be

$$f(t) \quad = \quad ue^{-at} \prod_{i=1}^{k} 2\cos^2\left(\frac{\omega t + \phi_i}{2}\right) = ue^{-at} \prod_{i=1}^{k} (1 + \cos(\omega t + \phi_i)).$$

If the order is $n = 2k + 1 = 7$, $a = 1$, $\omega = 0.884919$ $\phi_1 = 3.29263, \phi_2 = 3.90442, \phi_3 = 4.86219$ and u is set such that $\int f(t)dt = 1$ then the squared coefficient of variation of this distribution is $0.04288 < 1/23$ and its pdf is depicted in Figure 4.

The analytical treatment of this case is rather cumbersome, but it can be avoided by a numerical approach to obtain the associated matrix representation. The moments of the distribution can be computed from $m_i = \int t^i f(t)dt$ for $i = 1, \ldots, n$, Based on these moments a matrix representation of f(t) can be obtained in a two steps numerical method. In the first step we generate matrix $\mathbf{A}$ such that it exhibits the same

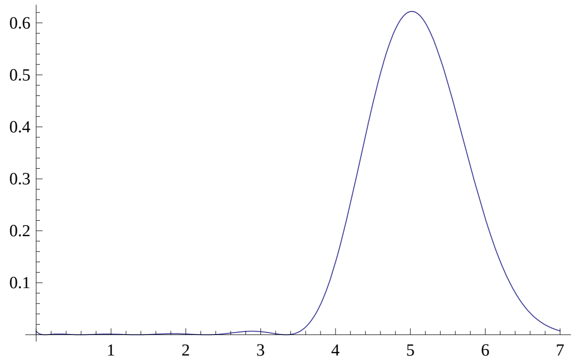

Figure 5: Probability density function of the order 11 ME with scv=0.017494

exponential coefficients, i.e., eigenvalues, as $f(t)$. We have

$$\mathbf{A} = \begin{pmatrix} -a & & & & & & \\ & -a & -\omega & & & & \\ & \omega & -a & & & & \\ & & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & & -a & -k\omega \\ & & & & & k\omega & -a \end{pmatrix},$$

i.e., the eigenvalues of $\mathbf{A}$ are $-a$ and $\{-a + i\omega, -a - i\omega\}$ for $i = 1, 2, \ldots, k$.

In the seconds step we obtain vector α by solving the linear system $i!\alpha(-\mathbf{A})^{-i}\mathbb{1} = m_i$ for $i = 0, \ldots, n-1$.

We applied this numerical procedure for $n = 7$ ($k = 3$) and obtained the ME distribution with matrix representation $(\alpha, \mathbf{A})$ where,

$$\alpha = \{9.77827, -4.71506, -6.56599, 0.172583, 2.40449, 0.213925, -0.288218\},$$

and $\mathbf{A}$ is defined by its structure.

3.3.2 $n = 11$

The approach works for $n = 11$ ($k = 5$). With $a = 1$, $\phi_1 = 3.61267$, $\phi_2 = 3.23369$, $\phi_3 = 4.21678$, $\phi_4 = 4.96537$, $\phi_5 = 5.81516$, $\omega = 0.800546$, and u set such that $\int f(t)dt = 1$, the scv of this distribution is $0.017494 < 1/57$; its pdf is depicted in Figure 5.

The initial vector of its matrix representation is

$$\alpha = \{25.312, -4.86661, -23.9613, -7.24198, 8.29355, 3.77278, 0.862056, \\ -0.214874, -0.938152, -0.0976547, 0.0802795\}.$$

3.3.3 $n = 15$

The same approach works also for $n = 15$ ($k = 7$). With $a = 1$, $\phi_1 = 3.47395$, $\phi_2 = 4.45106$, $\phi_3 = 3.20593$, $\phi_4 = 3.90857$, $\phi_5 = 5.06454$, $\phi_6 = 5.73043$, $\phi_7 = 6.4419$, $\omega = 0.745921$, and u set such that $\int f(t)dt = 1$, the scv of this distribution is $0.00931281 < 1/107$; its pdf is depicted in Figure 6.

The initial vector of its matrix representation is

$$\alpha = \{53.1771, 5.41155, -55.1427, -28.5444, 8.56841, 7.01384, 12.2088, 4.40013, -3.67725, \\ -1.432, -1.22016, -0.215864, 0.384661, 0.0534409, 0.0144613\}.$$

4 Stochastic Petri Nets with ME Distributed Firing Times

In this section we introduce stochastic Petri nets in which the firing times of the transitions are ME distributed. We consider first Petri nets and their reachability graph. Afterwards, the reachability graph is expanded by considering detailed state information to describe the age of the enabled transitions.

