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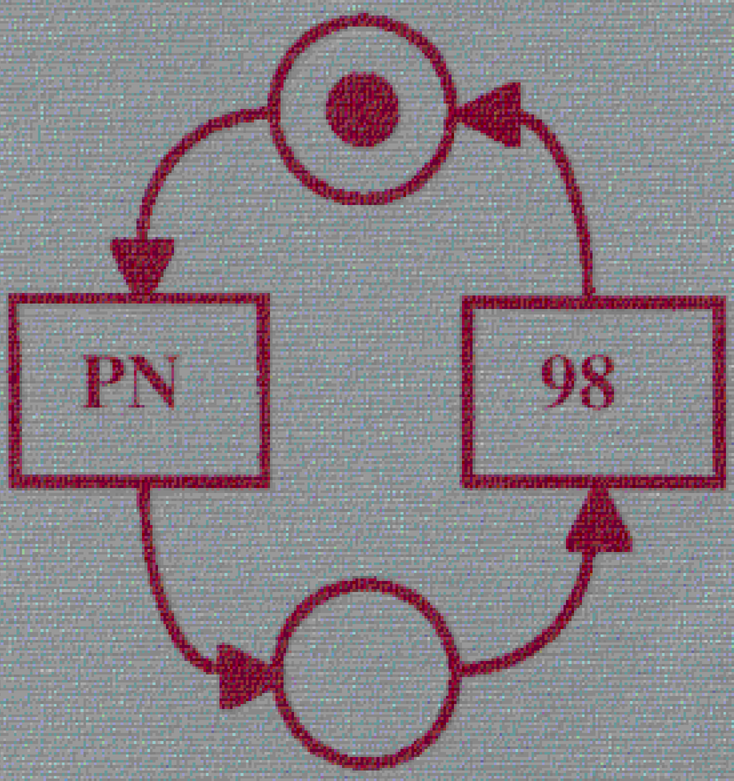

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On the Use of Structural Petri Net Analysis for Studying Product Form Equilibrium Distributions of Queueing Networks with Blocking

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Abstract. In this paper we investigate some relations between the Petri net formalism and the queueing networks with blocking. This type of queueing network models are used to represent systems with finite capacity resource constraints, such as production, communication and computer systems. Various blocking mechanisms have been defined in the literature to represent the different behaviours of real systems with limited resources.
We show that the representation of these queueing networks by means of Generalized Stochastic Petri Nets offers the possibility of using results developed within the Petri net framework. In particular, we investigate product form equilibrium distributions for queueing networks with blocking by means of structural Petri net results. More precisely, we use the notion of implicit places. With this concept we characterise a class of queueing networks with blocking having interesting properties. For each queueing network of this class there exists another model with the same performance measures and exhibiting product form equilibrium distribution.

1 Introduction

Queueing networks and Stochastic Petri nets are well known formalisms used to represent and analyse production, communication and computer systems and have been proved to be powerful tools for performance analysis and prediction. These formalisms have been developed with different purposes. Historically the queueing networks represent one of the first modeling paradigms proposed for performance analysis. The literature on this topic is full of results that allow to define the queueing network formalism one of the most used performance analysis tool.

One of the most important analytical results developed for calculating the equilibrium distribution describing the number of items at nodes in a performance model is the so called *product form* equilibrium distribution, introduced by Jackson [10], and nowadays found for a rather wide class of queueing models (see for instance [4] and other extensions). The main advantage of these product form distributions is their simplicity which makes them easy to use for computational issues as well as for theoretical reflections on performance models involving

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congestion as a consequence of queueing. However, practical performance models seldom satisfy the product form conditions. Nevertheless, results obtained via the theoretical product form distributions are used for practical applications since these results are found to be robust, that is models that violate the product form conditions are often found to behave in a way that is "qualitatively similar" to a product form counterpart. Also, various approximation and bounding techniques are based on product form results.

On the other hand the Stochastic Petri nets have been developed starting from the untimed Petri net formalism. A Stochastic Petri net is a Petri net in which a random variable, characterised by a negative exponential distribution function, is associated to any transition of the net. In this paper we consider the *Generalized Stochastic Petri nets (GSPN)* [1]. GSPNs are obtained by allowing transitions to belong to two different classes: immediate transitions and timed transitions.

