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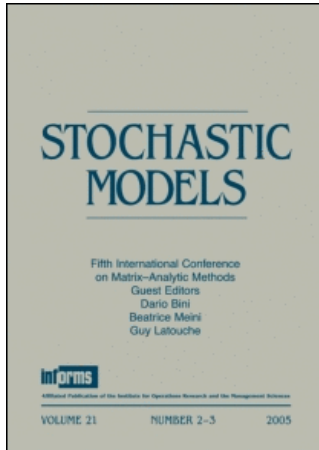
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MATCHING MORE THAN THREE MOMENTS WITH ACYCLIC PHASE TYPE DISTRIBUTIONS

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□ *This paper presents a matching procedure for generating an acyclic phase type distribution of order N given the first $2N - 1$ moments, if they are feasible. The matching procedure uses an iterative approach and, theoretically, it can be applied to match an arbitrary number of moments. The first step of the iterative procedure contains the solution of an equation of order N and the order is decreased by one in each consecutive step. Apart from these equations, the procedure makes use of explicit expressions. The practical applicability of the proposed procedure is limited by the numerical accuracy of the solution of these equations and the complexity of the involved expressions. We present examples for matching more than 10 moments with acyclic phase type distributions.*

Keywords Acyclic phase type distribution; Moment matching.

AMS Subject Classification Primary 60J20; Secondary 60E99.

1. INTRODUCTION

The applicability of phase type distributions in stochastic models of real systems is determined by the availability of appropriate phase type fitting procedures.¹ There are two main categories of phase type fitting algorithms: numerical optimization^[1,5] and procedures that make use of explicit expressions. Up to now, explicit expressions for matching only the first three moments of acyclic phase type (APH) distributions have been known^[2,7–9]. This paper presents an iterative

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¹A survey of fitting algorithms can be found in Reference^[6].

approach to match an arbitrary number of moments with acyclic phase type distributions.

The solution of matching an arbitrary number of moments with matrix exponential distributions, which is an even richer class of distributions containing the APH class, has been known for more than a decade; see Ref.^[10]. However, the applicability of this matching procedure is limited because the class of matrix exponential distributions and the properties of the matching method have the following disadvantages.

- The matrix exponential class contains the phase type and the acyclic phase type classes, but it also contains distributions which cannot be represented as the time to absorption in a Markov chain. These non-Markovian distributions might inhibit the use of the widely applied Markovian solution techniques.
- The matching procedure calculates a matrix for any set of moments without indicating if the matrix (together with the predefined initial and final vectors) corresponds to a matrix exponential distribution or not. When the procedure is called with the moments of a matrix exponential distribution of order N it results in a true matrix exponential distribution of order N , but when it is called with an invalid set of moments it results in a matrix which does not correspond to a real distribution. Unfortunately, it is hard to check if a given matrix corresponds to a real matrix exponential distribution; see Ref.^[4].

The analytical description of the moment bounds of matrix exponential distributions would avoid the second disadvantage, but they are not known beyond $N = 2$. The procedure presented in this paper avoids the above-mentioned disadvantages. APH matching results in a Markovian representation by definition and the feasibility of the solution is obvious (a solution with real, positive intensities and real probabilities between 0 and 1 is feasible).

With the procedure presented in this paper, matching the moments of an acyclic phase type distribution of order N (APH(N)) requires the solution of an equation of order N . As a consequence, the matching procedure has a symbolic solution up to $N = 4$ and numerical techniques are applicable for larger N . Here we present the symbolic solution for $N = 3$ (matching 5 moments). Because of the complexity of the symbolic solution for $N = 4$, we use a numerical technique for $N \geq 4$. The practical applicability of the proposed procedure is limited by the complexity of the involved expressions and by the numerical accuracy of the solution of the equation of order N . We present numerical results for matching more than 10 moments.

2. NOTATION AND THEORETICAL BACKGROUND

2.1. Normalized Moments

Based on the ordinary moments m_i we define the i th normalized moment as follows:

$$n_i = \frac{m_i}{m_{i-1}m_1}, \quad i \geq 1. \tag{1}$$

Note that throughout this paper we consider only non-defective distributions whose 0th moment is assumed to be 1. As a consequence the first normalized moment of the considered distributions is 1 (i.e., $m_0 = 1$ and $n_1 = 1$).

We introduce the normalized moments for two reasons.

- They simplify the problem of matching. With the use of normalized moments one can match the first moment and the normalized moments, n_i , $2 \leq i \leq N - 1$, separately.
- They simplify the obtained expressions, e.g., the i th normalized moment of an Erlang(N) distribution is $\frac{N+i-1}{N}$.

Based on (1), the ordinary moments can be calculated from the normalized moments using

$$m_i = n_i m_{i-1} m_1 = m_1^i \prod_{k=1}^i n_k. \tag{2}$$

2.2. Acyclic Phase Type Distributions

Definition 2.2.1. An acyclic phase type distribution of order N , denoted by APH(N), is the distribution of the time to absorption in an acyclic Markov chain with N transient states and an absorbing one.

Theorem 2.2.1. Any APH(N) distribution can be represented in the canonical form (CF) depicted in Figure 1 such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ ^[3].

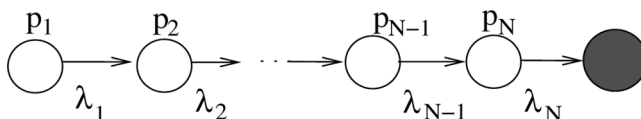


FIGURE 1 Canonical representation of APH(N) distributions.

Definition 2.2.2. The representation of an APH(N) in the structure of Figure 1 without any restrictions on the order of the intensities is called its series representation.

Both the series and the canonical representations of an APH(N) distribution are defined by two vectors $\underline{p} = \{p_1, \dots, p_N\}$ and $\underline{\lambda} = \{\lambda_1, \dots, \lambda_N\}$.

Corollary 2.2.1. The $\underline{\lambda}$ vectors of the different series representations of an APH(N) contains the same intensities. They differ only in the order of these intensities.

Proof. The corollary is a consequence of Cumani's result^[3], but it can be derived also from the fact that the eigenvalues of any triangular matrix are determined by the diagonal elements.

2.3. The $2N$ th and the $2N+1$ th Normalized Moments of APH(M) Distributions

As shown by Cumani^[3] an APH(N) distribution is characterized by $2N - 1$ parameters. According to Ref.^[10], the first $2N - 1$ moments of an APH(N) distribution characterize the distribution. As a consequence, one can calculate the higher moments of an APH(N) in terms of the first $2N - 1$ moments; see Ref.^[10]. This can be done as follows. For a given number of phases, N , the following two $N + 1 \times N + 1$ matrices of reduced moments, $r_i = m_i/i!$, are constructed

$$M_{2N} = \begin{vmatrix} r_0 & r_1 & \cdots & r_N \\ r_1 & r_2 & \cdots & r_{N+1} \\ \vdots & & & \vdots \\ r_N & r_{N+1} & \cdots & r_{2N} \end{vmatrix}, \quad M_{2N+1} = \begin{vmatrix} r_1 & r_2 & \cdots & r_{N+1} \\ r_2 & r_3 & \cdots & r_{N+2} \\ \vdots & & & \vdots \\ r_{N+1} & r_{N+2} & \cdots & r_{2N+1} \end{vmatrix} \quad (3)$$

where $r_0 = r_1 = 1$. Then r_{2N} can be obtained as a function of r_i , $0 \leq i \leq 2N - 1$ by solving the equation $\det(M_{2N}) = 0$. Similarly r_{2N+1} can be calculated from $\det(M_{2N+1}) = 0$.

It is straightforward to transform this result into the relation of the normalized moments

$$n_{2N} = \mathcal{F}_{2N}(n_2, \dots, n_{2N-1}), \quad n_{2N+1} = \mathcal{F}_{2N+1}(n_2, \dots, n_{2N}). \quad (4)$$

For $N = 2$, the second and third normalized moments determine all higher normalized moments. In this case, the fourth and fifth normalized

moments are

$$n_4 = \mathcal{F}_4(n_2, \dots, n_3) = \frac{2n_2(9n_2 + 2(n_3 - 6)n_3)}{3(n_2 - 2)n_3}, \tag{5}$$

$$n_5 = \mathcal{F}_5(n_2, \dots, n_4) = \frac{5n_3(4n_2(2n_3 - 3n_4) + 3n_4^2)}{6(2n_3 - 3n_2)n_4}. \tag{6}$$

For $N = 3$, n_2, n_3, n_4 , and n_5 determine all higher normalized moments. In this case

$$\begin{aligned} n_6 &= \mathcal{F}_6(n_2, \dots, n_5) \\ &= (n_3(25n_2(2n_2(16n_3^2 - 36n_3n_4 + 9n_4^2) - 3n_3(n_4 - 8)n_4^2)) \\ &\quad + n_3(120n_2(6n_2 + n_3(n_4 - 4) - 3n_4)n_4n_5 - 36(n_2 - 2)n_4^2n_5^2))/ \\ &\quad (10n_4(18n_2^2 + 6n_3n_4 + n_2n_3(4n_3 - 3(8 + n_4)))n_5), \end{aligned} \tag{7}$$

$$\begin{aligned} n_7 &= \mathcal{F}_7(n_2, \dots, n_6) \\ &= (7n_4(-8n_3n_5^2(3n_4(3n_5 - 5n_6) + 5n_6^2) \\ &\quad + 5n_2(12n_5^2n_6^2 + 12n_3n_5(6n_4n_5 - 5n_4n_6 - 4n_5n_6) \\ &\quad + n_3^2(75n_4^2 - 180n_4n_5 + 48n_5^2 + 80n_5n_6))))/ \\ &\quad (20n_5(5n_3(4n_2(2n_3 - 3n_4) + 3n_4^2) + 6(3n_2 - 2n_3)n_4n_5)n_6). \end{aligned} \tag{8}$$

Because of their complexity, we do not provide the expressions for higher order moments.

