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Fluid Stochastic Petri Nets Augmented with Flush-out Arcs: Modelling and Analysis

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Abstract. Fluid (or Hybrid) Petri Nets are Petri net based models with two classes of places: discrete places that carry a natural number of distinct objects (tokens), and fluid places that hold a positive amount of fluid, represented by a real number. With respect to previous formulations, the *FSPN* model presented in this paper, is augmented with a new primitive, called flush-out arc. A flush-out arc connects a fluid place to a timed transition, and has the effect of instantaneously emptying the fluid place when the transition fires. The paper discusses the modeling power of the augmented formalism, and shows how the dynamics of the underlying stochastic process can be analytically described by a set of integro-differential equations. A procedure is presented to automatically derive the solution equations from the model specifications. The whole methodology is illustrated by means of various examples.

Keywords: stochastic reward models, Petri nets, fluid stochastic Petri nets, performance analysis

1. Introduction

Fluid Stochastic Petri Nets (*FSPN*) or Hybrid Petri Nets (*HPN*) are Petri net based models, in which some places may hold a discrete number of tokens, and some places a continuous quantity represented by a non-negative real number. Places that hold continuous quantities are referred to as *fluid* or continuous places, and the non-negative real number is said to represent the fluid level in the place. Discrete tokens move along discrete arcs with the enabling and firing rules of standard *PN*, while the fluid moves along special continuous (or fluid) arcs according to an assigned instantaneous flow rate.

FSPN (Horton et al., 1998) and *HPN* (Alla and David, 1998) have been introduced in the literature mainly with the aim of providing a feasible approximation to discrete-state systems in which the number of objects to be considered (customers, packets, tasks, workpieces etc.) tends to become exceedingly large to be treated with the usual discrete state approach common to *SPN*.

In the formulation of this paper, we enlarge the above view, by showing that *FSPN* can be profitably employed to model actual continuous physical quantities (like the temperature)

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whose behaviour in time is influenced by a discrete controller. Moreover, we may as well assign to the continuous quantity the physical meaning of the flowing of the time, thus allowing the stochastic marking process associated to the model to have a complex time-dependent behaviour that cannot be mapped into a Continuous Time Markov Chain (*CTMC*) (Gribaudo et al., 1999).

It should, however, be noted that the analogy of the continuous part of the model with a physical fluid may be misleading, and we prefer to interpret the model semantic in a more abstract view (Gribaudo et al., 1999), where the instantaneous flow rate associated to fluid arcs represents an instantaneous reward rate associated to the underlying stochastic process, and the level of the fluid in the continuous places becomes the accumulated reward (Horton, 1996). Hence, *FSPN* can be thought as a graphical language to represent (non-Markovian) stochastic processes with rewards. It is shown that this view, provides a natural environment for the definition of performance measures related to the accumulation of the reward, that were not easily cast in previous models.

Two main research lines have appeared in the literature. The *FSPN* model was first proposed in (Trivedi and Kulkarni, 1993) and further elaborated in (Horton et al., 1998). The analytical solution of the model is presented in (Horton et al., 1998), while a simulative approach has been discussed in (Ciardo et al., 1999). For the *HPN*, a comprehensive review is in (Alla and David, 1998). The *HPN* defines, in addition, also continuous transitions as model primitives. In principle, timing can be either deterministic or stochastic, but extensive applications have been reported only with deterministic timing. Only a simulative approach is available.

The present paper assumes the basic model of *FSPN* (Bobbio et al., 1999; Gribaudo et al., 1999; Horton et al., 1998) by adding a new feature called *flush-out arc*. The flush-out arc connects a timed transition to a fluid place and has the effect of instantaneously emptying the fluid place as the transition fires. It is shown that flush-out arcs considerably increase the modeling power of the previously defined fluid models.

The *FSPN* model proposed in the present paper is introduced in Section 2. The firing rate of the timed discrete transitions, and the flow rate of the continuous arcs are function both of the discrete part of the marking (the number of tokens in the discrete places) as well as of the continuous part (the fluid levels in the continuous places). Flush-out arcs connect fluid places with timed transitions: the extension of allowing flush-out arcs to point to immediate transitions is possible but involves more cumbersome notation and does not enlarge the semantics of the model, and is therefore avoided in this paper. The dynamic behaviour of the marking process can be described analytically: Section 3 considers first the case of a single fluid place and then enlarges the formalism to any number of fluid places.

Section 4 introduces the performance measures that can be naturally specified on a *FSPN* and shows how continuous measures related to the accumulation of the fluid (reward) may become part of the standard specification. In this section we briefly discuss the numerical aspects related with the solution of FSPN models. Section 5 presents an illustrative example. Finally, Section 6 concludes the paper outlining possible future works on this topic.

2. Definitions and Notations

Throughout the paper calligraphic symbols are used to denote sets, boldface symbols to denote vectors and matrices (lowercase for vectors and uppercase for matrices), and uppercase symbols to denote functions. With $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ we denote natural, real, and non negative real numbers, respectively.

The definition of the FSPNs is derived from (Horton et al., 1998) with standard notation inherited from (Ajmone et al., 1995).

A FSPN is a tuple $\langle \mathcal{P}, \mathcal{T}, \mathcal{A}, B, F, W, R, M_0 \rangle$, where:

– $\mathcal{P}$ is the set of places partitioned into a set of discrete places $\mathcal{P}_d = \{p_1, \ldots, p_{|\mathcal{P}_d|}\}$, and a set of continuous places $\mathcal{P}_c = \{c_1, \ldots, c_{|\mathcal{P}_c|}\}$ (with $\mathcal{P}_d \cap \mathcal{P}_c = \emptyset$ and $\mathcal{P}_d \cup \mathcal{P}_c = \mathcal{P}$). The discrete places may contain a natural number of tokens, while the marking of a continuous place is a non negative real number. In the graphical representation a discrete place is drawn as a single circle while a continuous place is drawn with two concentric circles. The complete state (marking) of a FSPN is described by a pair of vectors $M = (\boldsymbol{m}, \boldsymbol{x})$, where the vector $\boldsymbol{m}$, of dimension $|\mathcal{P}_d|$ is the marking of the discrete part of the FSPN and the vector $\boldsymbol{x}$, of dimension $|\mathcal{P}_c|$, represents the fluid levels in the continuous places (with $x_l \geq 0$ for any $c_l \in \mathcal{P}_c$). We use $\mathcal{S}$ to denote the partially discrete and partially continuous state space. In the following we denote by $\mathcal{S}_d$ and $\mathcal{S}_c$ the discrete and the continuous component of the state space respectively. The marking $M = (\boldsymbol{m}, \boldsymbol{x})$ evolves in time. We denote the time by τ, and we can think the marking M at time τ as the stochastic marking process $\mathcal{M}(\tau) = \{(\boldsymbol{m}(\tau), \boldsymbol{x}(\tau)), \tau \geq 0\}$.

– $\mathcal{T}$ is the set of transitions partitioned into a set of stochastically timed transitions $\mathcal{T}_e$ and a set of immediate transitions $\mathcal{T}_i$ (with $\mathcal{T}_e \cap \mathcal{T}_i = \emptyset$ and $\mathcal{T}_e \cup \mathcal{T}_i = \mathcal{T}$). A timed transition $T_j \in \mathcal{T}_e$ is drawn as a rectangle and has an instantaneous firing rate associated to it. An immediate transition $t_h \in \mathcal{T}_i$ is drawn with a thin bar and has a constant zero firing time. We denote the timed transitions with uppercase letters and the immediate transitions with lowercase letters.

– $\mathcal{A}$ is the set of arcs partitioned into four subsets: $\mathcal{A}_d$, $\mathcal{A}_h$, $\mathcal{A}_c$, and $\mathcal{A}_f$. The subset $\mathcal{A}_d$ contains the discrete arcs which can be seen as a function[1] A_d: $((\mathcal{P}_d \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{P}_d)) \to \mathbb{N}$. The arcs $\mathcal{A}_d$ are drawn as single arrows. The subset $\mathcal{A}_h$ contains the inhibitor arcs, A_h: $(\mathcal{P}_d \times \mathcal{T}) \to \mathbb{N}$. These arcs are drawn with a small circle at the end. Given a transition $T_j \in \mathcal{T}$, we denote with ${}^\bullet T_j = \{p_i \in \mathcal{P}_d\colon A_d(p_i, T_j) > 0\}$ and with $T_j^\bullet = \{p_i \in \mathcal{P}_d\colon A_d(T_j, p_i) > 0\}$ the input and the output sets and with ${}^\circ T_j = \{p_i \in \mathcal{P}_d\colon A_h(p_i, T_j) > 0\}$ the inhibition set. The definition of ${}^\bullet T_j$, $T_j^\bullet$, and ${}^\circ T_j$ involves only discrete places and is the same as for the standard GSPNs.