In the literature there are many effort for the investigation of the relations between queueing networks and stochastic Petri nets. In several cases results developed within the framework of the queueing networks have been imported into the field of the stochastic Petri nets. Examples of this cross-fertilisation are, for instance, the product form solution, the approximate methods, the bounding techniques, and so on. In any of these techniques a result originally developed for queueing networks has been adapted for stochastic Petri nets. In all the cases the features of the Petri net formalism have been used for the exploitation of the potentialities of the methods.

In this paper we propose a different approach. We will use results developed in the framework of untimed Petri net formalism for the investigation of queueing network properties. We focus our attention on queueing networks with blocking. Queueing networks with limited capacity queues (FC-QNs) are used to represent systems with finite capacity resources and with resource constraints. Various blocking mechanisms have been defined in the literature to represent the different behaviours of real systems with limited resources (see [2,3,14,15] for details on these mechanisms).

In [9], a technique that allows to represent FC-QNs by means of GSPNs has been proposed. In this paper we use the Petri net representation of FC-QNs to derive new results for this class of queueing networks.

We investigate the utilisation of the *implicit places* theory for studying product form equilibrium distributions of FC-QNs. In particular using this notion we characterise a class of FC-QNs such that for each element of this class there exists a model with the same performance measures and with product form equilibrium distributions. For some of these FC-QNs the result that will presented in this paper are novel, in the sense that with the proposed method we characterise new product form cases.

These new results are also important because product form equilibrium distributions represent the starting points for studying non-product form models. Many "ad hoc" techniques have appeared in the literature in which the non-product form model is "transformed" into a product form one.

The balance of the paper is as follows. Section 2 provides some basic concepts of the FC-QNs and of the GSPNs. Section 3 reviews the definitions of the pertinent blocking mechanisms and their descriptions by means of GSPNs. Section 4 contains the main contribution of this paper. In this section we describe the utilisation of implicit places for studying product form distributions of FC-QNs. Finally, Section 5 presents some concluding remarks and direction for future work.

2 Queueing Networks with Finite Capacity and Generalized Stochastic Petri Nets: Definitions and Notation

Consider a closed queueing network with M finite capacity service centers (or nodes), and N customers in the network. The customers behaviour between nodes of the network is described by the routing matrix $\boldsymbol{P} = ||p_{ij}||$ $(1 \leq i, j \leq M)$, where p_{ij} denotes the probability that a job leaving node i tries to enter node j. In FC-QNs additional constrains on the number of customers are included to represent different types of resource constrains in real systems. This can be represented in the network by a maximum queue length constraint for a single node. We denote with b_i the maximum queue length admitted at node i (i.e., the buffer size).

Generalized Stochastic Petri Notation. We recall the basic notation on timed and untimed Petri nets that we are using in the paper. More comprehensive presentations of these concepts can be found in [1,13,18].

A *Generalized Stochastic Petri net* is a five-tuple $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, W, Q, \boldsymbol{m_0} \rangle$, where

$\mathcal{P}$	is the set of places;
$\mathcal{T}$	is the set of transitions;
$W : (\mathcal{P} \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{P}) \rightarrow \mathbb{N}$	defines the weighted flow relation;
$Q : \mathcal{T} \rightarrow \mathbb{R}^+$	is a function that associates rates of negative exponential distribution to timed transitions and weights to immediate transitions;
$\boldsymbol{m_0}$	is the initial marking of the GSPN.

With $\boldsymbol{Pre}$ and $\boldsymbol{C}$ we denote respectively the precondition and the incidence matrices. The row of $\boldsymbol{C}$ corresponding to place p_i is denoted by $\boldsymbol{C}[p_i, \cdot]$, while the column corresponding to transition t_j is denoted by $\boldsymbol{C}[\cdot, t_j]$.