3. THE MATCHING PROCEDURE

It is straightforward to compute the normalized moments, $n_i, 2 \leq i \leq 2N - 1$, of an APH(N) distribution in terms of its $2N - 1$ parameters. The inverse, i.e., to determine the parameters given the normalized moments, is non-trivial. In this section we present an iterative approach which, at each step, determines two parameters of the APH(N) distribution.

The section starts with the building blocks and then describes the procedure itself. Section 3.1 investigates the effects of extending an APH($N - 1$) distribution with an additional phase. Based on properties of the extension, the problem of matching $2N - 2$ normalized moments with an APH(N) distribution can be reduced to the problem of matching $2N - 4$ normalized moments with an APH($N - 1$) distribution. This reduction step is described in section 3.2. The complete algorithm is presented in section 3.4.

3.1. The APH($N - 1$)-EXP Structure

The extension of an APH($N - 1$) distribution with an additional phase, as depicted in Figure 2, is characterized by two parameters, p and λ . The following theorem and its corollary describe the effect of the extension on the moments and normalized moments, respectively.

Theorem 3.1.1. *The moments of the resulting APH($N - 1$)-EXP distribution, m'_i , can be expressed in terms of the moments of the APH($N - 1$) distribution, m_i , and the two parameters of the extension (p and λ), using*

$$m'_i = i! \lambda^{-i} \left(1 + p \sum_{j=1}^i \frac{\lambda^j m_j}{j!} \right). \quad (9)$$

Proof. Let $f_L(s)$ be the Laplace transform of the APH($N - 1$) distribution.

The Laplace transform of the APH($N - 1$)-EXP distribution, $g_L(s)$, is

$$g_L(s) = p f_L(s) \frac{\lambda}{s + \lambda} + (1 - p) \frac{\lambda}{s + \lambda} = \frac{\lambda}{s + \lambda} (p f_L(s) + 1 - p). \quad (10)$$

To obtain its moments, we first take the i th derivative of (10)

$$\begin{aligned} \frac{d^i}{ds^i} g_L(s) &= \sum_{j=0}^i \binom{i}{j} \frac{d^{i-j}}{ds^{i-j}} \left(\frac{\lambda}{s + \lambda} \right) \frac{d^j}{ds^j} (p f_L(s) + 1 - p) \\ &= \sum_{j=1}^i \binom{i}{j} \lambda (i - j)! (-1)^{i-j} (s + \lambda)^{-i+j-1} p \frac{d^j}{ds^j} f_L(s) \\ &\quad + \lambda i! (-1)^i (s + \lambda)^{-i-1} (p f_L(s) + 1 - p) \end{aligned}$$

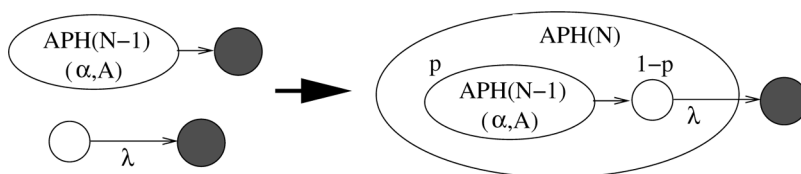


FIGURE 2 Realization of an APH(N) distribution by the composition of an APH($N - 1$) distribution and an additional phase.

then by the limit $s \rightarrow 0$

$$(-1)^i m'_i = \sum_{j=1}^i \binom{i}{j} (i-j)! (-1)^{i-j} \lambda^{-i+j} p (-1)^j m_j + i! (-1)^i \lambda^{-i}.$$

Multiplying both sides by $(-1)^i$ gives (9).

To simplify the following expressions and to eliminate the dependence on the first moment, we replace the parameters, p and λ , by

$$a = m_1 \lambda \quad \text{and} \quad b = ap. \tag{11}$$

Indeed $a = \frac{m_1}{1/\lambda}$ is the ratio of the means of the APH($N - 1$) and the EXP parts of the APH($N - 1$)-EXP distribution.

Corollary 3.1.1. *The normalized moments of the APH($N - 1$)-EXP distribution, n'_i , can be expressed in terms of the normalized moments of the APH($N - 1$) distribution, n_i , using*

$$n'_i = \frac{i \left(1 + b \sum_{j=1}^i \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}{(1+b) \left(1 + b \sum_{j=1}^{i-1} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)} \tag{12}$$

Proof. From the definition of the normalized moments, using (9) and (2) we have

$$n'_i = \frac{m'_i}{m'_{i-1} m'_1} = \frac{i! \lambda^{-i} \left(1 + p \sum_{j=1}^i \frac{\lambda^j}{j!} \prod_{k=1}^j n_k m_1^j \right)}{\left((i-1)! \lambda^{-i+1} \left(1 + p \sum_{j=1}^{i-1} \frac{\lambda^j}{j!} \prod_{k=1}^j n_k m_1^j \right) \right) \left(p m_1 + \lambda^{-1} \right)}.$$

Introducing the a and b parameters results in (12).

3.2. Reduction to One Phase Less

Given $2N - 2$ normalized moments, $n'_i, 2 \leq i \leq 2N - 1$, we aim to find $n_i, 2 \leq i \leq 2N - 3$, and a and b such that the extension of an APH($N - 1$) distribution whose normalized moments are n_i with an exponential phase according to Figure 2 with parameters a and b , results in an APH(N) distribution with normalized moments n'_i . To this end we derive the following two corollaries of Theorem 3.1.1.

Corollary 3.2.1. *The inverse function of (9), i.e., the moments of the APH($N - 1$) distribution as a function of the moments of the APH($N - 1$)-EXP distribution, have the following form*

$$m_i = \frac{\lambda m'_i - i m'_{i-1}}{\lambda p}. \quad (13)$$

Proof. We prove the corollary by induction. (13) holds for $i = 1$. Assuming that (13) holds for $j \in \{1, 2, \dots, i-1\}$ from (9) we have

$$\frac{\lambda^i}{i!} m'_i = 1 + p \sum_{j=1}^{i-1} \frac{\lambda^j}{j!} m'_j + p \frac{\lambda^i}{i!} m_i = 1 + p \sum_{j=1}^{i-1} \frac{\lambda^j}{j!} \frac{\lambda m'_j - j m'_{j-1}}{\lambda p} + p \frac{\lambda^i}{i!} m_i. \quad (14)$$

That is

$$\frac{\lambda^i}{i!} m'_i = 1 + \underbrace{\sum_{j=1}^{i-1} \left(\frac{\lambda^j m'_j}{j!} - \frac{\lambda^{j-1} m'_{j-1}}{(j-1)!} \right)}_{\frac{\lambda^{i-1} m'_{i-1}}{(i-1)!}} + p \frac{\lambda^i}{i!} m_i, \quad (15)$$

which gives (13) after some manipulation.

Corollary 3.2.2. *The normalized moments of the APH($N - 1$) distribution as a function of the normalized moments of the APH($N - 1$)-EXP distribution, have the following form*

$$n_i = \frac{n'_{i-1}(1+b)(n'_i(1+b) - i)}{a(n'_{i-1}(1+b) - (i-1))}, \quad i > 1. \quad (16)$$

Proof. From the definition of the normalized moments and (13) we have

$$\begin{aligned} n_i &= \frac{m_i}{m_{i-1} m_1} = \frac{\frac{\lambda m'_i - i m'_{i-1}}{\lambda p}}{\frac{\lambda m'_{i-1} - (i-1) m'_{i-2}}{\lambda p} \frac{a}{\lambda}} = \frac{(\lambda m'_i - i m'_{i-1}) \lambda}{(\lambda m'_{i-1} - (i-1) m'_{i-2}) a} \\ &= \frac{\left(\lambda \frac{m'_i}{m'_{i-1}} - i \right) \frac{m'_{i-1}}{m'_{i-2}} \lambda}{\left(\lambda \frac{m'_{i-1}}{m'_{i-2}} - (i-1) \right) a} = \frac{\left(\lambda n'_i \frac{1+b}{\lambda} - i \right) n'_{i-1} \frac{1+b}{\lambda} \lambda}{\left(\lambda n'_{i-1} \frac{1+b}{\lambda} - (i-1) \right) a}, \end{aligned} \quad (17)$$

where we applied $m'_i = (1+b)/\lambda$.