  The subsets $\mathcal{A}_c$ and $\mathcal{A}_f$ define arcs that are related with the continuous places. The subset $\mathcal{A}_c$ defines the continuous arcs. These arcs are drawn as double arrows to suggest a pipe. $\mathcal{A}_c$ is a subset of $(\mathcal{P}_c \times \mathcal{T}_e) \cup (\mathcal{T}_e \times \mathcal{P}_c)$, i.e., a continuous arc can connect a fluid place to a timed transition or it can connect a timed transition to a fluid place. The subset $\mathcal{A}_f$ contains the *flush-out* arcs. $\mathcal{A}_f$ is a subset of $(\mathcal{P}_c \times \mathcal{T}_e)$. These arcs connect continuous places to timed transitions, and describe the capability of a transition to empty in zero time the existing fluid from a continuous place when it fires. Flush-out arcs have been introduced in (Bobbio et al., 1999; Gribaudo et al., 1999) and represent an extension of the FSPN formalism proposed in (Horton et al., 1998) in which discrete jumps in the fluid level are not allowed. The arcs $\mathcal{A}_f$ are drawn as thick single arrows.

– The function $B\colon \mathcal{P}_c \to \mathbb{R}^+ \cup \{\infty\}$ describes the fluid upper bounds on each continuous place. This bound has no effect when it is set to infinity. Each fluid place has an implicit lower bound at level 0. From this, it follows that $\forall\, M = (\boldsymbol{m}, \boldsymbol{x}) \in \mathcal{S}$ and $c_l \in \mathcal{P}_c$, $0 \leq x_l \leq B(c_l)$.

– The firing rate function F is defined for timed transitions $\mathcal{T}_e$ so that $F\colon \mathcal{T}_e \times \mathcal{S} \to \mathbb{R}^+$. Therefore, a timed transition T_j enabled at time τ in a discrete marking $\boldsymbol{m}(\tau)$ with fluid level $\boldsymbol{x}(\tau)$, may fire with rate $F(T_j, \boldsymbol{m}(\tau), \boldsymbol{x}(\tau))$, that is:

$$\lim_{\Delta\tau \to 0} Pr\{T_j \text{ fires in } (\tau, \tau + \Delta\tau) | \mathcal{M}(\tau) = (\boldsymbol{m}(\tau), \boldsymbol{x}(\tau))\} = F(T_j, \boldsymbol{m}, \boldsymbol{x})\Delta\tau$$

We also use as a short hand notation $F(T_j, M)$, where $M = (\boldsymbol{m}, \boldsymbol{x})$).

– The weight function W for immediate transitions $\mathcal{T}_i$ ($W\colon \mathcal{T}_i \times \mathcal{S}_d \to \mathbb{R}^+$) has the usual meaning and it may depend only on the discrete part (Ajmone et al., 1995).

– The function $R\colon \mathcal{A}_c \times \mathcal{S} \to \mathbb{R}^+ \cup \{0\}$ is called the *flow rate function* and describes the marking dependent flow of fluid across continuous arcs.

– The initial state of the FSPN is denoted by the pair $M_0 = (\boldsymbol{m_0}, \boldsymbol{x_0})$.

The role of the previous sets and functions will be clarified by providing the enabling and firing rules.
Let us denote by m_i the i-th component of the vector $\boldsymbol{m}$, i.e., the number of tokens in place p_i when the discrete marking is $\boldsymbol{m}$. We say that a transition $T_j \in \mathcal{T}$ has concession in marking $M = (\boldsymbol{m}, \boldsymbol{x})$ iff

$$\forall\, p_i \in {}^\bullet T_j, \quad m_i \geq A_d(p_i, T_j) \qquad \text{and} \qquad \forall\, p_i \in {}^\circ T_j, \quad m_i < A_h(p_i, T_j).$$

If an immediate transition has concession in $M = (\boldsymbol{m}, \boldsymbol{x})$, it is enabled and the marking is vanishing. Otherwise, the marking is tangible and any timed transition with concession is enabled in it. Note that the previous definition is exactly the one of standard GSPNs (Ajmone et al., 1995), i.e., the concession and the enabling conditions depend only on the discrete part of the FSPN. Let $\mathcal{E}(M)$ denote the set of enabled transitions in marking $M = (\boldsymbol{m}, \boldsymbol{x})$, we have that $\mathcal{E}(M) = \mathcal{E}(M')$, for any marking $M' = (\boldsymbol{m}, \boldsymbol{x'})$.

For the firing rule we must distinguish two cases.

1. An immediate transition $t_j \in \mathcal{T}_i$ enabled in marking $M = (\boldsymbol{m}, \boldsymbol{x})$ yields a new marking $M' = (\boldsymbol{m'}, \boldsymbol{x})$, i.e., the firing of an immediate transition does not change the continuous part of the marking. We can write $(\boldsymbol{m}, \boldsymbol{x}) \xrightarrow{t_j} (\boldsymbol{m'}, \boldsymbol{x})$, where

$$\forall\, p_i \in \mathcal{P}_d, \quad m'_i = m_i + A_d(t_j, p_i) - A_d(p_i, t_j).$$

2. If marking $M = (\boldsymbol{m}, \boldsymbol{x})$ is tangible, the firing of a timed transition $T_j \in \mathcal{T}_e$ enabled in $M = (\boldsymbol{m}, \boldsymbol{x})$ yields a new marking $M' = (\boldsymbol{m'}, \boldsymbol{x'})$, i.e., $(\boldsymbol{m}, \boldsymbol{x}) \xrightarrow{T_j} (\boldsymbol{m'}, \boldsymbol{x'})$, where

$$\forall\, p_i \in \mathcal{P}_d,\ \ m'_i = m_i + A_d(T_j, p_i) - A_d(p_i, T_j) \quad \text{and}$$
$$\forall\, c_l \in \mathcal{P}_c,\quad x'_l = \begin{cases} 0 & \text{if } (c_l, T_j) \in \mathcal{A}_f \\ x_l & \text{otherwise.} \end{cases}$$

In other words, the firing of a timed transition T_j immediately empties (flushes out) all the continuous places that are connected with flush-out arcs to T_j.

Let us see how enabled transitions may influence the continuous part of the marking. Each continuous arc that connects a fluid place $c_l \in \mathcal{P}_c$ to an enabled timed transition $T_j \in \mathcal{T}_e$ (resp. an enabled transition T_j to a fluid place c_l), causes a “change” in the fluid level of place c_l. Let $\mathcal{M}(\tau)$ be the marking process, i.e., $\mathcal{M}(\tau) = M$ if at time τ the marking of the FSPN is $M = (\boldsymbol{m}, \boldsymbol{x})$. Thus, when the FSPN marking is $\mathcal{M}(\tau)$, for each (timed) transition T_j enabled in $\mathcal{M}(\tau)$, fluid can leave place $c_l \in \mathcal{P}_c$ along the arc $(c_l, T_j) \in \mathcal{A}_c$ at rate $R((c_l, T_j), \mathcal{M}(\tau))$ and can enter the continuous place c_l at rate $R((T_j, c_l), \mathcal{M}(\tau))$. The potential rate of change of fluid level for the continuous place c_l in marking $\mathcal{M}(\tau)$ is:

$$r_l^p(\mathcal{M}(\tau)) = \sum_{T_j \in \mathcal{E}(\mathcal{M}(\tau))} R((T_j, c_l), \mathcal{M}(\tau)) - R((c_l, T_j), \mathcal{M}(\tau)).$$

We require that for every discrete marking $\boldsymbol{m}$ and continuous arc (c_l, T_j) (resp. (T_j, c_l)), the rate $R((c_l, T_j), (\boldsymbol{x}, \boldsymbol{m}))$ (resp. $R((T_j, c_l), (\boldsymbol{x}, \boldsymbol{m}))$) is a piecewise continuous function of $\boldsymbol{x}$.