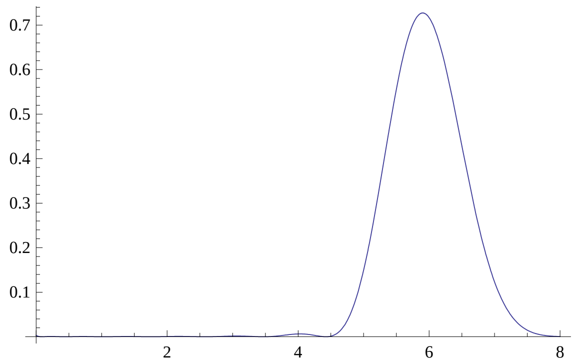

Figure 6: Probability density function of the order 15 ME with scv=0.00931281

4.1 Basic Notations and Definitions

We briefly present some basic definitions and results for Petri nets; further results and details can be found in the literature [11]. The introduction here follows [6].

Definition 4. *A Petri net is a five tuple* $PN = (P, T, I, O, M_0)$ *where*

- P *is a set of places,*
- T *is a set of transitions such that* $P \cap T = \emptyset$*,*
- $I : P \times T \to \mathbb{N}$ *is the input function,*
- $O : T \times P \to \mathbb{N}$ *is the output function, and*
- $M_0 : P \to \mathbb{N}$ *is the initial marking.*

We assume an ordering the set of transitions such that for $t, t' \in T$ with $t \neq t'$ either $t < t'$ or $t > t'$ holds. Denote by $\bullet t = \{p | p \in P \wedge I(p,t) > 0\}$ and $t\bullet = \{p | p \in P \wedge O(t,p) > 0\}$ the input and output bag of transition t, respectively. A marking M is a vector of length $|P|$ whose elements represent the token population of each place. $M(p)$ denotes the p-th element of this vector. Marking M_0 defines the initial token population. Transition t is enabled in marking M if and only if $M(p) \geq I(p,t)$ for all $p \in \bullet t$. If t is enabled in marking M and fires, then a new marking M' with $M'(p) = M(p) - I(p,t) + O(t,p)$ is generated. For this event we use the notation $M \xrightarrow{t} M'$. We assume that $M \xrightarrow{t} M'$ implies $M \neq M'$. However, the extension to $M \xrightarrow{t} M'$ is straightforward but requires a more complicate notation. The set of markings available from M_0 with repeated application of relation $\xrightarrow{t}$ defines the reachability set $\mathcal{RS}$ of the Petri net. The reachability graph $\mathcal{RG}$ is a directed and labeled graph with vertex set $\mathcal{RS}$ and an arc labeled with t between $M, M' \in \mathcal{RS}$ if and only if $M \xrightarrow{t} M'$. Further assumptions about $\mathcal{RS}$ and $\mathcal{RG}$, like finiteness or strong connectivity will be made later when necessary.

Let $Ena(M) = \{t | t \in T \text{ and for all } p \in P : \ M(p) \geq I(p,t)\}$ be the set of enabled transitions in marking M. The concept underlying our definition of newly enabled transitions is denoted as enabling memory in [1]. The general approach is applicable for age memory policy as well. Only the structure of the state descriptor and the definition of the resetting or maintaining the memory in (5) has to be modified in that case. Furthermore, we assume single server semantics for all transitions.

4.2 Flow interpretation of SPN with ME distributed firing times

Hereinafter we show that the behaviour of a PN with ME timings can be described through the behaviour of the levels of the liquids associated with the ME distributed transitions of the net. This is done by associating each marking M with a vector $v(u, M)$ providing at time u the joint state (i.e., the joint liquid levels) of the ME distributions of the transitions that are enabled in marking M. We denote by n_t, α_t, $\mathbf{A}_t$ and $a_t = (-\mathbf{A}_t)\mathbb{1}$ the size, the initial vector, the generator and the closing vector of the ME distribution associated with transition t. Using these notations we can present the main theorem of the paper.