Based on (16), the $n_i, 2 \leq i \leq 2N - 3$, normalized moments of the APH($N - 1$) distribution are explicitly known as a function of $n'_i, 2 \leq i \leq$

$2N - 3$, and a and b . The two other unknowns, a and b , can be determined from the remaining two normalized moments n'_{2N-2} and n'_{2N-1} .

An equation to determine a is constructed by the following steps.

Step 1: Take (12) for $i = 2N - 2$ which results in

$$n'_{2N-2} = \frac{(2N - 2) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}{(1 + b) \left(1 + b \sum_{j=1}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}. \tag{18}$$

Step 2: In (18) substitute n_{2N-2} by $\mathcal{F}_{2N-2}(n_2, \dots, n_{2N-3})$, which can be computed as described in section 2.3. We have that

$$n'_{2N-2} = \frac{(2N - 2) \left(1 + b \sum_{j=1}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k + \frac{ba^{2N-3}}{(2N-2)!} \mathcal{F}_{2N-2}(n_2, \dots, n_{2N-3}) \prod_{k=1}^{2N-3} n_k \right)}{(1 + b) \left(1 + b \sum_{j=1}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}. \tag{19}$$

Step 3: Apply (16) to $n_i, 2 \leq i \leq 2N - 3$ in (19), i.e. substitute $n_i, 2 \leq i \leq 2N - 3$ by the right-hand side of (16).

Step 3 results in

$$n'_{2N-2} = (2N - 2) \left(1 + b + b \sum_{j=2}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \right) \tag{20}$$

$$+ \frac{ba^{2N-3}}{(2N - 2)!} \mathcal{F}_{2N-2} \left(\frac{n'_1(1 + b)(n'_2(1 + b) - 2)}{a(n'_1(1 + b) - 1)}, \dots, \frac{n'_{2N-4}(1 + b)(n'_{2N-3}(1 + b) - (2N - 3))}{a(n'_{2N-4}(1 + b) - (2N - 4))} \right) \tag{21}$$

$$\prod_{k=2}^{2N-3} \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \Bigg/ \tag{22}$$

$$\left((1 + b) \left(1 + b + b \sum_{j=2}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \right) \right). \tag{23}$$

Theorem 3.2.1. Equation (20–23) has two unknowns, a and b , and it is linear in a . Therefore, it can be rearranged to have an explicit expression for a as a function of $n'_i, 2 \leq i \leq 2N - 2$, and b .

In a similar manner, an equation to determine b is constructed by the following steps.

Step 1: Take (12) for $i = 2N - 1$ which results in

$$n'_{2N-1} = \frac{(2N - 1) \left(1 + b \sum_{j=1}^{2N-1} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}{(1 + b) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}. \tag{24}$$

Step 2: In (24) substitute n_{2N-1} by $\mathcal{F}_{2N-1}(n_2, \dots, n_{2N-2})$, which can be computed as described in section 2.3. We have that

$$n'_{2N-1} = \frac{(2N - 1) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k + \frac{ba^{2N-2}}{(2N-1)!} \mathcal{F}_{2N-1}(n_2, \dots, n_{2N-2}) \prod_{k=1}^{2N-2} n_k \right)}{(1 + b) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}. \tag{25}$$

Step 3: Substitute $n_i, 2 \leq i \leq 2N - 2$ by the right-hand side of (16). After Step 3 we have

$$n'_{2N-1} = (2N - 1) \left(1 + b + b \sum_{j=2}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \right) \tag{26}$$

$$+ \frac{ba^{2N-2}}{(2N - 1)!} \mathcal{F}_{2N-1} \left(\frac{n'_1(1 + b)(n'_2(1 + b) - 2)}{a(n'_1(1 + b) - 1)}, \dots, \frac{n'_{2N-3}(1 + b)(n'_{2N-2}(1 + b) - (2N - 2))}{a(n'_{2N-3}(1 + b) - (2N - 3))} \right) \tag{27}$$

$$\prod_{k=2}^{2N-2} \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \Big/ \tag{28}$$

$$\left((1 + b) \left(1 + b + b \sum_{j=2}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1 + b)(n'_k(1 + b) - k)}{a(n'_{k-1}(1 + b) - (k - 1))} \right) \right). \tag{29}$$

Theorem 3.2.2. Equation (26–29) has a single unknown, b , and it is of order N in b . Hence, it can be rearranged to have a polynomial equation of order N to determine b .

A consequence of Theorem 3.2.2 is that one can look for symbolic solution up to $N = 4$ (matching 7 moments). Fitting more than 7 moments requires the use of numerical techniques.

The proofs of Theorems 3.2.1 and 3.2.2 are in Appendix A. We demonstrate the procedure for obtaining $n_i, 2 \leq i \leq 2N - 3$, and a and b given $n'_i, 2 \leq i \leq 2N - 1$, for $N = 3$ in section 4.

3.3. Feasibility

The parameters of the APH($N - 1$)-EXP structure, a and b , are positive real numbers with the following bounds, $0 < a < \infty$ and $0 < b = ap \leq a$ (since $0 < p \leq 1$). $p = 0$ is not feasible because it results in an exponential distribution, and $a = 0$ is not feasible because it implies $m_1 = 0$ or $\lambda = 0$.

Theorem 3.3.1. A set of normalized moments $\{n'_2, \dots, n'_{2N-1}\}$ is feasible with an APH(N) distribution if and only if there exists a solution of the reduction step, $n_i, 2 \leq i \leq 2N - 3$, and a and b such that

- a and b are real numbers such that $0 < b (=ap) \leq a$,
- the normalized moments $\{n_2, \dots, n_{2N-3}\}$ are feasible with an APH($N - 1$) distribution.

Proof. The theorem is a consequence of Cumani’s seminal result (Theorem 2.2.1)^[3]. Since any APH(N) can be represented in canonical form, there is an APH(N) with the given moments if and only if there is an APH($N - 1$)-EXP structure with the given constraints.

The first condition of the theorem can be checked easily. The second condition has to be checked by repeated applications of the reduction step. In each step the number of moments is decreased by two. The whole procedure is described in the next section.

Theorem 3.3.1 has an important consequence. It allows us to check if a given set of moments is inside the moments bounds of the APH(N) class or not. Indeed, the procedure described hereinafter directly constructs an APH(N) distribution, if possible. To the best of our knowledge it is the first method to check this property for more than three moments.

Due to the simple stochastic interpretation of the APH distribution, it is easy to check if the result of the procedure is a proper APH distribution. This property is valuable with respect to the matrix exponential case where the matrix representation, calculated from the set of moments^[10], does not indicate the feasibility of the solution.

3.4. Iterative Approach of Moments Matching

Based on the reduction to one phase less presented in section 3.2, the following iterative algorithm generates an APH(N) distributions given the first $2N - 1$ moments.

1. We calculate the normalized moments, $n_i, 2 \leq i \leq 2N - 1$, based on the first $2N - 1$ ordinary moments, $m_i, 1 \leq i \leq 2N - 1$, according to (1).

2. The reduction step can be applied according to the tree scheme depicted in Figure 3. At level zero we have $n_i, 2 \leq i \leq 2N - 1$, i.e., the normalized moments. At the first level the reduction step gives at most N different solutions (Theorems 3.2.1 and 3.2.2). The reduction step can be applied again to any of these solutions leading to at most $N(N - 1)$ nodes at the second level. In order to avoid cumbersome notation, indices of the different solutions for b, a and n_i are indicated on the top of the nodes. At the last level there are at most $N!$ nodes.

3. Taking any path from the root to a leaf gives a sequence of values for b and a . This sequence can be used to construct an APH(N) distribution with normalized moments $n_i, 2 \leq i \leq 2N - 1$. We start with an APH(1) distribution with rate 1 and in each step the distribution is extended by one phase. The values for a and b are taken from bottom to top and are applied according to (11). Note that, as it will be illustrated later in this section, not all the paths lead to a proper APH(N) distribution.

4. Finally, we set the first ordinary moment (the mean) by properly scaling the intensities of the result.

All the routes provide a series form (with coefficients which are not necessarily feasible). Due to the fact that they solve equations (16) and (4), they all provide the given $2N - 1$ ordinary moments.