Now let $X_l(\tau)$ be the random variable representing the fluid level at time τ in a continuous place $c_l \in \mathcal{P}_c$. The fluid level, in each continuous place c_l can never become negative or exceed the bound $B(c_l)$, so the (actual) rate of change over time, τ, when the marking is $\mathcal{M}(\tau)$, is

$$r_l(\mathcal{M}(\tau)) = \frac{dX_l(\tau)}{d\tau} = \begin{cases} r_l^p(\mathcal{M}(\tau)) & \text{if } X_l(\tau) = 0 \text{ and } r_l^p(\mathcal{M}(\tau)) \geq 0 \\ r_l^p(\mathcal{M}(\tau)) & \text{if } X_l(\tau) = B(c_l) \text{ and } r_l^p(\mathcal{M}(\tau)) \leq 0 \\ 0 & \text{if } X_l(\tau) = 0 \text{ and } r_l(\mathcal{M}(\tau)) < 0 \\ 0 & \text{if } X_l(\tau) = B(c_l) \text{ and } r_l^p(\mathcal{M}(\tau)) > 0 \\ r_l^p(\mathcal{M}(\tau)) & \text{if } 0 < X_l(\tau) < B(c_l) \text{ and} \\ & \quad r_l^p(\mathcal{M}(\tau^-)) r_l^p(\mathcal{M}(\tau^+)) \geq 0 \\ 0 & \text{if } 0 < X_l(\tau) < B(c_l) \text{ and} \\ & \quad r_l^p(\mathcal{M}(\tau^-)) r_l^p(\mathcal{M}(\tau^+)) < 0. \end{cases} \tag{1}$$

The first two cases of Equation (1) concern situations when $X_l(\tau) = 0$ (resp. $X_l(\tau) = B(c_l)$) and the potential rate is $r_l^p(\mathcal{M}(\tau)) \geq 0$ (resp. $r_l^p(\mathcal{M}(\tau)) \leq 0$). In both cases the actual rate is equal to the potential rate. The third and the fourth cases prevent the fluid level to overcome the lower and the upper bound. The last two cases require a deeper explanation (a reference for a complete discussion is (Elwalid and Mitra, 1994)). As it has been assumed in (Horton et al., 1998) the flow rate function $R(\cdot, \cdot)$ is a piecewise continuous function of the continuous part of the marking. The meaning of the last case is that a sign change (from $+$ to $-$) in $r_l^p(\mathcal{M}(\tau))$ will "trap" $X_l(\tau)$ to be constant. With this assumption, the analysis of the stochastic process $\mathcal{M}(\tau)$ is simplified (see (Elwalid and Mitra, 1994) for a discussion on this type of situation). The fifth case, which is the most common one, accounts for the fact that there is no sign change from $+$ to $-$ in $r_l^p(\mathcal{M}(\tau))$ and hence the actual rate is equal to the potential rate.

3. Analysis

In this section, we derive the equations for the joint process $\mathcal{M}(\tau) = (\boldsymbol{m}(\tau), \boldsymbol{x}(\tau))$ that describes the dynamic behaviour of the FSPN model as a function of the time. First, we introduce the infinitesimal generator, then we present the complete equations for the case in which the FSPN has only a single fluid place. Finally, we extend the results to FSPNs with more than a single fluid place.

3.1. The Infinitesimal Generator

The marking process $\mathcal{M}(\tau)$ is characterised by a matrix $\boldsymbol{Q}$, that we call *infinitesimal generator*. The set $\mathcal{S} = (\mathcal{S}_d \times \mathcal{S}_c)$ of all the states is decomposed in two parts, where $\mathcal{S}_d$ represents the discrete component of the state space and $\mathcal{S}_c$ the continuous component.

In order to derive the complete equations we start investigating the behaviour of the discrete part of the system. Since fluid arcs and flush-out arcs do not change the enabling condition of a transition, standard analysis techniques can be applied to the discrete marking process $\boldsymbol{m}(\tau)$ (Ajmone et al., 1995). These techniques split the discrete state space into two disjoint subsets, called respectively the *tangible marking* set and the *vanishing marking* set. Since the process spends no time in vanishing markings, they can be removed and their effect can be included in the transitions between tangible markings. From this point on, we will consider only tangible markings. In GSPNs, the underlying stochastic process is a CTMC, whose infinitesimal generator is a matrix $\boldsymbol{Q}$. Each entry q_{ij} represents the rate of transition from a tangible state $\boldsymbol{m}_i$ to a tangible state $\boldsymbol{m}_j$, that is:

$$q_{ij} = \sum_{T_k \in \mathcal{E}(\boldsymbol{m}_i) | \boldsymbol{m}_i \xrightarrow{T_k} \boldsymbol{m}_j} F(T_k, \boldsymbol{m}_i),$$

where $\mathcal{E}(\boldsymbol{m}_i)$ represents the set of enabled transitions in marking $\boldsymbol{m}_i$, and $\boldsymbol{m}_i \xrightarrow{T_k} \boldsymbol{m}_j$ means that the firing of transition T_k changes the discrete state of the system from $\boldsymbol{m}_i$ to $\boldsymbol{m}_j$.

In the FSPN model defined in (Horton et al., 1998), the firing rate of each timed transition can be dependent on the continuous component of the state. With this extension, the infinitesimal generator matrix must be also dependent on the fluid component of the state, that is: $\boldsymbol{Q}(\boldsymbol{x})$. The addition of flush-out arcs requires a further extension. We must differentiate between transitions that cause flush-out of continuous places, and transitions that do not. Moreover, since a transition may flush-out more than one single fluid place, all the possible combinations of fluid places must be treated separately. We do this by making matrix $\boldsymbol{Q}$ dependent also on the power set of the fluid places: $\boldsymbol{Q}(\boldsymbol{x}, \mathsf{S})$, where $\mathsf{S} \in 2^{\mathcal{P}_c}$. The matrix $\boldsymbol{Q}(\boldsymbol{x}, \emptyset)$ accounts for the transition rates among tangible states when no flush-out occurs, and $\boldsymbol{Q}(\boldsymbol{x}, \{c_l\})$ accounts for the transition rates among tangible states when flush-out of place c_l does occur. If a transition flushes out two fluid places c_l and c_k, its effect is included in $\boldsymbol{Q}(\boldsymbol{x}, \{c_l, c_k\})$ and so on. In particular, we define:

$$\boldsymbol{Q}(\boldsymbol{x}, \mathsf{S}) = [q_{ij}(\boldsymbol{x}, \mathsf{S})]$$

where $q_{ij}(\boldsymbol{x}, \mathsf{S})$ represents the transition rate from state $\boldsymbol{m_i}$ to state $\boldsymbol{m_j}$ when the level of the fluid is $\boldsymbol{x}$ and the considered transition flushes out the fluid places belonging to the (possibly empty) set S, that is:

$$q_{ij}(\boldsymbol{x}, \mathsf{S}) = \sum_{\substack{T_k \in \mathcal{E}(\boldsymbol{m}_i) \mid \\ \boldsymbol{m}_i \xrightarrow{T_k} \boldsymbol{m}_j \wedge c_l \in \mathsf{S} \Leftrightarrow (c_l, T_k) \in \mathcal{A}_f}} F(T_k, \boldsymbol{m}_i, \boldsymbol{x}).$$

The summation considers the transition rate of all the transitions T_k that bring the net from state $\boldsymbol{m}_i$ to $\boldsymbol{m}_j$, flushing out exactly all the fluid places specified in the (possibly empty) set S. In the standard equations that describe a CTMC, the terms on the diagonal of the infinitesimal generator accounts for the probability of exiting from a state. Here we have to consider not only standard transitions, but also changes of state that cause a flush-out. We denote by

$$q_i(\boldsymbol{x}) = \sum_{\boldsymbol{m}_j \in \mathcal{S}_d} \sum_{\mathsf{S} \in 2^{\mathcal{P}_c}} q_{ij}(\boldsymbol{x}, \mathsf{S}) \tag{2}$$

the total exit rate from state $\boldsymbol{m_i}$, when the fluid level is $\boldsymbol{x}$. This function takes into account the sum of the rates from state i to any state $\boldsymbol{m_j}$, with any combination of flush-outs. The diagonal element defined in (2) is included in the matrix $\boldsymbol{Q}(\boldsymbol{x}, \emptyset)$ and hence:

$$q_{ii}(\boldsymbol{x}, \emptyset) = -q_i(\boldsymbol{x}). \tag{3}$$

The above defined matrix $\boldsymbol{Q}(\boldsymbol{x}, \mathsf{S})$ of dimensions $|\mathcal{S}_d| \times |\mathcal{S}_d|$ is an infinitesimal generator, that is, each row sum of $\sum_{\mathsf{S} \in 2^{\mathcal{P}_c}} Q(\boldsymbol{x}, \mathsf{S})$, is equal to zero.