Theorem 1. *$v(u,M)$ satisfies the vector differential equation*

$$\frac{dv(u,M)}{du} = v(u,M) \bigoplus_{t\in Ena(M)} \mathbf{A}_t + \sum_{t:M'\xrightarrow{t}M} v(u,M') \bigotimes_{t'\in T} \mathbf{R}_{t',t}(M',M) \tag{4}$$

where

$$\mathbf{R}_{t',t}(M',M) = \begin{cases} \mathbf{I}_{n_{t'},n_{t'}} & \textit{if } t' \neq t \textit{ and } t' \in Ena(M') \cap Ena(M), \\ \alpha_{t'} & \textit{if } t' \neq t,\ t' \notin Ena(M') \textit{ and } t' \in Ena(M), \\ \mathbb{1}_{n_{t'}} & \textit{if } t' \neq t,\ t' \in Ena(M') \textit{ and } t' \notin Ena(M), \\ a_t & \textit{if } t' = t,\ \textit{ and } t \notin Ena(M), \\ a_t\alpha_t & \textit{if } t' = t \textit{ and } t \in Ena(M), \\ 1 & \textit{otherwise,} \end{cases} \tag{5}$$

with initial condition

$$v(0,M_0) = \bigotimes_{t\in Ena(M_0)} \alpha_t \quad \textit{and} \quad v(0,M) = 0 \ \textit{ for } \ \forall M \neq M_0.$$

Proof. To prove the theorem we present the scalar equations governing the system behaviour. Unfortunately, it requires the introduction of complex notations referring to the elements of complex vectors and matrices. We denote by K_M the number of active ME distributions in marking M, and by $n_{M,i}, \alpha_{M,i}$, $\mathbf{A}_{M,i}$ and $a_{M,i} = (-\mathbf{A}_{M,i})\mathbb{1}$ the size and the descriptors of the ith active ME distribution in marking M. The entries of $\alpha_{M,i}$, $\mathbf{A}_{M,i}$ and $a_{M,i}$, in order to avoid heavy subscripting, will be indicated in parenthesis, i.e., for example, the jth entry of $\alpha_{M,i}$ as $\alpha_{M,i}(j)$ and the entry (j,k) of $\mathbf{A}_{M,i}$ as $\mathbf{A}_{M,i}(j,k)$. The index of transition t in marking M will be denoted by $p_{M,t}$, i.e., if the ith active ME distribution in marking M is t then $p_{M,t} = i$.

The vector $v(u,M)$ is of length $\prod_{t\in Ena(M)} n_t$ and its entries are organised according to a lexicographical order which is also referred to as the mixed-base scheme. The lexicographical order is naturally generated by the Kronecker product operation of the vectors representing the level of the containers associated with the active transitions in marking M at time u. For the elements of the vector the lexicographical order means that, having a vector of indices $l = |l_1, l_2, ..., l_{K_M}|$ with $1 \leq l_i \leq n_{M,i}, 1 \leq i \leq K_M$ identifying a given container for each enabled transition of marking M, the entry of $v(u,M)$ that describes the joint state of these containers is in position $(...((l_1-1)n_{M,2}+l_2-1)n_{M,3}...)n_{M,K_M}+l_{K_M}-1 = \sum_{k=1}^{K_M}(l_k-1)\prod_{i=k+1}^{K_M} n_{M,k}$ (where, for simplicity of notation, empty product equals to 1, from which $\prod_{i=K_M+1}^{K_M} = 1$ follows). A given entry of the vectors $v(u,M)$ will be a container itself and the vectors $v(u,M)$ provide the expanded state space of the containers of the individual transitions. The entry of $v(u,M)$ corresponding to the vector of containers $|l_1, l_2, ..., l_{K_M}|$ will be denoted by $v(u,M,|l_1,...,l_{K_M}|)$.

The entries of $v(u,M)$ will be such that the level of the jth container of the ith enabled transition of marking M can be recovered by the sum

$$v(u,M,i,j) = \sum_{l_1=1}^{n_{M,1}} \sum_{l_2=1}^{n_{M,2}} \cdots \sum_{l_{i-1}=1}^{n_{M,i-1}} \sum_{l_{i+1}=1}^{n_{M,i+1}} \cdots \sum_{l_{K_M}=1}^{n_{M,K_M}} v(u,M,|l_1,l_2,..,l_{i-1},j,l_{i+1},...,l_{K_M}|)\ . \tag{6}$$

Further, the probability of marking M at time u will be given by the total amount of liquid present in $v(u,M)$ as

$$\pi(u,M) = \sum_{l_1=1}^{n_{M,1}} \cdots \sum_{l_{K_M}=1}^{n_{M,K_M}} v(u,M,|l_1,...,l_{K_M}|)\ . \tag{7}$$

The initial condition for the vectors $v(u,M)$ are given as

$$v(0,M_0) = \bigotimes_{t\in Ena(M_0)} \alpha_t \quad \text{and} \quad \forall M \neq M_0 : v(0,M) = \underline{0}. \tag{8}$$

with which it is easy to see that

$$v(0,M_0,i,j) = \alpha_{M_0,i}(j),\ \forall i,j : 1 \leq i \leq K_{M_0}, 1 \leq j \leq n_{M_0,i},$$
$$v(0,M,i,j) = 0,\ \forall M,i,j : M \neq M_0, 1 \leq i \leq K_M, 1 \leq j \leq n_{M,i},$$

i.e., (8) provides correct initial setting of the levels of the liquids.