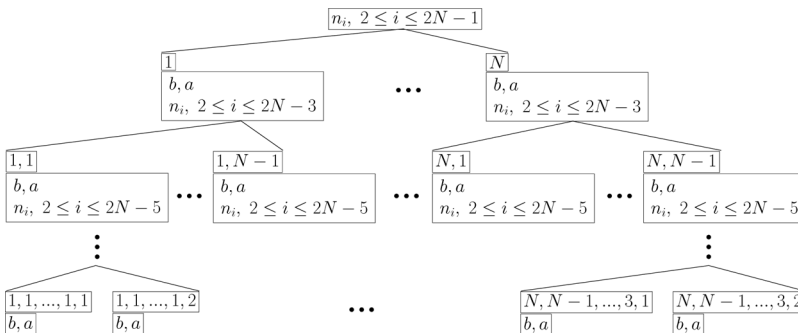


FIGURE 3 Tree scheme of the application of the reduction step.

Corollary 3.4.1. *The λ vectors of the $N!$ solutions contain the same intensity values with all possible permutations.*

Proof. The eigenvalues of the generator of the PH are determined by the first $2N - 1$ moments of the distribution^[10]. In the case of APH distributions, the eigenvalues are real and they determine the departure rates of the phases. The $N!$ solutions are obtained by the permutations of these eigenvalues.

Let $N = 3$, $m_1 = 1$, $m_2 = 42/25 \cong 1.68$, $m_3 = 61614/15625 \cong 3.94$, $m_4 = 4654584/390625 \cong 11.92$, $m_5 = 17249112/390625 \cong 44.16$. The normalized moments are $n_2 = 42/25 \cong 1.68$, $n_3 = 1467/625 \cong 2.35$, $n_4 = 86196/28525 \cong 3.02$, $n_5 = 26619/7183 \cong 3.71$. Application of the iterative procedure is summarized in Table 1. The first column gives the three different solutions for a, b, n_2 and n_3 at the first application of the reduction step (first level of the tree in Figure 3). Then we apply the reduction step to all of these three solutions. At this level the procedure results in two solutions for a and b . These solutions, which are the leaves of the tree depicted in Figure 3, are listed in the second column. In the first two columns the indices of the solutions are indicated as in Figure 3. The corresponding APH(3) distributions are given in the third column by the initial probability vector and the vector of intensities.

As indicated by Corollary 3.4.1, the application of the procedure provides all permutations of the same set of intensities. In this particular case, 4 permutations of the $3! = 6$ possible permutations result in a proper APH(3) distribution. These 4 different series representations correspond to the same APH(3) distribution, whose (unique) canonical representation is in the last row of the table.

The following corollary, whose proof is in Appendix 8, indicates how to choose among the different solutions.

TABLE 1 The $N!$ different solutions of the iterative procedure for $N = 3$

a, b, n_2, n_3	a, b	$\{p_1, p_2, p_3\}, \{\lambda_1, \lambda_2, \lambda_3\}$
1	1,1	$\left\{ \frac{3}{4}, -\frac{1}{8} \right\}, \left\{ \frac{-1}{9}, \frac{7}{9}, \frac{1}{3} \right\}, \left\{ \frac{25}{9}, \frac{25}{12}, \frac{25}{18} \right\}$
	1,2	$\left\{ \frac{4}{3}, \frac{1}{6} \right\}, \left\{ \frac{1}{12}, \frac{7}{12}, \frac{1}{3} \right\}, \left\{ \frac{25}{12}, \frac{25}{9}, \frac{25}{18} \right\}$
2	2,1	$\left\{ \frac{1}{2}, -\frac{1}{14} \right\}, \left\{ \frac{-1}{9}, \frac{8}{9}, \frac{2}{9} \right\}, \left\{ \frac{25}{9}, \frac{25}{18}, \frac{25}{12} \right\}$
	2,2	$\left\{ 2, \frac{6}{7} \right\}, \left\{ \frac{1}{3}, \frac{4}{9}, \frac{2}{9} \right\}, \left\{ \frac{25}{18}, \frac{25}{9}, \frac{25}{12} \right\}$
3	3,1	$\left\{ \frac{2}{3}, \frac{1}{15} \right\}, \left\{ \frac{1}{12}, \frac{3}{4}, \frac{1}{6} \right\}, \left\{ \frac{25}{12}, \frac{25}{18}, \frac{25}{9} \right\}$
	3,2	$\left\{ \frac{3}{2}, \frac{3}{5} \right\}, \left\{ \frac{1}{3}, \frac{1}{2}, \frac{1}{6} \right\}, \left\{ \frac{25}{18}, \frac{25}{12}, \frac{25}{9} \right\}$

Corollary 3.4.2. *If the given moment set is feasible, selecting the largest solution for b in each step of the iterative procedure results in an APH distribution in CF, i.e., with non-decreasing intensities. If the given moment set is not feasible, the solutions provided by the procedure are improper for an APH distribution.*

Corollary 3.4.2 allows us to reduce the $N!$ possibilities to one. This clearly simplifies the procedure. Furthermore, this solution is in CF which is a minimal unique representation of the APH distributions.

4. CONSTRUCTING AN APH(3) WITH GIVEN SECOND TO FIFTH NORMALIZED MOMENTS

We apply the iterative procedure for $N = 3$ with normalized moments n'_2, n'_3, n'_4 , and n'_5 . In order to construct an equation for a , as a first step, we take (12) for $i = 4$:

$$n'_4 = \frac{4\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{1}{24}a^3n_2n_3n_4\right)\right)}{(1+b)\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3\right)\right)}.$$

Then n_4 is substituted by $\mathcal{F}_4(n_2, n_3)$ (given in (5)) which results in

$$n'_4 = \frac{4\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{a^3n_2^2(9n_2+2(n_3-6)n_3)}{36(n_2-2)}\right)\right)}{(1+b)\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3\right)\right)}.$$

By applying (16) to n_2 and n_3 , and simplifying the right-hand side we obtain

$$\begin{aligned} n'_4 = & (2(9a(-2 + n'_2 + bn'_2))(4 + n'_2(-4 + (1+b)^2n'_2)) \\ & - 12abn'_2(-2 - b + (1+b)^2n'_2)n'_3 \\ & + 2b(1+b)n'_2(-6n'_3 + n'_2(9 + (1+b)n'_3(-3 + n'_3 + bn'_3)))))/ \\ & ((3b(1+b)n'_2(-2(1+b+ab) + (1+b)^2n'_2)n'_3)) \end{aligned} \quad (30)$$

from which we have an expression for a as a function of n'_2, n'_3, n'_4 and b .

In order to construct the third order equation for b , we take (12) for $i = 5$:

$$n'_5 = \frac{5\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{1}{24}a^3n_2n_3n_4 + \frac{1}{120}a^4n_2n_3n_4n_5\right)\right)}{(1+b)\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{1}{24}a^3n_2n_3n_4\right)\right)}.$$

Then n_5 is substituted by $\mathcal{F}_5(n_2, n_3, n_4)$ (given in (6)), which results in

$$n'_5 = \frac{5\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{1}{24}a^3n_2n_3n_4 + \frac{a^4n_2n_3^2(4n_2(2n_3-3n_4)+3n_4^2)}{144(2n_3-3n_2)}\right)\right)}{(1+b)\left(1 + b\left(1 + \frac{1}{2}an_2 + \frac{1}{6}a^2n_2n_3 + \frac{1}{24}a^3n_2n_3n_4\right)\right)}.$$

By applying (16) to $n_2, n_3,$ and $n_4,$ and simplifying the right-hand side we have

$$\begin{aligned} n'_5 = & ((5(-72n'_3n'_4 + 2n_2^2(4(-3 + n'_3 + bn'_3))(9 + (1+b)^2n_3^2) \\ & - 3(1+b)^2n'_3(-3 + 2(1+b)n'_3)n'_4) \\ & + 3n'_2n'_3(96 - 12bn'_4 + n'_3(-16(2+b) + 4(1+b)(2+b)n'_4 \\ & + b(1+b)^2n_4^2)))))) / \\ & ((6(1+b)n'_3(-12 - 3(1+b)^2n_2^2 + 2n'_2(6 + b(3 + n'_3 + bn'_3)))n'_4)) \end{aligned} \tag{31}$$

which provides a third order polynomial equation to determine b .

Solving (31) and substituting the solutions into (30) and (16) results in three solutions for n_2, n_3, a and b . According to Corollary 3.4.2, we can take that with the largest b , check the feasibility of a and b and look for an APH(2) with normalized moments n_2 and n_3 by applying the reduction step for $N = 2$.

5. IMPLEMENTATION AND NUMERICAL ACCURACY

The presented method is implemented in Mathematica, which allows us to check its properties by applying either symbolic calculations (with exact rational representation of the involved quantities) or numerical calculations with different precision. The experiments were carried out on a computer with 1.5GHz processor and 524MB RAM.

The exact rational arithmetic is applicable only to moment sets that correspond to solution vectors composed of rational numbers. We generated such moment sets by starting from an APH distribution with rational initial probabilities and intensities. Based on the obtained rational moments, the procedure calculated exactly the starting APH distribution for 8 phases (matching 15 moments) in 165 seconds.