3.2. Equations for the Case of a Single Fluid Place

Let us denote by c_l the single fluid place of the net and let us use the shorthand notation $r(i, x) = r_l(M)$, where $M = (\boldsymbol{m_i}, x)$ (the subscript index that identifies the fluid place is removed).

For each state $\boldsymbol{m_i} \in \mathcal{S}_d$ we compute $\pi_i(\tau, x) = Pr\{\mathcal{M}(\tau) = (\boldsymbol{m_i}, x)\}$ which is the probability density of finding the system in state $\boldsymbol{m_i}$ with fluid level x in place c_l at time τ. Theorem 1 describes the transient behaviour of the FSPN.

THEOREM 1 *For each* $\boldsymbol{m}_i \in \mathcal{S}_d$ *the probability* $\pi_i(\tau, x)$ *is given by:*

$$\frac{\partial \pi_i(\tau, x)}{\partial \tau} + \frac{\partial \left(r(i, x)\pi_i(\tau, x)\right)}{\partial x}$$

$$= \sum_{m_j \in \mathcal{S}_d} \left(\pi_j(\tau, x) q_{ji}(x, \emptyset) + \delta(x) \int_0^\infty \pi_j(\tau, x') q_{ji}(x', \{c_l\}) dx' \right), \qquad (4)$$

where $\delta(x)$ is the Dirac's delta function. Equation (3), may be written in vector notation. If we denote by $\boldsymbol{\pi}(\tau, x)$ the vector whose i-th component is $\pi_i(\tau, x)$, and by $\boldsymbol{R}(x) = \text{diag}(r(i, x))$ the diagonal matrix whose components account for the actual flow rate out of c_l, Equation (3) becomes:

$$\frac{\partial \boldsymbol{\pi}(\tau, x)}{\partial \tau} + \frac{\partial \left(\boldsymbol{R}(x) \boldsymbol{\pi}(\tau, x)\right)}{\partial x} = \boldsymbol{\pi}(\tau, x) \boldsymbol{Q}(x, \emptyset) + \delta(x) \int_0^\infty \boldsymbol{\pi}(\tau, x') \boldsymbol{Q}(x', \{c_l\}) dx'. \qquad (5)$$

No boundary conditions are needed, since they are included in the definition of the potential flow rate (Equation (1)). Dirac's delta functions in the solution, represent cases where there is a non zero probability of finding the system in a particular marking (both discrete and continuous).

The proof of Theorem 1 is given in Appendix. Here we give a more intuitive interpretation of the theorem. Equation (3) is composed of four terms. The first term accounts for the time that elapses in the state. The second term instead, is related to the fluid flow: matrix $\boldsymbol{R}(x)$ represents the actual flow rate in each discrete state. It is defined in such a way that the flow is stopped (i.e., $\boldsymbol{R}(x) = 0$) whenever a boundary is reached. Setting $\boldsymbol{R}(x) = 0$, has also the effect of generating Dirac's deltas in the solution. These are equivalent to probability masses generated at the boundaries. The third term (on the right hand side), accounts for both the transitions into the state that do not flush out the place, and for the transitions out from the state (this comes out from the definition of $q_{ii}(x, \emptyset)$, i.e., Equation (2)). The last term, accounts for the entrance into the state, caused by transitions that flush out the fluid place. The Dirac's delta that multiply the term, has the effect of considering the entrance only at zero level (due to the flush-out), and the integral means that an entrance can happen from every fluid level. The Dirac's delta includes the boundary condition inside the equation.

3.3. Steady State Solution

Steady state exists only if the system is stable. Unfortunately, stability conditions on FSPNs are still a research topic. In FSPN with flush-out, we guess that standard stability condition (as expressed in (Horton et al., 1998)) can be applied. Moreover, the addition of flush-out arcs, limiting the growth of the fluid level in the continuous places seems to provide even more stable systems. However, if the system is stable, we know that, for every $\pi_i(\tau, x)$,

$$\lim_{\tau \to \infty} \frac{\partial \pi_i(\tau, x)}{\partial \tau} = 0.$$

If we denote by $\pi_i(x) = \lim_{\tau \to \infty} \pi_i(\tau, x)$, the steady state equation can be written in the following manner:

$$\frac{\partial \left(\boldsymbol{R}(x) \boldsymbol{\pi}(x)\right)}{\partial x} = \boldsymbol{\pi}(x) \boldsymbol{Q}(x, \emptyset) + \delta(x) \int_0^\infty \boldsymbol{\pi}(x') \boldsymbol{Q}(x', \{c_l\}) dx'. \qquad (6)$$

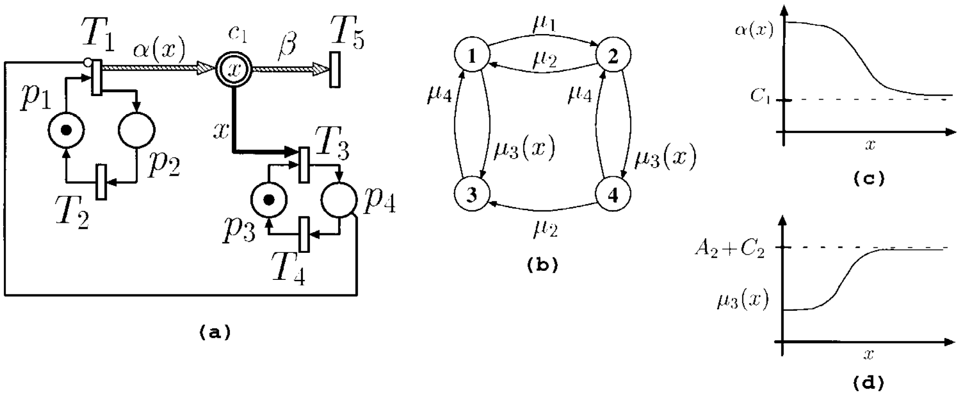


Figure 1. FSPN for a producer/consumer system with failures and with flush-out of the buffer content (a), discrete state space diagrams (b), example of flow rate function $\alpha(x)$ (c), and example of firing rate function $\mu_3(x)$ (d).

EXAMPLE 1 We consider a pharmaceutical manufacturing system. In this field, common policies in manufacturing lines are generally determined by specific rules. For example, if the equipment fails during the sterilisation of a given product contained in a buffer, the product is no longer "safe" and all the content of the buffer should be discarded. We consider a producer/consumer system, in which we also model the occurrence of failures in the sterilisation phase. In the case of failure, all the content of the buffer has to be discarded. We model such a system with the FSPN of Figure 1(a).

The producer alternates between two independent, exponentially distributed states that represent the conditions "producing" (token in place p_1) and "non-producing" (token in place p_2). The buffer is represented by the fluid place. When the producer is in service it is able to produce $\alpha(x)$ unit of product (represented in term of fluid) per unit time. The production rate is driven by the buffer level and is confined between two thresholds: when the buffer is empty the production is at maximum rate; when the buffer fills up the production rate decreases and attains a minimum constant rate. An example of such a function can be

$$\alpha(x) = A_1 \left(1 - \frac{1}{1 + exp(B_1 - x)}\right) + C_1,$$

where A_1, B_1, and C_1 are constants values. Figure 1(c) plots this fluid dependent flow rate function.

The consumer is able to consume β units of product per unit time. The sterilisation process alternates between two states that represent the conditions "normal" (token in place p_3) and "abnormal" (token in place p_4). The inhibitor arc from place p_4 to transition T_1 stops the producer when the system is in the abnormal state. When transition T_3 fires (occurrence of a fault in the sterilisation process) the content of the buffer is discarded. This is represented by the flush-out arc from place c_1 to transition T_3. The discrete state space of the FSPN is presented in Figure 1(b). It is composed by four states representing the possible combinations of the producer and the sterilisation processes: (producing, normal) (state

1), (non-producing, normal) (state 2), (producing, abnormal) (state 3), (non-producing, abnormal) (state 4). When the FSPN is in state 3, due to the presence of the inhibitor arc, the production is stopped even if the producer is in the working state. According to the FSPN formalism we can also model faults of the sterilisation process that depend on the quantity of fluid present in the buffer. With μ_1, μ_2, and μ_4 we denote the (constant) firing rate of transitions T_1, T_2, and T_4 respectively, while the transition rate of T_3 depends on the buffer level. A possible fluid dependent firing rate is

$$\mu_3(x) = A_2 \frac{1}{1 + exp(B_2 - x)} + C_2$$

where A_2, B_2, and C_2 are constants values. Figure 1(d) plots this fluid dependent firing rate function. It can be used to reflect the fact that the probability of having a fault (chance of being infected) increases with the quantity of product present in the buffer.