In order to describe correctly the evolution of the PN, the evolution of $v(u, M)$, $\forall M$ and $u \geq 0$, has to be such that the level of the jth container of the ith enabled transition of M given by $v(u, M, i, j)$, $1 \leq i \leq K_M, 1 \leq j \leq n_{M,i}$, and computed according to (6) satisfies the following conditions.

1. The level $v(u, M, i, j)$ is decreased at rate $\mathbf{A}_{M,i}(j, j)$.
2. There is an exchange of liquids from containers $v(u, M, i, k), k \neq j$, to container $v(u, M, i, j)$ with rate $\mathbf{A}_{M,i}(k, j)$.
3. For $\forall t, t' : M' \xrightarrow{t} M, t' \neq t$
   - the firing potential of t at time u has to be equal to $\sum_{j=1}^{n_t} v(u, M', p_{M',t}, j)a_t(j)$ and, accordingly, in the interval $[u, u + du]$ the amount of liquid flowing from the containers of $v(u, M')$ to the containers of $v(u, M)$ is $du \sum_{j=1}^{n_t} v(u, M', p_{M',t}, j)a_t(j)$;
   - if $t \in Ena(M)$ then the liquid flowing from the containers of $v(u, M')$ to the containers of $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t}, i), 1 \leq i \leq n_t$ according to α_t;
   - if $t' \in Ena(M')$ and $t' \in Ena(M)$ then the liquid flowing from $v(u, M')$ to $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t'}, i), 1 \leq i \leq n_{t'}$ as it was distributed among the levels $v(u, M', p_{M',t'}, i), 1 \leq i \leq n_{t'}$, i.e., the state (age) of t' has to be maintained;
   - if $t' \notin Ena(M')$ and $t' \in Ena(M)$ then the liquid flowing from $v(u, M')$ to $v(u, M)$ has to be distributed among the levels $v(u, M, p_{M,t'}, i), 1 \leq i \leq n_{t'}$ according to $\alpha_{t'}$, i.e., the state of t' has to be initialised;
   - if $t' \notin Ena(M)$ then t' has no impact on the flow from $v(u, M')$ to $v(u, M)$.

In the following we provide a set of ODEs which describes the evolution of each container of $v(u, M)$ for every marking M of the PN. The fact that these ODEs satisfy the conditions listed above can be seen by straightforward but cumbersome algebraic steps based on the summation provided in (6).

For a vector $l = |l_1, l_2, ..., l_{K_M}|$ of marking M and an index $i, 1 \leq i \leq K_M$, we will denote by $f(M, l, i)$ the set of vectors which differs from l at most in position i, i.e., $f(M, l, i) = \{|l_1, ..., l_{i-1}, k, l_{i+1}, ..., l_{K_M}| : 1 \leq k \leq n_{M,i}, k \neq l_i\}$; note that $l \notin f(M, l, i)$.

Within a given marking M, liquids flow to the container $|l_1, ..., l_{K_M}|$ from another container $|k_1, ..., k_{K_M}|$ of marking M if $\exists i : k \in f(M, l, i)$ and at rate $\mathbf{A}_{M,i}(k_i, l_i)$. Liquid flows away instead from container $|l_1, ..., l_{K_M}|$ of marking M at rate $-\sum_{i=1}^{K_M} \mathbf{A}_{M,i}(l_i, l_i)$.

The containers of other markings, $M' \neq M$, from which liquids flow toward container $l = |l_1, ..., l_{K_M}|$ of marking M can be identified as follows. From a container $k = |k_1, ..., k_{K_{M'}}|$ of marking M' fluid flows to $|l_1, ..., l_{K_M}|$ of marking M if the following conditions hold

1. $\exists t : M' \xrightarrow{t} M$;
2. $\forall t' \neq t$: if $t' \in Ena(M') \cap Ena(M)$ then we must have $l_{p_{M,t'}} = k_{p_{M',t'}}$;
3. $\forall t' \neq t$: if $t' \notin Ena(M')$ and $t' \in Ena(M)$ then we must have $\alpha_{t'}(l_{p_{M,t'}}) \neq 0$;
4. if $t \notin Ena(M)$ then we must have $a_t(k_{M',t}) \neq 0$;
5. if $t \in Ena(M)$ then we have must $a_t(k_{M',t}) \neq 0$ and $\alpha_t(l_{M,t}) \neq 0$.