We found that the standard floating point precision of Mathematica (16 digits) is usually sufficient up to $N = 5$, but the round-off error is already visible in the 8th digit of the normalized moments. Fitting higher order APH distributions requires higher numerical precision. For $N = 5$, using 32 digit accuracy floating point arithmetic, the computation

TABLE 2 Normalized moments and descriptors of APH distributions for Example 6.1

	n_2	n_3	n_4	n_5	$\underline{\mu}$	$\hat{\lambda}$
D1	2.5	10	34.16	52.23	$\{2.412 \times 10^{-3}, 9.975 \times 10^{-1}\}$	$\{9.352 \times 10^{-2}, 1.026\}$
D2	2.5	10	100	200	$\{3.349 \times 10^{-5}, 1.205 \times 10^{-1}, 8.794 \times 10^{-1}\}$	$\{2.436 \times 10^{-2}, 5.380 \times 10^{-1}, 1.291\}$
D3	2.5	10	50	200	$\{4.383 \times 10^{-7}, 2.884 \times 10^{-3}, 9.971 \times 10^{-1}\}$	$\{1.228 \times 10^{-2}, 1.024 \times 10^{-1}, 1.029\}$
D4	2.5	10	100	400	$\{9.187 \times 10^{-7}, 4.958 \times 10^{-3}, 9.950 \times 10^{-1}\}$	$\{1.041 \times 10^{-2}, 1.335 \times 10^{-1}, 1.038\}$
D5	2.5	10	50	400	$\{9.500 \times 10^{-9}, 2.575 \times 10^{-3}, 9.974 \times 10^{-1}\}$	$\{4.828 \times 10^{-3}, 9.654 \times 10^{-2}, 1.027\}$

time is approximately the same and the round-off error disappears. The computational complexity of matching higher order APH distributions increases exponentially with the order. The calculations for 5, 6, 7 and 8 phases takes about 2.5, 5.5, 32 and 165 seconds, respectively. For $N > 8$ we cannot perform the procedure because with the computer we use we are not able to provide the functions $\mathcal{F}_{2N}(n_2, \dots, n_{2N-1})$, and $\mathcal{F}_{2N+1}(n_2, \dots, n_{2N})$.

6. EXAMPLES

We demonstrate the properties of moments based APH distribution matching through some simple numerical examples.

Example 6.1. We construct APH distributions with $m_1 = 1$ and the sets of normalized moments listed in Table 2. Distribution D1 is the result of matching normalized moments $n_2 = 2.5$ and $n_3 = 10$ with an APH(2). All higher order moments of this distribution are determined by n_2 and n_3 . All other cases are obtained by APH(3) distributions. Figure 4 presents the body and the tail of the pdf of the distributions. The queue length distribution of the M/APH/1 queue with $\rho = 0.8$ utilization is provided in Figure 5. Even if the second and third normalized moments are identical, the distributions and the associated queue length distributions can differ significantly. In particular, we can recognize the difference between the role of the odd and the even moments. Comparing, for example, D2 and D4 ($\{n_2, n_3\} = \{100, 200\}$ and $\{n_2, n_3\} = \{100, 400\}$) indicates that a higher fifth moment results in a longer tail for the pdf and also for the queue length distribution. Instead, comparing D2 and D3 ($\{n_2, n_3\} = \{100, 200\}$ and $\{n_2, n_3\} = \{50, 200\}$) indicates that a higher fourth moment results in a shorter tail.

Example 6.2. We match an APH distribution with $m_1 = 1$ to sets of normalized moments listed in Table 3. Similar to Example 6.1, the first

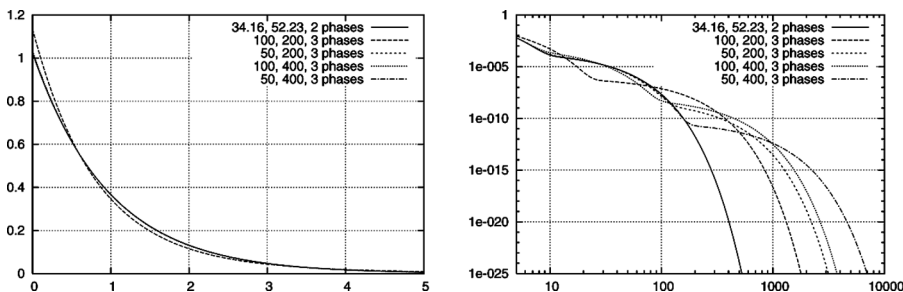


FIGURE 4 The body and the tail of the pdf of the distributions D1–D5.

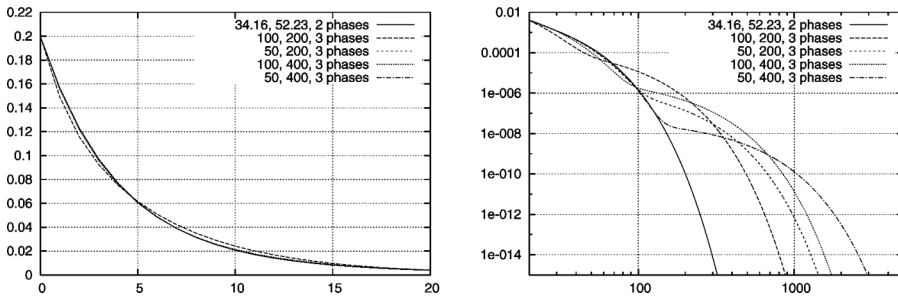


FIGURE 5 The body and the tail of the queue length distribution of the M/APH/1 queue for distributions D1–D5.

TABLE 3 Normalized moments and descriptors of APH distributions of Example 6.2

	n_2	n_3	n_4	n_5	n_6	n_7
D6	2.5	10	50	200	429.46	560.28
D7	2.5	10	50	200	1000	2000
D8	2.5	10	50	200	600	2000
D9	2.5	10	50	200	1000	2800
D10	2.5	10	50	200	600	2800

set of moments of this example (D6) can be realized with an APH(3) distribution. The other cases require 4 phases. The pdf and the queue length distribution of the M/APH/1 queue with $\rho = 0.8$ utilization are provided in Figures 6 and 7, respectively. The different behaviour of the odd and the even moments appears also in this case.

Example 6.3. We match APH distributions to a distribution with Pareto tail whose pdf is given by

$$f(t) = \begin{cases} \alpha B^{-1} e^{-\frac{\alpha}{B}t} & \text{for } t \leq B, \\ \alpha B^\alpha e^{-\alpha} t^{-(\alpha+1)} & \text{for } t > B, \end{cases}$$

where $\alpha = 3.5$, $B = 4$ and the distribution is truncated at 10^4 to ensure finite higher moments. The matching is performed based on the distribution's first 3, 5 or 7 moments.

The pdf and the resulting queue length distributions are depicted in Figures 8 and 9, respectively. The figures show that the approximate APH(3) and APH(4) distributions (matching 5 and 7 moments) captures the slowly decaying tail behaviour of the Pareto-like distribution.

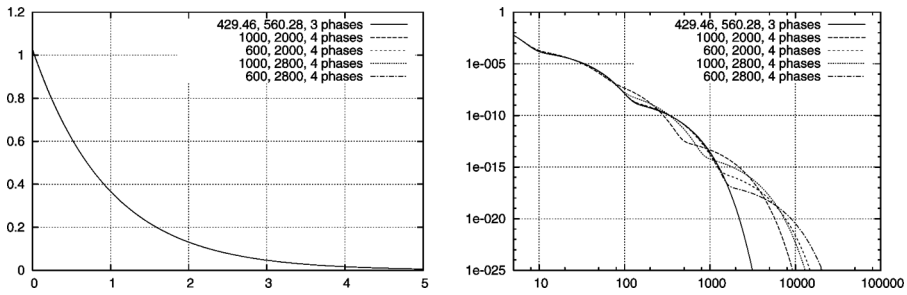


FIGURE 6 The body and the tail of the pdf of the distributions D6–D10.

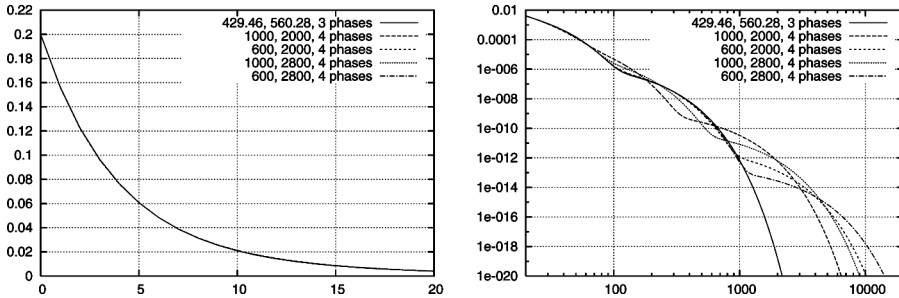


FIGURE 7 The body and the tail of the queue length distribution of the M/APH/1 queue for distributions D6-D10.