The potential rate of change of the fluid place, is $r_1^p = \alpha(x) - \beta$ in state 1, since both transition T_1 and T_5 are enabled, and $r_1^p = -\beta$ in states 2, 3 and 4 since only T_5 is enabled. If we suppose that $\beta < C_1$, the actual rate of change of the fluid place is equal to the potential rate of change in state 1, since no boundary can be reached (being $\alpha(x) - \beta > 0$, for any x). In states 2, 3 and 4 instead, the actual rate of change is defined as:

$$r_1(x) = \begin{cases} 0 & x = 0 \\ -\beta & x > 0 \end{cases}$$

The matrices entering in Equation (4) have the following form:

$$\boldsymbol{Q}(x, \emptyset) = \begin{pmatrix} -\mu_1 - \mu_3(x) & \mu_1 & 0 & 0 \\ \mu_2 & -\mu_2 - \mu_3(x) & 0 & 0 \\ \mu_4 & 0 & -\mu_4 & 0 \\ 0 & \mu_4 & \mu_2 & -\mu_4 - \mu_2 \end{pmatrix}$$

$$\boldsymbol{Q}(x, \{c_1\}) = \begin{pmatrix} 0 & 0 & \mu_3(x) & 0 \\ 0 & 0 & 0 & \mu_3(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{R}(x) = \begin{pmatrix} \alpha(x) - \beta & 0 & 0 & 0 \\ 0 & r_1(x) & 0 & 0 \\ 0 & 0 & r_1(x) & 0 \\ 0 & 0 & 0 & r_1(x) \end{pmatrix}.$$

In Section 5 we present some numerical results obtained for the FSPN presented in this example.

3.4. Generalisation to More Than One Fluid Place

We extend Equation (4) to FSPNs having more than one fluid place. Recall that $\boldsymbol{x} = (x_1, x_2, \ldots, x_{|\mathcal{P}_c|})$ is the vector whose component x_l represents the fluid level in the continuous place c_l. We denote by $2^{\mathcal{P}_c}$ the power set of the fluid place set. Let $\mathsf{s} \in 2^{\mathcal{P}_c}$ be a subset (possibly the empty set) of $\mathcal{P}_c$.

Since we have more than one fluid place, we collect all the possible actual flow rates in a diagonal matrix $\boldsymbol{R}(\{c_l\}, \boldsymbol{x})$, with $\{c_l\} \in \mathcal{P}_c$. The element $r_{jj}(\{c_l\}, \boldsymbol{x})$ of $\boldsymbol{R}(\boldsymbol{x})$, represents the fluid flow rate of continuous place c_l, in discrete state j, conditioned to the fluid level $\boldsymbol{x}$, that is: $r_{jj}(\{c_l\}, \boldsymbol{x}) = r_l(M = (\boldsymbol{m}_j, \boldsymbol{x}))$.

Matrix $\boldsymbol{Q}$, together with matrix $\boldsymbol{R}$, describe completely the stochastic process. Since we need to consider also the case of multiple flush-outs induced by a single transition, the equation that describes the system, should have an integral term for each possible combination of fluid places flushed out together. For example, with two fluid places $\mathcal{P}_c = \{c_1, c_2\}$, and $\boldsymbol{x} = (x_1, x_2)$, the power set of $\mathcal{P}_c$ is $\{\{\emptyset\}, \{c_1\}, \{c_2\}, \{c_1, c_2\}\}$ and the equation becomes:

$$\begin{aligned}
&\frac{\partial \boldsymbol{\pi}(\tau, x_1, x_2)}{\partial \tau} + \frac{\partial\, (\boldsymbol{R}(c_1, x_1, x_2)\boldsymbol{\pi}(\tau, x_1, x_2))}{\partial x_1} + \frac{\partial\, (\boldsymbol{R}(c_2, x_1, x_2)\boldsymbol{\pi}(\tau, x_1, x_2))}{\partial x_2} \\
&\qquad = \boldsymbol{\pi}(\tau, x_1, x_2)\boldsymbol{Q}(x_1, x_2, \emptyset) + \delta(x_1) \int_0^\infty \boldsymbol{\pi}(\tau, x_1', x_2)\boldsymbol{Q}(x_1', x_2, \{c_1\}) dx_1' \\
&\qquad + \delta(x_2) \int_0^\infty \boldsymbol{\pi}(\tau, x_1, x_2')\boldsymbol{Q}(x_1, x_2', \{c_2\}) dx_2' \\
&\qquad + \delta(x_1)\delta(x_2) \int_0^\infty \int_0^\infty \boldsymbol{\pi}(\tau, x_1', x_2')\boldsymbol{Q}(x_1', x_2', \{c_1, c_2\}) dx_1' dx_2'.
\end{aligned}$$

In order to write the equation in a more compact form, and extend it to an arbitrary number of fluid places, a special notation is introduced. First we define a Dirac's delta extended to a set:

$$\boldsymbol{\delta}(\boldsymbol{x}, \mathsf{s}) = \begin{cases} 1 & \mathsf{s} = \emptyset \\ \prod_{c_l \in \mathsf{s}} \delta(x_l) & \mathsf{s} \neq \emptyset. \end{cases}$$

This special function corresponds to a product of Dirac's deltas, one for each element of the set. It is used to allow a special entrance at level zero, for all the fluid places that are simultaneously flushed out by the firing of a transition.

We define also the integral extended to a set:

$$\int_0^\infty F(\cdots) d\mathsf{s} = \begin{cases} F(\cdots) & \mathsf{s} = \emptyset \\ \int_0^\infty \int_0^\infty \cdots \int_0^\infty F(\cdots) dx_{i_1}' dx_{i_2}' \cdots dx_{i_{|\mathsf{s}|}}' & \mathsf{s} \neq \emptyset. \end{cases} \tag{7}$$

This symbol is a short-hand notation, used to describe the fact that each flush-out may happen at any level of the fluid places that will be emptied with the transition. This behaviour is caught by integrating the solution over each fluid component which represents a continuous place involved in the flush-out. We also need a projection operator:

$$\boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s}) = (\sigma_1, \sigma_2, \ldots, \sigma_{|\mathcal{P}_c|}) \quad \sigma_l = \begin{cases} x_l & c_l \notin \mathsf{s} \\ x_l' & c_l \in \mathsf{s}. \end{cases} \tag{8}$$

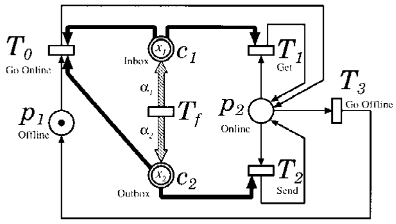


Figure 2. FSPN with all the possible combination of flush-out.

The purpose of this operator is just to select the correct integration variables in the extended integral notation defined in (7).

Using this notation, we can collapse all the integrals into a single summation term. If we denote by $\boldsymbol{\pi}(\tau, \boldsymbol{x})$ the probability density vector at time τ with level $\boldsymbol{x}$, the equation for the general case is:

$$\frac{\partial \boldsymbol{\pi}(\tau, \boldsymbol{x})}{\partial \tau} + \sum_{c_l \in \mathcal{P}_c} \frac{\partial (\boldsymbol{R}(c_l, \boldsymbol{x}) \boldsymbol{\pi}(\tau, \boldsymbol{x}))}{\partial x_l} = \sum_{\mathsf{s} \in 2^{\mathcal{P}_c}} \boldsymbol{\delta}(\boldsymbol{x}, \mathsf{s}) \int_0^\infty \boldsymbol{\pi}(\tau, \boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s})) \boldsymbol{Q}(\boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s}), \mathsf{s}) d\mathsf{s}. \tag{9}$$