Condition (1) simply states that there must be a transition that takes the system from marking M' to marking M. As described by condition (2), if t' is not the transition that fires and it is enabled both in M and M' then the liquid describing the state of the ME distribution of transition t' flows from a container of M' to the corresponding one in M in such a way that the age (state) of the transition is maintained. The remaining three conditions have an effect also on the rate at which liquid flows from $|l_1, ..., l_{K_M}|$ of marking M to $|k_1, ..., k_{K_{M'}}|$ of marking M'. In particular, as condition (3) states it, if t' is not enabled in M' but it is enabled in M then liquid flows only toward those containers of M that corresponds to local containers of t' that have to be initialised to a non-zero level. The effect of t' on the rate of the flow is given by $\alpha_{t'}(l_{p_{M,t'}})$. If transition t is not enabled in marking M then it contributes to the flow, according to condition (4), only if its local container has a flow toward its fictitious container representing the termination of the activity associated with t. The associated rate is $a_t(k_{M',t})$. Condition (5) states that, if transition t is enabled in M then it contributes to the flow if its local container $k_{M',t}$ has a flow toward its fictitious container and its local container $l_{M,t}$ is with non-zero initial liquid level. The associated rate is $a_t(k_{M',t})\alpha_t(l_{M,t}) \neq 0$.

We will denote by $g(M,l)$ the set of couples $(M',k), M' \neq M$ for which there is a flow from container $k = |k_1, ..., k_{K_{M'}}|$ of marking M' to container $l = |l_1, ..., l_{K_M}|$ of marking M.

Based on the above description we can write the change of the level of the liquid present in container $l = |l_1, ..., l_{K_M}|$ of marking M as

$$
\begin{aligned}
\frac{v(u,M,|l_1,...,l_{K_M}|)}{du} = v(u,M,|l_1,...,l_{K_M}|)\sum_{i=1}^{K_M}\mathbf{A}_{M,i}(l_i,l_i) + \\
\sum_{i=1}^{K_M}\sum_{k\in f(M,l,i)} v(u,M,|k_1,...,k_{K_M}|)\mathbf{A}_{M,i}(k_i,l_i) + \\
\sum_{(M',k)\in g(M,l),M'\xrightarrow{t}M}\Bigg(v(u,M',|k_1,...,k_{K_{M'}}|)\cdot a_t(k_{M',t})\cdot(\alpha_t(l_{M,t}))_{t\in Ena(M)}\cdot \\
\prod_{t'\neq t,t'\notin Ena(M'),t'\in Ena(M)}\alpha_{t'}(l_{p_{M,t'}})\Bigg)
\end{aligned}
\tag{9}
$$

where $(\alpha_t(l_{M,t}))_{t\in Ena(M)}$ gives $\alpha_t(l_{M,t})$ if $t \in Ena(M)$ and 1 otherwise.

In matrix notation (9) can be written as (4), which completes the proof. □

A positive consequence of Theorem 1 is that the transient behaviour of a PN with ME distributed firing times can be analysed based on similar differential equations as in case of Markovian PN models, but due to the more general structure of ME distributions the constant coefficients of the differential equation do not obey sign restrictions.

4.3 Stationary behaviour

The stationary solution of the PN is obtained by solving the balance equations. Let $w(M) = \lim_{u\to\infty} v(u,M)$.

Theorem 2. *$w(M)$ satisfies following balance equation*

$$
0 = w(M)\bigoplus_{t\in Ena(M)}\mathbf{A}_t + \sum_{t:M'\xrightarrow{t}M} w(M')\bigotimes_{t'\in T}\mathbf{R}_{t',t}(M',M) \tag{10}
$$

where $\mathbf{R}_{t',t}(M',M)$ is defined in (5).

Proof. The balance equation is obtained from the transient differential equation by taking the $u \to \infty$ limit. □

To compute the stationary marking distribution the set of linear equations, (10), needs to be solved, where the elements of the constant matrices can contain positive and negative elements at any position.

5 Two SPN examples

We now consider two simple example nets with ME distributed firing times to present some numerical results.

5.1 SPN with synchronised activities

The first net is intended to compare Erlang and ME distributions in modeling low coefficients of variation. The basic net is shown in Fig. 7. We assume that the initial marking is given by putting 10 tokens at the places $p1$ and $p2$.