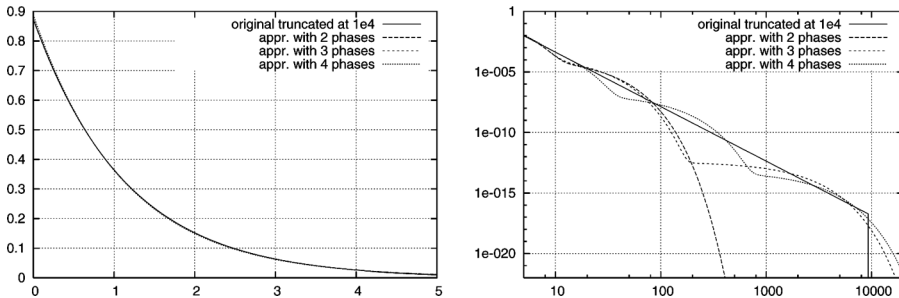


FIGURE 8 The pdf of the Pareto-like and the matching APH distributions.

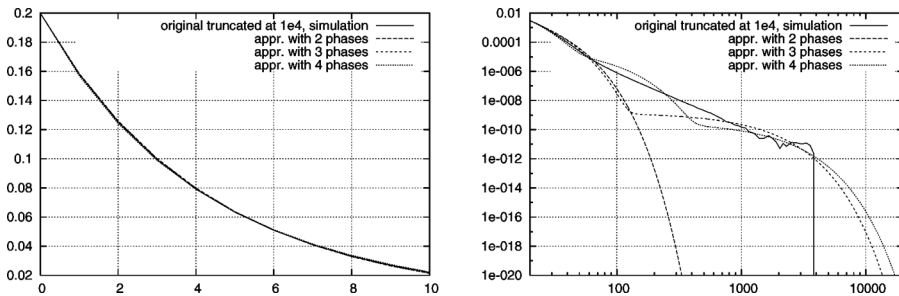


FIGURE 9 The queue length distribution of the M/•/1 queue with the Pareto-like and the matching APH distributions.

Example 6.4. The normalized moments of the APH(5) distribution with

$$\underline{p} = \left\{ \frac{7}{50}, \frac{2}{25}, \frac{11}{50}, \frac{9}{25}, \frac{1}{5} \right\} \quad \text{and} \quad \underline{\lambda} = \left\{ \frac{6}{5}, \frac{3}{2}, \frac{21}{5}, \frac{11}{2}, \frac{33}{5} \right\}$$

are

$$\begin{aligned} n_2 &= \frac{591837100}{236021769}, & n_3 &= \frac{127015173550}{30307977891}, \\ n_4 &= \frac{73234144433000}{13008894074991}, & n_5 &= \frac{23269903941935150}{3375288482772537}, \\ n_6 &= \frac{19293234891794731100}{2383303561732998063}, & n_7 &= \frac{9169582135866851564950}{988006558808808179631}, \\ n_8 &= \frac{12639577877112590464966000}{1207476774457049490791373}, \\ n_9 &= \frac{125792074401828042303577430}{10787879718115595961848481} \end{aligned}$$

and its first moment is 5121/7700. If performed symbolically, i.e., the involved quantities are calculated exactly by rational fractions, the procedure returns an APH(5) distribution with exact rational fractions in 2.5 seconds. This distribution is the canonical representation of the original one and naturally has exactly the same moments. When performed numerically, with accuracy set to 16 effective digits (it is the normal machine precision), the procedure returns an APH(5) distribution whose normalized moments differ from the original normalized moments at most by 10^{-7} . Using 32 effective digits the error disappears.

Example 6.5. The normalized moments of the APH(6) distribution with

$$\underline{p} = \left\{ \frac{7}{50}, \frac{2}{25}, \frac{11}{50}, \frac{9}{25}, \frac{3}{25}, \frac{2}{25} \right\} \quad \text{and} \quad \underline{\lambda} = \left\{ \frac{6}{5}, \frac{3}{2}, \frac{16}{5}, \frac{21}{5}, \frac{11}{2}, \frac{33}{5} \right\}$$

are

$$\begin{aligned} n_2 &= \frac{13910465950}{7071632649}, & n_3 &= \frac{24046145958125}{7798485420889}, \\ n_4 &= \frac{4466727822468564}{1078460027763523}, & n_5 &= \frac{1435435363741658131625}{281715407081136714339}, \\ n_6 &= \frac{1927035881489971461789950}{321893509448339352700643}, \\ n_7 &= \frac{7403583591151809030412389625}{1080334855880907800908681769}, \end{aligned}$$

$$\begin{aligned}
 n_8 &= \frac{16466099579460034164965782200920}{2134592759762499691866751133949}, \\
 n_9 &= \frac{98971954701929165646426693349164975}{11539030932796105441953896021849713}, \\
 n_{10} &= \frac{349390344841677602778873382799288523250}{36990438163330352563133155216939245523}, \\
 n_{11} &= \frac{3638345221897932321274699737112766471711375}{352575387225254335805805592556886837427947}
 \end{aligned}$$

and its first moment is 28031/30800. With symbolic computations, the procedure returns an APH(6) distribution with exact rational fractions in 5.5 seconds. When performed numerically with low accuracy (16 or 32 effective digits) the procedure fails. With 48 effective digits the algorithm returns an APH(6) whose normalized moments are equal to the original normalized moments (with respect to the first 48 digits) and whose parameters differ from the parameters of the original APH(6) at most by 10^{-15} .

Example 6.6. The normalized moments of the APH(7) distribution with

$$\underline{p} = \left\{ \frac{7}{50}, \frac{2}{25}, \frac{3}{100}, \frac{9}{25}, \frac{3}{25}, \frac{2}{25}, \frac{19}{100} \right\} \quad \text{and} \quad \underline{\lambda} = \left\{ \frac{6}{5}, \frac{3}{2}, \frac{16}{5}, \frac{21}{5}, \frac{11}{2}, \frac{33}{5}, 7 \right\}$$

are

$$\begin{aligned}
 n_2 &= \frac{231370003400}{106993755801}, & n_3 &= \frac{22323601720100}{6638675152819}, \\
 n_4 &= \frac{6126228628514858000}{1387385281818168081}, & n_5 &= \frac{5357085754374288325900}{1001941629079290768471}, \\
 n_6 &= \frac{36371002969685395640867400}{5840991310566917790378547}, \\
 n_7 &= \frac{140685558226672418118570735100}{19828197833968538714553476121}, \\
 n_8 &= \frac{1571218174089671270601023873684000}{197220451758798519832141580916321}, \\
 n_9 &= \frac{378861818429988484204018121271493140}{42828657793879781911860359004846893}, \\
 n_{10} &= \frac{20103343383358824457998073326051014657000}{2065422032443846719910835390829735576681}, \\
 n_{11} &= \frac{69887740898933010255011787166095371032412900}{6575783517353288121386711786877960843290043},
 \end{aligned}$$

$$n_{12} = \frac{79867798947162511649089160314693417481753706000}{6927336412212148097395181990982615081615523387},$$

$$n_{13} = \frac{18049786717034286902032065423855354281464368290060}{1451370953767661688772523069431994564714675304383}$$

and its first moment is 109033/123200. As in the previous example, with symbolic computations the procedure returns an APH(7) distribution with exact rational fractions in 32 seconds. With low accuracy, the procedure returns incorrect values, while with higher accuracy the memory requirement of the computation is over the available amount of memory (524MB RAM).

Example 6.7. If the input of the procedure is the set of normalized moments of the APH(8) distribution with

$$\underline{p} = \left\{ \frac{7}{50}, \frac{2}{25}, \frac{3}{100}, \frac{9}{25}, \frac{3}{25}, \frac{2}{25}, \frac{11}{100}, \frac{2}{25} \right\} \quad \text{and}$$

$$\underline{\lambda} = \left\{ \frac{6}{5}, \frac{3}{2}, \frac{16}{5}, \frac{21}{5}, \frac{11}{2}, \frac{33}{5}, 7, \frac{36}{5} \right\}$$

then symbolic computations returns the original APH(8) in 165 seconds. As in the case of the APH(7) distribution, numerical computations fail either for low accuracy or for too high memory requirement.

7. CONCLUSION

The paper presents a moment matching method for the APH(N) class. According to the authors present knowledge, this is the first procedure that provides a “Markovian” distribution based on more than 3 moments, which has practical importance in applied PH fitting.

The procedure can provide symbolic results up to order 4 and numeric results for higher orders. The practical applicability of the numerical procedure is limited by floating point errors and the complexity of the involved expressions. According to our experience, the procedure becomes instable around order 5–6 with standard floating point arithmetic and requires the use of higher numerical precision.

The procedure always provides a solution when the given $2N - 1$ moments are the moments of an APH(N) distribution (apart from numerical errors in the case of numerical computation). In the future, this will allow us to investigate the symbolic/numeric moment bounds of the APH(N) class.