The first term of Equation (9) represents the time, and the second accounts for the fluid flow in all the fluid places. Each continuous place has a term in the summation which represents its instantaneous fluid change in each discrete state. The term in the right hand side of Equation (9), accounts for the probability change due to state change. Each term of the summation corresponds to transitions that flush-out a particular subset of fluid places. If the system is also stable, its steady state solution $\boldsymbol{\pi}(\boldsymbol{x})$ may be computed by means of the following equation:

$$\sum_{c_l \in \mathcal{P}_c} \frac{\partial (\boldsymbol{R}(c_l, \boldsymbol{x}) \boldsymbol{\pi}(\boldsymbol{x}))}{\partial x_l} = \sum_{\mathsf{s} \in 2^{\mathcal{P}_c}} \boldsymbol{\delta}(\boldsymbol{x}, \mathsf{s}) \int_0^\infty \boldsymbol{\pi}(\boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s})) \boldsymbol{Q}(\boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s}), \mathsf{s}) d\mathsf{s}.$$

EXAMPLE 2 Consider a person who uses a mailer. During his work, he writes mails offline. Sometimes he goes on-line, sends the queued mails and downloads the new ones. While he is on-line, he continues writing mails and has three possible choices: send the newly written mails, get the mails received in the meantime, or go off-line. Figure 2, models this system. Fluid place c_1 contains the mails on the server that are going to be received, and fluid place c_2 contains the mails queued to be sent. The two fluid arcs model the arrival of mails to the server (arc connecting T_f to c_1) and the process of writing new mails by the user (arc connecting T_f to c_2). Transition T_f, which is always enbaled, has the only purpose of keeping the fluid flowing along the two fluid arcs. Discrete place p_1 models the off-line state, and discrete place p_2 models the on-line state. Timed transition T_0 models the action of going on-line: the two flush-out arcs that empty both fluid places, represent the action of sending the queued mails and getting the ones present on the server. Transition T_3 represents the action of going off-line. Transition T_1 models the action of getting the mails while on-line: it does this by flushing out the server buffer when it fires (through the flush-out arc that connects c_1 to T_1). Transition T_2 corresponds to the action of sending the newly written mails. It does this by flushing out place c_2. We assume that transition T_i has a constant transition rate λ_i, with $i = 0 \cdots 3$. This system has two fluid places and has only two discrete states. The problem is that all the combinations of flush-outs of the two places are possible. The set of all the possible combinations of flush-outs is: $2^{\mathcal{P}_c} = \{\emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}\}$. The equations that describe this system are:

$$\frac{\partial \boldsymbol{\pi}(\tau, \boldsymbol{x})}{\partial \tau} + \frac{\partial (\boldsymbol{R}(c_1, \boldsymbol{x})\boldsymbol{\pi}(\tau, \boldsymbol{x}))}{\partial x_1} + \frac{\partial (\boldsymbol{R}(c_2, \boldsymbol{x})\boldsymbol{\pi}(\tau, \boldsymbol{x}))}{\partial x_2}$$
$$= \sum_{\mathsf{s} \in 2^{\mathcal{P}_c}} \boldsymbol{\delta}(\boldsymbol{x}, \mathsf{s}) \int_0^{\infty} \boldsymbol{\pi}(\tau, \boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s}))\boldsymbol{Q}(\boldsymbol{\sigma}(\boldsymbol{x}, \mathsf{s}), \mathsf{s}) d\mathsf{s},$$

where

$$\boldsymbol{Q}(\boldsymbol{x}, \emptyset) = \begin{pmatrix} -\lambda_0 & 0 \\ \lambda_3 & -\lambda_1 - \lambda_2 - \lambda_3 \end{pmatrix}, \ \boldsymbol{Q}(\boldsymbol{x}, \{c_1\}) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$
$$\boldsymbol{Q}(\boldsymbol{x}, \{c_2\}) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \boldsymbol{Q}(\boldsymbol{x}, \{c_1, c_2\}) = \begin{pmatrix} 0 & \lambda_0 \\ 0 & 0 \end{pmatrix},$$
$$\boldsymbol{R}(c_1, \boldsymbol{x}) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \ \boldsymbol{R}(c_2, \boldsymbol{x}) = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

3.5. Generation of Matrices Q and R

The previous example allows to point out that the key point in the derivation of equations that describe a given FSPN is the generation of the matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$.

The first step of this generation is the elimination of the immediate transitions, this can be done using the procedure described in (Ajmone et al., 1995). In this manner, we can assume that the input for the algorithm presented in this section is a FSPN without immediate transitions.

Procedure_Generate_Matrices(FSPN)
  Generate the discrete state space $\mathcal{S}_d$
  for all $\boldsymbol{m_i} \in \mathcal{S}_d$ do
    for all fluid place $c_l \in \mathcal{P}_c$ do
      $r_{i\,i}(\{c_l\}, \boldsymbol{x}) \leftarrow r_l(\boldsymbol{m_i}, \boldsymbol{x})$
    end for
    for all each transition $T_k \in \mathcal{E}(\boldsymbol{m_i})$ do
      Let $\boldsymbol{m_j}$ the state after the firing of T_k ($\boldsymbol{m_i} \xrightarrow{T_k} \boldsymbol{m_j}$)
      Let s be the set of fluid places flushed out with the firing of T_k

$q_{ij}(\boldsymbol{x}, \mathsf{s}) \leftarrow q_{ij}(\boldsymbol{x}, \mathsf{s}) + F(T_k, \boldsymbol{m}_i, \boldsymbol{x})$
end for
end for
End Procedure_Generate_Matrices

4. Application of FSPN

In order to make the technique useful and appealing from an application point of view, two issues need to be addressed: *i)* - how to define performance measures at the net level; *ii)* - how to numerically solve the model and compute the specified measures.

4.1. Performance Measures Defined on FSPN Models

In general, the set of performance measures that can be evaluated from a FSPN encompasses the set of measures that can be evaluated in discrete SPN models. In fact, in addition, we can define new measures that are specifically related to the fluid (or continuous) part of the net. The measures connected to the discrete part of the FSPN are referred to as *discrete performance measures* and those connected to the continuous part as *continuous performance measures*.

Discrete performance measures can still be classified as *discrete state measures* (when the measure refers to the probability of occurrence of some condition on the discrete markings) and *throughput measures* (when the measure refers to the passage of tokens through the net or to the number of firings of a transition). Similarly, *continuous performance measures* can be classified as *fluid state measures* and *flow measures*. Fluid state measures give the probability of a condition connected to the fluid levels in the net, while flow measures can be considered as the continuous counterpart of discrete throughput measures.

A very elegant and unifying way to define and to compute discrete performance measures in discrete SPNs is by means of the concept of reward (Howard, 1971; Ciardo et al., 1991; Bobbio et al., 1998). In FSPN, the flow rate assigned to a continuous arc may be interpreted as a reward rate, and hence the reward specification is directly associated with the graphical representation of the model. In this view, the continuous performance measures can be defined at the net level as a function of the model primitives (Horton, 1996).

In Markov Reward Models, like those generated from Discrete SPNs, only the expected instantaneous reward and the expected accumulated reward can be evaluated at the same cost as the solution of the standard Markov equation (Bobbio et al., 1998). The evaluation of the *cdf* of the reward accumulated over a finite time interval (sometimes called the *performability*) requires a considerable additional computational effort and is usually not offered in SPN packages. On the contrary, FSPNs allow to define and to evaluate these distribution measures within the default structural specifications.

4.1.1. Discrete Performance Measures

A typical example of *discrete state measure* is the distribution of the tokens in a discrete place p_j. This measure can be computed as:

$$P\{\#(p_j) = i\} = \sum_{\boldsymbol{m}:\ m_j=i} \int_0^\infty \cdots \int_0^\infty \pi(\boldsymbol{m}, x_1, \ldots, x_{|\mathcal{P}_c|})\, dx_1 \cdots dx_{|\mathcal{P}_x|}.$$

Using the extended integral notation defined in Equations (7) and (8), the above expression becomes:

$$P\{\#(p_j) = i\} = \sum_{\boldsymbol{m}:\ m_j=i} \int_0^\infty \pi(\boldsymbol{m}, \boldsymbol{\sigma}(\mathbf{x}, \mathcal{P}_c))\, d\mathcal{P}_c.$$

If we denote by $\mathcal{E}(T_j)$ the set of discrete markings in which transition T_j is enabled, and recalling that $F(T_j, M)$ is the firing rate of T_j in marking M, then the throughput of transition T_j can be expressed as:

$$\chi(T_j) = \sum_{\boldsymbol{m}\in\mathcal{E}(T_j)} \int_0^\infty \pi(\boldsymbol{m}, \boldsymbol{\sigma}(\mathbf{x}, \mathcal{P}_c))\, F(T_j, \boldsymbol{m}, \boldsymbol{\sigma}(\mathbf{x}, \mathcal{P}_c))\, d\mathcal{P}_c.$$