First, we consider a configuration where transition $t1$ has an exponentially distributed firing time with mean 1.0 and the transitions $t2$ and $t3$ have identically distributed ME or Erlang distributed firing times.

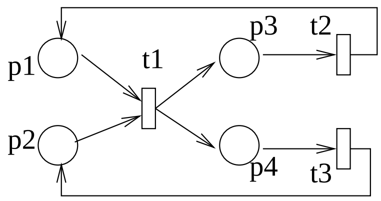

Figure 7: SPN with synchronised activities

No. Phases	states	Erlang non zeros	ME non zeros
3	961	3605	8309
7	5041	19077	82077
11	12321	46741	290821
15	22801	86597	703661

Table 2: Number of states and non-zero elements for different number of phases for the distribution of $t2$ and $t3$.

We apply the ME distributions with 3, 7, 11 and 15 phases which have been defined above and compare them with Erlang distributions with the same number of phases. In all cases we assume that the mean firing time of the distributions is 1 and the coefficient of variation is as low as possible. Table 2 contains the number of states and the number of non-zero elements in the overall generator matrix. The number of states depends only on the number of phases and not on the non-zero structure of the matrices for the distributions. The number of non-zero elements in the resulting generating matrix depends on the number of non-zero elements in the matrix and the initial vector of the distributions. Since the ME distributions have more non-zero elements in its vector and matrix, the generator matrix becomes more dense when using ME instead of Erlang distributions.

As the measure of interest we consider first the token distribution in place $p3$. For deterministically distributed firing times of the transitions $t2$ and $t3$ with the same mean, the model is equivalent to a M/D/1/10 queueing model with mean inter-arrival and mean service time equal to 1. Figures 8 and 9 contain the results for different configurations and include for comparison the results for an M/D/1/10 queue and for the same net with exponentially distributed firing times. It can be clearly seen that with the same number of phases, ME distributions approximate deterministic distributions much better than Erlang distributions do. In particular, the ME distributions with 11 and 15 phases that have a very small coefficient

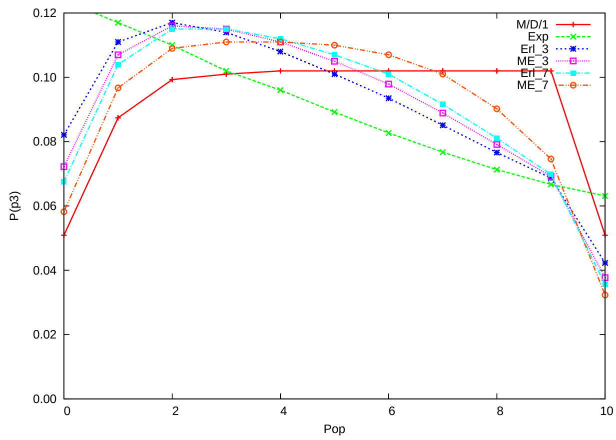

Figure 8: Token distribution at place $p3$ for the Erlang and ME distributions with 3 and 7 phases

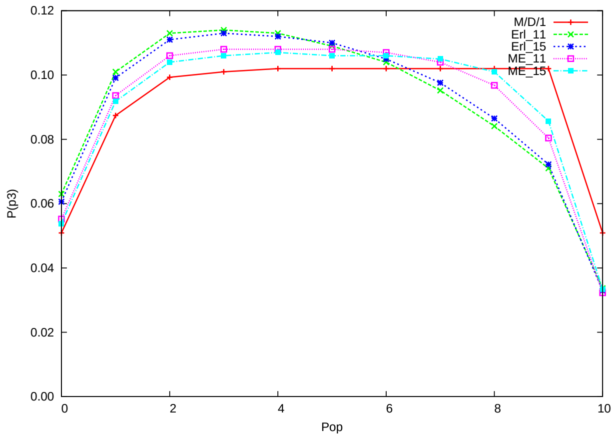

Figure 9: Token distribution at place $p3$ for the Erlang and ME distributions with 11 and 15 phases

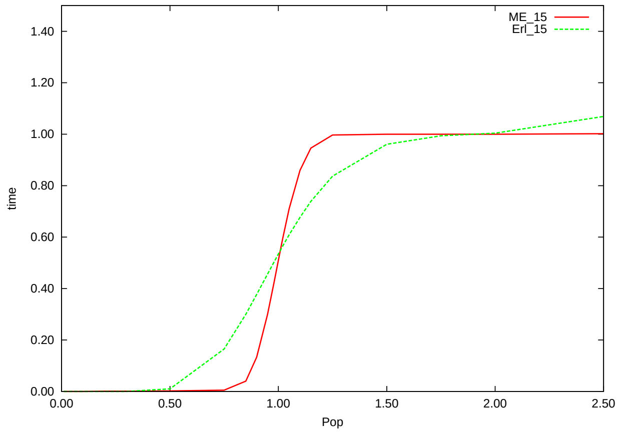

Figure 10: Transient token population in place $p3$

of variation approximate the token distribution of the model with deterministic distributions quite well, only for population 9 and 10 the probabilities are underestimated.