APPENDIX A: PROOFS OF THEOREMS 9 AND 10

We start with discussing properties of the determinant of the matrices M_{2N} and M_{2N+1} given in (3). It follows from the definition of the determinant that

$$\begin{aligned} \det(M_{2N}) &= \sum_{i=1}^{N+1} r_{N+i-1} \det_{i,N+1}(M_{2N}) \\ &= r_{2N} \det(M_{2N-2}) + \sum_{i=1}^N r_{N+i-1} \det_{i,N+1}(M_{2N}), \end{aligned}$$

where $\det_{i,j}(M_{2N})$ is the (signed) subdeterminant of the element i, j . We have that

$$r_{2N} = - \frac{\sum_{i=1}^N r_{N+i-1} \det_{i,N+1}(M_{2N})}{\det(M_{2N-2})}. \tag{32}$$

and, similarly, one can obtain

$$r_{2N+1} = - \frac{\sum_{i=1}^N r_{N+i} \det_{i,N+1}(M_{2N+1})}{\det(M_{2N-1})}. \tag{33}$$

Corollary A1. *The determinant of the $n \times n$ matrix,*

$$\begin{vmatrix} r_{a_1+1} & r_{a_2+1} & \cdots & r_{a_n+1} \\ r_{a_1+2} & r_{a_2+2} & \cdots & r_{a_n+2} \\ \vdots & & & \vdots \\ r_{a_1+n} & r_{a_2+n} & \cdots & r_{a_n+n} \end{vmatrix}, \tag{34}$$

is a sum of products $r_{i_1}^{p_1} r_{i_2}^{p_2} \cdots r_{i_m}^{p_m}$, where each product has the following properties

$$\sum_{j=1}^m p_j = n, \quad \text{and} \quad \sum_{j=1}^m i_j p_j = \sum_{j=1}^n a_j + \frac{n(n+1)}{2} \tag{35}$$

Proof. The corollary holds for $n = 1$. Assuming that the corollary holds for $k > 1$ we evaluate the determinant of the matrix of (34) for $n = k + 1$ using the subdeterminants of the last row and we get (35) for $k + 1$.

Corollary A1 has the following consequences. The terms of $\det(M_{2N})$, $r_{i_1}^{p_1} r_{i_2}^{p_2} \cdots r_{i_m}^{p_m}$, as well as the terms of the numerator of (32) satisfy $\sum_{j=1}^m p_j = N + 1$, and $\sum_{j=1}^m i_j p_j = N(N + 1)$. The terms of $\det(M_{2N-2})$, which give the the denominator of (32), satisfy $\sum_{j=1}^m p_j = N$, and $\sum_{j=1}^m i_j p_j = N(N - 1)$.

The terms of $\det(M_{2N+1})$ and the terms of the numerator of (33) satisfy $\sum_{j=1}^m p_j = N + 1$, and $\sum_{j=1}^m i_j p_j = (N + 1)^2$. The terms of $\det(M_{2N-1})$, which give the denominator of (33) satisfy $\sum_{j=1}^m p_j = N$, and $\sum_{j=1}^m i_j p_j = N^2$.

As an example, for $N = 2$, (33) results in $r_5 = \frac{-r_3^3 + 2r_2 r_3 r_4 - r_1 r_4^2}{r_2^2 - r_1 r_3}$, where for any product in the numerator we have $\sum_{j=1}^m p_j = 3$, and $\sum_{j=1}^m i_j p_j = 9$, while for the denominator we have $\sum_{j=1}^m p_j = 2$, and $\sum_{j=1}^m i_j p_j = 4$.

From (32) and (33) one can obtain $\mathcal{F}_{2N}(n_2, \dots, n_{2N-1})$ and $\mathcal{F}_{2N+1}(n_2, \dots, n_{2N})$ using $r_0 = 1, r_i = \frac{1}{i!} \prod_{j=1}^i n_j$, for $1 \leq i \leq 2N - 1$. Based on the above discussed properties of r_{2N} and r_{2N+1} , we have the following properties of $\mathcal{F}_{2N}(n_2, \dots, n_{2N-1})$ and $\mathcal{F}_{2N+1}(n_2, \dots, n_{2N})$.

Property A1. $\mathcal{F}_{2N}(n_2, \dots, n_{2N-1})$ is a fraction of sums of products of normalized moments. For any product of the numerator, $n_{i_1}^{p_1} n_{i_2}^{p_2} \dots n_{i_n}^{p_n}$, there exists a constant K such that $K \leq \sum_{j=1}^n p_j \leq K + 1$. For any product of the denominator, $n_{i_1}^{p_1} n_{i_2}^{p_2} \dots n_{i_n}^{p_n}$, we have $K - 1 \leq \sum_{j=1}^n p_j \leq K$.

As an example, consider (5) in which the sum of the exponents of any product of the numerator (denominator) is either two or three (one or two).

Property A2. $\mathcal{F}_{2N+1}(n_2, \dots, n_{2N})$ is a fraction of sums of products of normalized moments. For any product of the numerator $n_{i_1}^{p_1} n_{i_2}^{p_2} \dots n_{i_n}^{p_n}$, there exists a constant K such that $\sum_{j=1}^n p_j = K$. For any product of the denominator, $n_{i_1}^{p_1} n_{i_2}^{p_2} \dots n_{i_n}^{p_n}$, we have $\sum_{j=1}^n p_j = K - 1$.

As an example, consider (6) in which the sum of the exponents of any product of the numerator (denominator) is three (two).

The above two properties allows us to determine the structure of the two equations that are used to determine a and b .

Proof of Theorem 3.2.1. According to Property A1, $\mathcal{F}_{2N-2}(n_2, \dots, n_{2N-3})$ is of the form

$$\frac{\sum_i \prod_{j=2}^{2N-3} n_j^{e_{i,j}}}{\sum_k \prod_{l=2}^{2N-3} n_l^{f_{k,l}}}. \tag{36}$$

The actual values of $e_{i,j}$ and $f_{k,l}$ depend on N and can be derived easily based on $\mathcal{F}_{2N-2}(n_2, \dots, n_{2N-3})$. From Property A1 we know also that there exists a constant, K such that

$$K \leq \sum_{j=2}^{2N-2} e_{i,j} \leq K + 1, \quad \forall i, \quad \text{and} \quad K - 1 \leq \sum_{l=2}^{2N-2} f_{k,l} \leq K, \quad \forall k.$$

By applying (36) in (20–23) we obtain

$$n'_{2N-2} = (2N - 2) \left(1 + b + b \sum_{j=2}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1+b)(n'_k(1+b) - k)}{a(n'_{k-1}(1+b) - (k-1))} \right) \tag{37}$$

$$+ \frac{ba^{2N-3}}{(2N - 2)!} \frac{\sum_i \prod_{j=2}^{2N-3} \left(\frac{n'_{j-1}(1+b)(n'_j(1+b)-j)}{a(n'_{j-1}(1+b)-(j-1))} \right)^{e_{i,j}}}{\sum_k \prod_{l=2}^{2N-3} \left(\frac{n'_{l-1}(1+b)(n'_l(1+b)-l)}{a(n'_{l-1}(1+b)-(l-1))} \right)^{f_{k,l}}} \times \prod_{k=2}^{2N-3} \frac{n'_{k-1}(1+b)(n'_k(1+b) - k)}{a(n'_{k-1}(1+b) - (k-1))} \Bigg) / \tag{38}$$

$$\left((1+b) \left(1 + b + b \sum_{j=2}^{2N-3} \frac{a^{j-1}}{j!} \prod_{k=2}^j \frac{n'_{k-1}(1+b)(n'_k(1+b) - k)}{a(n'_{k-1}(1+b) - (k-1))} \right) \right). \tag{39}$$

In the above equation, parts (37) and (39) do not depend on a . Part (38) instead depends on a in such a way that the whole equation (37–39) can be rearranged into the form

$$n'_{2N-2} = \frac{C_1 + a \frac{C_2}{a^{\frac{1}{K}} + a^{\frac{1}{K+1}}} + \frac{C_3}{a^{\frac{1}{K-1}} + a^{\frac{1}{K}}}}{C_6} = \frac{a(C_2 + C_1 C_4) + C_3 + C_1 C_5}{a C_4 C_6 + C_5 C_6} \tag{40}$$

where the constants, $C_i, 1 \leq i \leq 6$ are independent of a . It is easy to see that (40) can be rearranged to an equation which is linear in a .