4.1.2. Continuous Performance Measures

The continuous measure, corresponding to the distribution of tokens in a place, is the probability density of the fluid level in a fluid place. Let $X(c_l)$ be the random fluid level in place c_l, the probability density of $X(c_l)$ can be computed from:

$$P\{X(c_l) = x_l\} = \sum_{\boldsymbol{m}\in\mathcal{S}_d} \int_0^\infty \pi(\boldsymbol{m}, \boldsymbol{\sigma}(\mathbf{x}, \mathcal{P}_c/\{c_l\}))\, d(\mathcal{P}_c/\{c_l\}).$$

The computation of the flow rate across a continuous arc (the continuous throughput), requires a more detailed analysis. In fact, a problem arises when the fluid level reaches one of the boundary conditions (Eq. (1)). Consider the fluid place c_1 with fluid level x_1 in Figure 3(a). If $\alpha_1 + \alpha_2 < \beta_1 + \beta_2$, the fluid level as function of the time decreases linearly with rate $(\alpha_1+\alpha_2) - (\beta_1+\beta_2)$ as in Figure 3(b), until it reaches the boundary value $x_1 = 0$. From this point on, the potential rate of change of Equation (1) is set to 0 and place c_1 remains constantly empty. The equations of Section 3 are still completely defined, but, in order to compute the continuous throughput along the arcs, the individual flow rates along each arc must be redefined in such a way that the difference between input and output rates remains equal to 0. By renaming the new output flow rates β_1' and β_2', different semantics can be adopted (Alla and David, 1998) to redistribute the flow rate such that the following relation is verified: $\alpha_1 + \alpha_2 = \beta_1' + \beta_2'$. Under a *priority semantic*, the input flows are used to saturate the output flows in a predefined order. If for example the arc labeled β_1 has

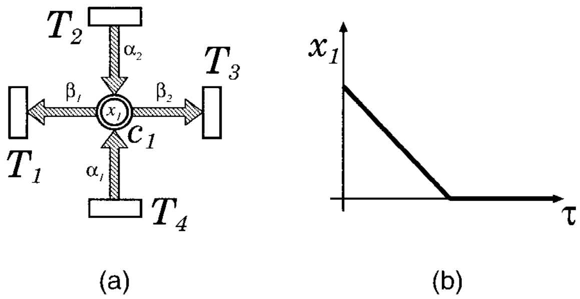


Figure 3. A fluid place with multiple inputs and outputs.

priority over the arc labeled β_2, then:

– if $\alpha_1 + \alpha_2 \leq \beta_1$, then $\beta'_1 = \alpha_1 + \alpha_2$ and $\beta'_2 = 0$.

– if $\beta_1 < \alpha_1 + \alpha_2 \leq \beta_1 + \beta_2$, then $\beta'_1 = \beta_1$ and $\beta'_2 = \alpha_1 + \alpha_2 - \beta_1$.

Under a *proportional semantic* instead, the input flow is partitioned in a way that keeps constant the ratio between the output flows, that is:

$$\beta'_i = \beta_i \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2}.$$

In order to respect these semantics, and the potential rate of change established in Equation (1), the individual flow rates $R((c_l, T_j), M)$ of each fluid arc must be changed to an actual flow rate $R^*((c_l, T_j), M)$ that differs from the original at the boundary points and reflects the chosen semantic. For example, with the proportional semantic defined above, $R^*((c_1, T_1), x)$ of the net defined in Figure 3(a), becomes:

$$R^*((c_1, T_1), x) = \begin{cases} \beta_1 & x > 0 \\ \beta_1 \frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} & x = 0 \end{cases}$$

With this consideration, the mean flow rate across a continuous arc that connects place c_l to transition T_j, can be defined as:

$$\Phi(c_l, T_j) = \sum_{\boldsymbol{m} \in \mathcal{E}(T_j)} \int_0^\infty \pi(\boldsymbol{m}, \boldsymbol{\sigma}(\boldsymbol{x}, \mathcal{P}_c)) R^*((c_l, T_j), \boldsymbol{m}, \boldsymbol{\sigma}(\boldsymbol{x}, \mathcal{P}_c)) d\mathcal{P}_c.$$

Mean fluid flow across continuous arcs that connects transitions to places, may be computed in a similar way.

The mean fluid flushed out by a flush-out arc that connects place c_l to transition T_j, can be computed as:

$$\Psi(c_l, T_j) = \sum_{\boldsymbol{m} \in \mathcal{E}(T_j)} \int_0^\infty x_l \, \pi(\boldsymbol{m}, \boldsymbol{\sigma}(\boldsymbol{x}, \mathcal{P}_c)) F(T_j, \boldsymbol{m}, \boldsymbol{\sigma}(\boldsymbol{x}, \mathcal{P}_c)) d\mathcal{P}_c.$$

4.2. Numerical Solution Methods

When both the fluid rates and the transition rates do not depend on the continuous part of the marking, even simple explicit time-domain discretisation schemes (as those described in (Horton et al., 1998)) provide high accuracy. When the flow and/or transition rates are marking dependent, these schemes may become inaccurate since the the error may accumulate step by step. In this case, more sophisticated discretisation techniques should be applied where both the fluid level and time are discretised using different stepsizes.

When the system has more than one fluid place, the size of the equidistant discretised state space grows geometrically with the number of continuous variables and setting the stepsizes small enough for accurate results makes the computation very time consuming. Finding a general method to solve FSPN models with more than one fluid place is an open problem. Possible directions for future investigation could be based on finite elements or finite volumes techniques (Patankar, 1980). Laplace transform methods (Cox, 1955) may be efficiently used only when the distributions associated with the process have rational Laplace transform. Yet it is possible to solve FSPN models using "ad hoc" discretisation schemes based on the understanding of the dynamics of the model as in (Bobbio et al., 1999).

A possible alternative is the simulative approach. Because of the mixed discrete and continuous state space, when the transition rates depend on the fluid level, the marking process is no longer time homogeneous, and simulation of FSPN poses some interesting challenges which are addressed in (Ciardo et al., 1999; Gribaudo and Sereno, 2000).

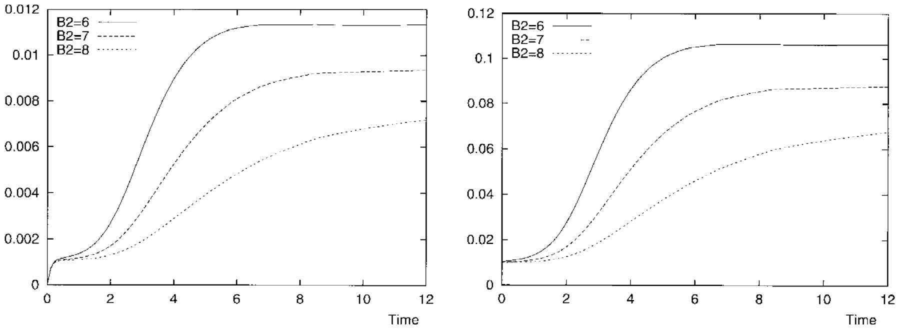
5. FSPN Formalism: An Illustrative Example

Some numerical results for the FSPN described in Example 1 are presented. The example exhibits two interesting features that motivates the use of FSPNs. The need of modeling a continuous quantity (the plant production) suggests the use of a formalism able to manage both discrete and continuous entities. On the other hand, the discarding action of the buffer when the system fails, due to the nature of the sterilisation process, motivates the use of a flush-out arc. The example was analysed using the following set of parameters:

$$\begin{gathered}\mu_1 = 0.05, \quad \mu_2 = 5, \quad \mu_4 = 10, \quad \beta = 1, \\ A_1 = 2, \quad B_1 = 5, \quad C_1 = 1, \\ A_2 = 0.2, \quad B_2 = 6, 7 \text{ and } 8, \quad C_2 = 0.01.\end{gathered}$$

Various performance measures have been evaluated. As an example of *discrete state measures*, the probability that the sterilisation process is stopped is depicted in Figure 4(a) as a function of the time. Since the sterilisation process is stopped when place p_4 is marked, the above measure was computed by the equation:

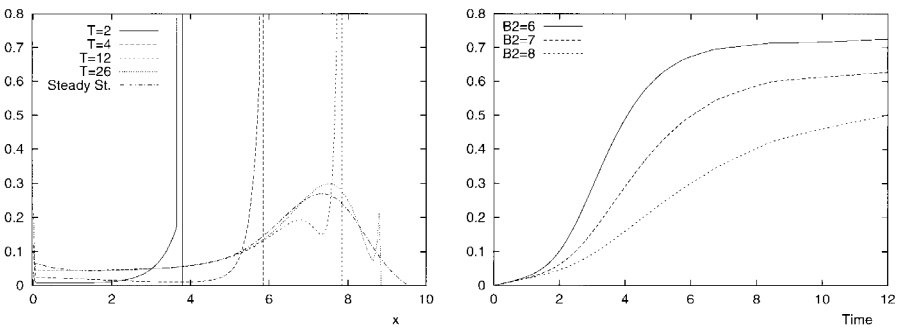
$$P\{\text{sterilisation stopped at time } \tau\} = \sum_{\boldsymbol{m}_i :\, m_4=1} \int_0^{\infty} \pi_i(\tau, x_1)\, dx_1.$$



(a) Crash probability

(b) Frequency of faults

Figure 4. Discrete state measure.