Additionally, we analyse the transient behaviour of the net when all transitions have ME or Erlang distributed firing times with 15 phases and mean 1. Results are shown in Fig. 10. The ME distribution results in a good approximation of a step function which would occur in a deterministic system.

5.2 SPN model of production cells

The second example we consider has been taken from [13]. The net is shown in Fig. 11. The net describes two consecutive production cells with two types of material to be processed. The first type is processed by a machine that is subject to failures and repairs. Failures are modeled by transitions $t10$, $t12$, repair operations by transitions $t9$ and $t11$. For further explanations of the model we refer to [13]. For our analysis we assume that the transitions $t9$, $t10$, $t11$ and $t12$ have an exponentially distributed firing time with rate 1.0 for transitions $t9$ and $t11$ and rate 0.2 for transitions $t10$ and $t12$. The remaining transitions have ME distributed firing times with mean 1.0 for $t1, \ldots, t6$ and mean 0.5 for $t7$ and $t8$.

In Fig. 12 the transient population of place $p2$ (that is the mean number of tokens at p_2) in the interval $[0, 12]$ is shown for ME distributions with 3 and 15 phases. Furthermore, the net has been analysed with exponentially distributed firing times at all transitions. It can be seen that with exponentially distributed firing times for all transitions the population converges quickly towards the steady state, whereas the ME distributions show a cyclic behaviour which is smoothed by the firing of the exponential distributions $t10$ and $t12$, i.e., by failures during the processing step. For the ME distribution with 15 states the transient popula-

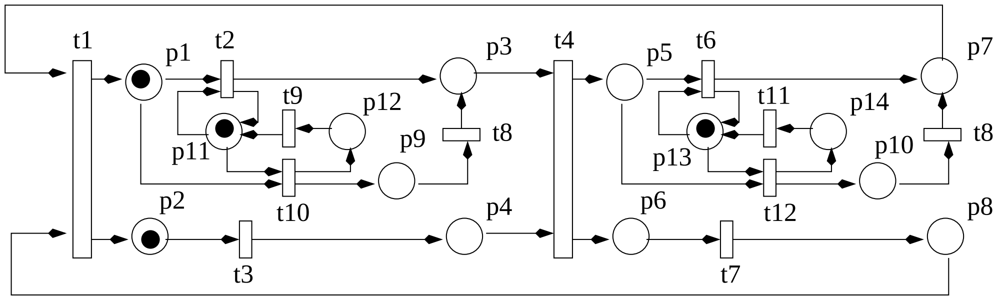

Figure 11: SPN model of production cells

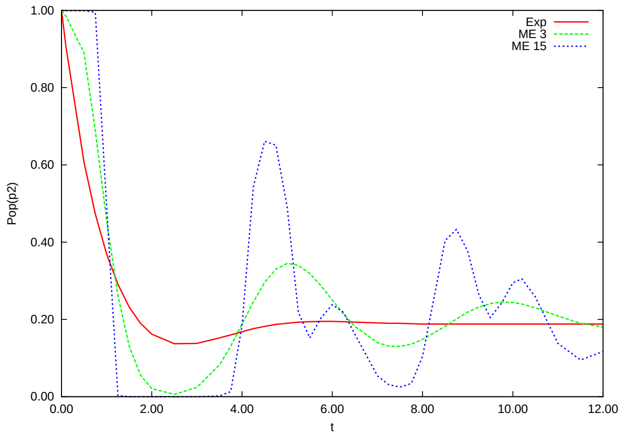

Figure 12: Transient token population in place $p2$

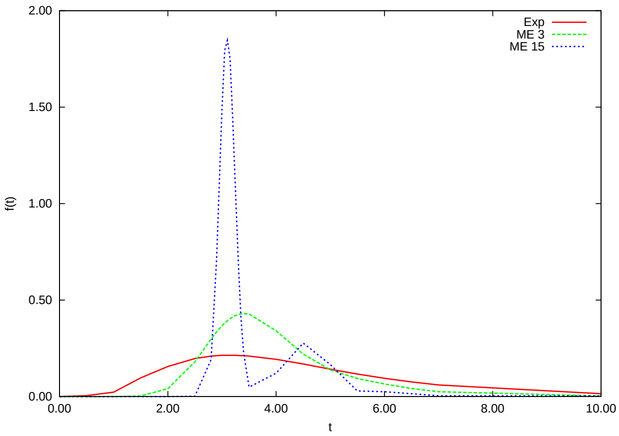

Figure 13: Sojourn time density of a single processing step

tion contains peaks that describe cycles without failures and those with a single failure. (The probability of two failures during a cycle is so small that there is not a corresponding peak in the transient probabilities.) This behaviour is not visible if exponential distributions or with ME distributions with 3 phases are used.