Proof of Theorem 3.2.2. Step 2 of the procedure to construct the equation for b leads to

$$n'_{2N-1} = \frac{(2N - 1) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k + \frac{ba^{2N-2}}{(2N-1)!} \mathcal{F}_{2N-1}(n_2, \dots, n_{2N-2}) \prod_{k=1}^{2N-2} n_k \right)}{(1+b) \left(1 + b \sum_{j=1}^{2N-2} \frac{a^{j-1}}{j!} \prod_{k=1}^j n_k \right)}.$$

After application of Step 3, by similar steps—as in the proof of Theorem 3.2.1 and considering Property A1, A2, we obtain an equation of

the form

$$n'_{2N-1} = \frac{D_1(1+b)^{2N-2} + b(1+b)^{2N-2} \frac{\sum_i \frac{\prod_{j=2}^{2N-2} (n'_j(1+b)-j)^{g_{i,j}}}{D_{2,i} b^{\sum_{j=2}^{2N-2} g_{i,j}}}}{\sum_k \frac{\prod_{l=2}^{2N-2} (n'_l(1+b)-l)^{h_{k,l}}}{D_{3,k} b^{\sum_{l=2}^{2N-2} h_{k,l}}}}}{D_4(1+b)^{2N-1}} \quad (41)$$

where the constants, $D_1, D_{2,i}, D_{3,k}, D_4$ are independent of b and a . The actual values of $g_{i,j}$ and $h_{k,l}$ depend on N and can be derived easily based on $\mathcal{F}_{2N-2}(n_2, \dots, n_{2N-3})$. From Property A2 we know also that

$$\sum_{j=2}^{2N-2} g_{i,j} \leq N, \quad \forall i, \quad \text{and} \quad \sum_{l=2}^{2N-2} h_{k,l} \leq N-1, \quad \forall k.$$

From (41) by cancelling common factors and putting terms in the two sums over a common denominator we have

$$n'_{2N-1} = \frac{D_1 + b \frac{P_1(b,N)}{b^N}}{D_4(1+b)} = \frac{D_1 + \frac{P_1(b,N)}{P_2(b,N-1)}}{D_4(1+b)} = \frac{D_1 P_2(b, N-1) + P_1(b, N)}{D_4(1+b) P_2(b, N-1)} \quad (42)$$

where $P_i(b, l)$ denotes a polynomial of b of degree l . It is easy to see that from (41) we can obtain a polynomial equation for b of order N .

8. APPENDIX B: SELECTING THE SOLUTION OF INTEREST

To be unique, the canonical representation of an acyclic phase type distributions requires that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. The following theorem allows us to select the solution of b , which results in an APH(N) distribution in CF, i.e., with non-decreasing intensities.

Theorem B1. *Among the APH($N-1$)-EXP representations of an APH(N) distribution, the largest b value is associated with the one where the intensity of the additional phase is the largest intensity.*

Proof. Starting from an APH*($N-1$)-EXP representation of an APH(N) distribution where the intensity of the additional phase (λ^*) is not the largest intensity, we apply an equivalent transformation into an APH($N-1$)-EXP structure such that the intensity of the additional phase becomes the largest intensity ($\hat{\lambda}$). The steps of the transformation are depicted in Figure 10. In the first step, we replace the APH*($N-1$) distribution with its CF (Fig. 10b). Consequently $\tilde{\lambda}_{N-1} = \hat{\lambda}$ is the

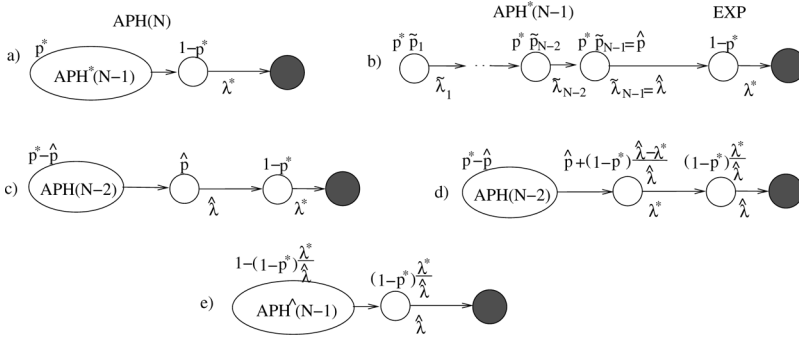


FIGURE 10 Equivalent transformation of APH(N - 1)-EXP forms.

largest intensity of the APH(N) distribution. The obtained structure is decomposed into an APH(N - 2) part and two additional exponential phases (Fig. 10c). After that, the last two phases are interchanged such that the APH(N) distribution remains the same (Fig. 10d). This transformation is due to the fact that

$$\begin{aligned} & \hat{p} \frac{\hat{\lambda}}{s + \hat{\lambda}} \frac{\lambda^*}{s + \lambda^*} + (1 - p^*) \frac{\lambda^*}{s + \lambda^*} \\ &= \left(\hat{p} + (1 - p^*) \frac{\hat{\lambda} - \lambda^*}{\hat{\lambda}} \right) \frac{\lambda^*}{s + \lambda^*} \frac{\hat{\lambda}}{s + \hat{\lambda}} + (1 - p^*) \frac{\lambda^*}{\hat{\lambda}} \frac{\hat{\lambda}}{s + \hat{\lambda}}. \end{aligned}$$

The b parameter of the original APH*(N - 1)-EXP representation, b^* (Fig. 10a), and the final APH^(N - 1)-EXP representation, \hat{b} (Fig. 10e), are

$$b^* = p^* m_1^* \lambda^* \quad \text{and} \quad \hat{b} = \left(1 - (1 - p^*) \frac{\lambda^*}{\hat{\lambda}} \right) \hat{m}_1 \hat{\lambda},$$

where m_1^* is the mean of APH*(N - 1) and \hat{m}_1 is the mean of APH^(N - 1).

$$\begin{aligned} m_1^* &= \sum_{i=1}^{N-1} \tilde{p}_i \sum_{j=i}^{N-1} \frac{1}{\tilde{\lambda}_j} = (1 - \tilde{p}_{N-1}) m'_1 + \frac{1}{\tilde{\lambda}_{N-1}} \\ &= \left(1 - \frac{\hat{p}}{p^*} \right) m'_1 + \frac{1}{\hat{\lambda}}, \\ \hat{m}_1 &= \frac{p^* - \hat{p}}{1 - (1 - p^*) \frac{\lambda^*}{\hat{\lambda}}} m'_1 + \frac{1}{\hat{\lambda}}, \end{aligned}$$

where m'_1 is the mean of the APH($N - 2$) and $\hat{p} = p^* \tilde{p}_{N-1}$. Finally, substituting m'_1 and \hat{m}_1 results:

$$\begin{aligned} \hat{b} - b^* &= \left(\hat{\lambda}(p^* - \hat{p})m'_1 + 1 - (1 - p^*)\frac{\lambda^*}{\hat{\lambda}} \right) - \left(\lambda^*(p^* - \hat{p})m'_1 + p^*\frac{\lambda^*}{\hat{\lambda}} \right) \\ &= \underbrace{(\hat{\lambda} - \lambda^*)}_{>0} \underbrace{(p^* - \hat{p})}_{p^*(1 - \tilde{p}_{N-1}) > 0} m'_1 + \underbrace{1 - \frac{\lambda^*}{\hat{\lambda}}}_{>0} + \underbrace{p^*\frac{\lambda^*}{\hat{\lambda}} - p^*\frac{\lambda^*}{\hat{\lambda}}}_0 > 0. \end{aligned}$$

Corollary 3.4.2 is a consequence of Theorems 3.3.1 and B1, and Corollary 3.4.1.

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REFERENCES

1. Asmussen, S.; Nerman, O. Fitting phase-type distributions via the EM algorithm. In *Proceedings: Symposium i Advent Statistik, Copenhagen, 1991*, 335–346.
2. Bobbio, A.; Horváth, A.; Telek, M. Matching three moments with minimal acyclic phase type distributions. *Stochastic Models* **2005**, *21*, 303–326.
3. Cumani, A. On the canonical representation of homogeneous Markov processes modelling failure-time distributions. *Microelectronics and Reliability* **1982**, *22*, 583–602.
4. Fackrell, M. Characterization of Matrix-Exponential Distributions. PhD thesis, The University of Adelaide, School of Applied Mathematics, 2003.
5. Horváth, A.; Telek, M. PhFit: A general purpose phase type fitting tool. In *Tools 2002*; Springer: New York, 2002, 82–91.
6. Lang, A.; Arthur, J.L. Parameter approximation for phase-type distributions. In *Matrix-Analytic Methods in Stochastic Models*; Lecture Notes in Pure and Applied Mathematics. Chakravarty, S.R.; Alfa, A.S., Eds.; Marcel Dekker, Inc. 1996, 151–206.
7. Osogami, T.; Harchol-Balter, M. A closed form solution for mapping general distributions to minimal PH distributions. In *International Conference on Performance Tools – TOOLS 2003*. Springer: New York, 2003, 200–217.
8. Osogami, T.; Harchol-Balter, M. Necessary and sufficient conditions for representing general distributions by Coxians. In *International Conference on Performance Tools – TOOLS 2003*. Springer: New York, 182–199.
9. Telek, M.; Heindl, A. Matching moments for acyclic discrete and continuous phase-type distributions of second order. *International Journal of Simulation Systems, Science & Technology* **2002**, *3* (3–4), 47–57.
10. van de Liefvoort, A. The moment problem for continuous distributions (Technical report WP-CM-1990-02). University of Missouri: Kansas City, 1990.