(a) Buffer distribution

(b) Mean quantity of product wasted

Figure 5. Fluid state measure

As an example of *throughput measures*, Figure 4(b) reports the throughput of transition T_3 (representing the frequency of faults) as a function of the time. Such a measure is obtained from:

$$\chi(T_3) = \sum_{\substack{T_3 \text{ is enabled in } \boldsymbol{m_i} \\ \boldsymbol{m_i} \in \mathcal{S}_d \text{ such that}}} \int_0^\infty \pi_i(\tau, x_1)\, \mu_3(x_1)\, dx_1.$$

The probability density function (*pdf*) of the distribution of the fluid level in place c_1 at different time instants and in steady state is depicted in Figure 5(a) for $B_2 = 7$. This measure, which is an example of a *fluid state measure*, represents the pdf of the accumulated reward (production) at time τ and is calculated from:

$$\sum_{\boldsymbol{m_i} \in \mathcal{S}_d} \pi_i(\tau, x_1).$$

The above equation is solved by means of a discretisation technique, and the continuous pdf curves are plotted by interpolating the discrete values. The peaks in the curves of 5(a) reflect the discontinuities of the pdf. The discontinuities at non-zero fluid levels represent the probability mass that the system does not leave the initial state up to the given time τ. The discontinuity at fluid level 0, instead, is the probability of having an empty buffer.

As a last example of *flow measures*, the throughput of the flush-out arc connecting place c_1 to transition T_3 is evaluated. This measure represents the mean quantity of wasted production due to system failures as a function of the time. This measure is plotted in Figure 5(b), and is computed from:

$$\Psi(c_1, T_3) = \sum_{\substack{\boldsymbol{m}_i \in \mathcal{S}_d \text{ such that} \\ T_3 \text{ is enabled in } \boldsymbol{m}_i}} \int_0^\infty x_1\, \pi_i(\tau, x_1)\, \mu_3(x_1)\, dx_1$$

The previous numerical results agree with the ones obtained with a FSPN simulator (Gribaudo and Sereno, 2000).

6. Conclusions

With respect to previous formulations, the paper has developed an augmented *FSPN* formalism by adding a novel primitive called flush-out arc that instantaneously resets the reward accumulated in a continuous place. This novel primitive makes it possible to extend the use of the formalism to applications in which the accumulation of a continuous quantity (reward, work, time) can be preempted and restarted.

Despite the increased modeling power and the major complexity of the augmented model, an analytical description of the stochastic marking process is feasible, and the paper has indicated a general procedure to automatically derive the integro-differential equations from the model specification. Moreover, a noticeable new feature offered by the presented *FSPN* formulation, is that the output performance measures related to the continuous part of the model (for instance the cdf of the accumulated reward) can be specified by the user at the net level as a function of the structural primitives. The above two points lead back the proposed model in the main stream of *PN* based model, where the solution complexity is hidden from the modeler who interacts only through a specification interface.

Numerical discretisation schemes have been analysed and implemented. These schemes are highly efficient with only one fluid place (one dimension in the continuous part of the state space). Multi–dimensional discretisation patterns, able to handle more than one fluid place at the time, are an open research topic to which further effort will be addressed. In parallel, a simulative solution has been explored (Gribaudo and Sereno, 2000).

Proof of Theorem 1

Here we put the proof for Equation (3). Extension to the case with more than one continuous place, can be obtained following the proposed scheme. Suppose that each time a flush-out occurs, the level of fluid place c_l is set to a random value according to a probability density function $b(x)$. We know that:

$$
\begin{aligned}
&\pi_i(\tau+\Delta\tau, x+r(i,x)\Delta\tau+o(\Delta\tau^2))\\
&= +\pi_i(\tau,x)Pr\{\text{‘not leave } \boldsymbol{m_i} \text{ in } \Delta\tau\text{'}\} + \sum_{\substack{\boldsymbol{m_i}\in\mathcal{S}_d\\ \boldsymbol{m_j}\neq\boldsymbol{m_i}}} \pi_j(\tau,x)Pr\left\{\begin{matrix}\text{jump to } \boldsymbol{m_i} \text{ / we came from } \boldsymbol{m_j}\\ \text{with } x \text{ unit of fluid, in } \Delta\tau\text{'}\end{matrix}\right\}\\
&\quad + Pr\left\{\begin{matrix}\text{‘set the fluid level to } x+o(\Delta\tau)/\\ \text{a flush-out occured in } \Delta\tau\text{'}\end{matrix}\right\} Pr\{\text{‘a flush-out occured in } \Delta\tau\text{'}\} + o(\Delta\tau^2)\\
&= \pi_i(\tau,x)\left(1-\sum_{\substack{\boldsymbol{m_j}\neq\boldsymbol{m_i}\\ \boldsymbol{m_j}\in\mathcal{S}_d}} q_{ij}(x,\emptyset)\Delta\tau - \sum_{\boldsymbol{m_j}\in\mathcal{S}_d} q_{ij}(x,\{c_l\})\Delta\tau\right)\\
&\quad + \sum_{\substack{\boldsymbol{m_j}\neq\boldsymbol{m_i}\\ \boldsymbol{m_j}\in\mathcal{S}_d}} \pi_j(\tau,x)q_{ji}(x,\emptyset)\Delta\tau\\
&\quad + b\,(x+o(\Delta\tau))\sum_{\boldsymbol{m_j}\in\mathcal{S}_d}\int_0^\infty \pi_j(\tau,x')Pr\left\{\begin{matrix}\text{jump to } \boldsymbol{m_i} \text{ with a flush out / we came}\\ \text{from } \boldsymbol{m_j} \text{ with } x' \text{ unit of fluid in } \Delta\tau\text{'}\end{matrix}\right\}dx'\\
&\quad + o(\Delta\tau^2)\\
&= \pi_i(\tau,x)\,(1-q_i(x)\Delta\tau) + \sum_{\substack{\boldsymbol{m_j}\neq\boldsymbol{m_i}\\ \boldsymbol{m_j}\in\mathcal{S}_d}} \pi_j(\tau,x)q_{ji}(x,\emptyset)\Delta\tau\\
&\quad + b\,(x+o(\Delta\tau))\sum_{\boldsymbol{m_j}\in\mathcal{S}_d}\int_0^\infty \pi_j(\tau,x')q_{ji}(x',\{c_l\})dx'\Delta\tau + o(\Delta\tau^2).
\end{aligned}
$$

Equation (2) has been used at the end of the previous derivation. If we rearrange the terms and divide by $\Delta\tau$, and observe that in our case $b(x) = \delta(x)$ (since after a flush-out the fluid level is always set to 0), we obtain:

$$
\begin{aligned}
&\frac{\pi_i(\tau+\Delta\tau, x+r(i,x)\Delta\tau+o(\Delta\tau^2)) - \pi_i(\tau,x)}{\Delta\tau}\\
&\quad = -\pi_i(\tau,x)q_i(x) + \sum_{\substack{\boldsymbol{m_j}\neq\boldsymbol{m_i}\\ \boldsymbol{m_j}\in\mathcal{S}_d}} \pi_j(\tau,x)q_{ji}(x,\emptyset)\\
&\qquad + \delta(x+o(\Delta\tau))\sum_{\boldsymbol{m_j}\in\mathcal{S}_d}\int_0^\infty \pi_j(\tau,x')q_{ji}(x',\{c_l\})dx' + o(\Delta\tau).
\end{aligned}
$$

If we take the limit $\Delta\tau \to 0$, remembering Equation (3) we obtain Equation (3).

Notes

1. Note that when the arcs are defined as a function we use uppercase symbols.

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