Finally, the density of the sojourn time for a single run of the net is computed. The run starts in the initial marking and ends when both tokens reached the places $p7$ and $p8$. Fig. 13 shows the density function which is smooth for exponentially distributed firing times. For the ME distribution with 15 phases we can again observe the two modes with and without failure.

6 Conclusions and future work

In this paper we discussed a methodology to evaluate the transient and the stationary parameters of stochastic Petri nets with firing times whose coefficient of variation is very low. In particular, we proposed the use of matrix exponential distributions. We presented order 3, 7, 11 and 15 ME distributions with low coefficient of variations. These distributions can be used to represent distributions with low coefficient of variation and still allow for using ordinary differential equations for transient analysis and linear equations for stationary analysis.

We proved the widespread conjecture that a similar extension approach can be applied for SPNs with ME distributed firing time as the one for SPNs with PH distributed firing time and it is independent of the fact if the density function of the ME distributed firing time becomes 0 at some points in $(0, \infty)$.

Numerical examples demonstrate the benefit of using ME distributions with low coefficient of variation instead of Erlang distributions. Much better accuracy/computational complexity ratio is obtained with ME distributions. For example the presented order 15 ME distribution has lower coefficient of variation than the Erlang distribution of order 107.

The stable and efficient numerical analysis of the system of differential equations (for transient analysis) and linear equations (for steady state analysis) of SPNs with ME distributed firing time is still an open research problem. The results presented in the paper are obtained from standard numerical methods (Runge-Kutta, generalised minimal residual) without utilising any structural property of the analysed system.

References

[1] M. Ajmone-Marsan, G. Balbo, A. Bobbio, G. Chiola, G. Conte, and A. Cumani. The effect of execution policies on the semantics and analysis of stochastic Petri nets. *IEEE Transactions on Software Engineering*, 15(7):832–845, 1989.

[2] S. Asmussen and M. Bladt. Point processes with finite-dimensional conditional probabilities. *Stochastic Processes and their Application*, 82:127–142, 1999.

[3] S. Asmussen and C. A. O'Cinneide. Matrix-exponential distributions – distributions with a rational Laplace transform. In S. Kotz and C. Read, editors, *Encyclopedia of Statistical Sciences*, pages 435–440, New York, 1997. John Wiley & Sons.

[4] N.G. Bean and B.F. Nielsen. Quasi-birth-and-death processes with rational arrival process components. *Stochastic Models*, 26(3):309–334, 2010.

[5] M. Bladt and M. F. Neuts. Matrix-exponential distributions: Calculus and interpretations via flows. *Stochastic Models*, 19(1):113–124, 2003.

[6] P. Buchholz and M. Telek. Stochastic petri nets with matrix exponentially distributed firing times. *Performance Evaluation*, 67:1373 – 1385, 2010.

[7] Hoon Choi, V.G. Kulkarni, and K. Trivedi. Markov regenerative stochastic Petri nets. *Performance Evaluation*, 20:337–357, 1994.

[8] T. Éltető, S. Rácz, and M. Telek. Minimal coefficient of variation of matrix exponential distributions. In *2nd Madrid Conference on Queueing Theory*, Madrid, Spain, July 2006. abstract.

[9] R. German. Markov Regenerative Stochastic Petri Nets with general execution policies: Supplementary variable analysis and a prototype tool. In *Computer Performance Evaluation*, volume 1469. Springer, LNCS, 1998.

[10] L. Lipsky. *Queueing Theory: A Linear Algebraic Approach*. Springer, 2008.

[11] T. Murata. Petri nets: properties, analysis and applications. *Proc. of the IEEE*, 77:541–580, 1989.

[12] M. F. Neuts. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approachg*. Dover, 1981.

[13] E. Vicario, L. Sassoli, and L. Carnevali. Using stochastic state classes in quantitative evaluation of dense-time reactive systems. *IEEE Trans. Software Eng.*, 35(5):703–719, 2009.