3 Blocking Mechanisms and their GSPN Interpretation

In [9], a technique that allows to represent FC-QNs by means of GSPNs has been proposed. In this section we review the definitions only of the blocking mechanisms that will be used in this paper and their descriptions by means of GSPNs, in particular we only consider the following blocking mechanisms: *Blocking After Service*, and *Blocking Before Service*. Interested readers can find the description of other blocking mechanisms in [2,3,14].

For each node i we use $\mathsf{PREV}(i)$ to denote the set of nodes j such that $p_{ji} > 0$.

Blocking Before Service (BBS). In this blocking mechanism a customer at node i declares its destination node j before it starts receiving its service. If node j is full, node i becomes blocked. When a departure occurs from the destination node j, node i is unblocked and its server starts serving the customer. If the destination node j becomes full during the service of a customer at node i, the service is interrupted and node i is blocked. The service is resumed from the interruption point as soon as a space becomes available at the destination node. As discussed in [14], two different subcategories can be introduced depending on whether the server can be used to hold a customer when the node is blocked:
Blocking Before Service - Server Occupied (BBS-SO). In this case the server of a blocked node is used to hold a customer.
Blocking Before Service - Server Not Occupied (BBS-SNO). A server of a blocked node cannot be used to hold a customer. In this blocking mechanism, if a node i has a buffer capacity b_i, when it becomes blocked, its capacity must be decrease to $b_i - 1$. This type of blocking can only be implemented in some special topology network. In particular it cannot be implemented in a position in which its upstreams nodes may become full due to an arrival of a customer from a different node.
The distinction between BBS-SO and BBS-SNO blocking mechanisms is meaningful when modeling different types of systems. For example, in communication networks, a server corresponds to a communication channel. If there is no space in the downstream node, then the message cannot be transmitted. Furthermore, the channel itself cannot be used to store messages due to physical constraints of the channel. On the other hand, BBS-SO blocking arises if the service facility can be used to hold the blocked customer. BBS-SO blocking has been used to model manufacturing systems, terminal concentrators, mass storage systems, disk-to-tape backup systems, window flow control mechanisms, and communication systems (for further details see [14] and the references therein).

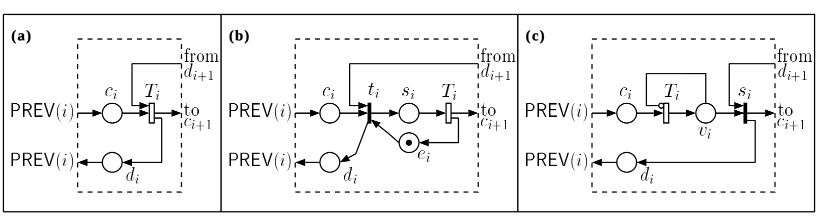

Fig. 1. GSPN subnets representing nodes with BBS-SO (a), BBS-SNO (b), and BAS (c) blocking mechanisms.

GSPN Subnets of a BBS-SO Node. In Figure 1(a) we present the GSPN subnets modeling a node with BBS-SO mechanism that has only one destination node. If the buffer of the destination node is full, transition T_i is blocked. When a departure occurs from the destination node, transition T_i is unblocked and starts serving the customers. The number of tokens in place c_i represents the number of customers in the node i, while the buffer capacity is given by the sum of tokens in place c_i and place d_i. The throughput of the node is given by the throughput of transition T_i.

GSPN Subnets of a BBS-SNO Node. Figure 1(b) shows a BBS-SNO node having one possible destination node. In this GSPN subnet place s_i represents the position in front of the server and hence it can be used as storage room only if the node is not blocked. When the buffer of the destination node is full the immediate transition t_i cannot be enabled because place d_{i+1} is empty. When a departure occurs from the destination node transition t_i can be enabled and hence the position in front of the server becomes again available. The actual capacity of node i is represented by the number of tokens in places c_i, d_i, e_i, and s_i. In the initial marking we must have that $b_i = \boldsymbol{m}[c_i] + \boldsymbol{m}[d_i] + \boldsymbol{m}[e_i] + \boldsymbol{m}[s_i]$ and $\boldsymbol{m}[s_i] + \boldsymbol{m}[e_i] = 1$. The average number of customers in the node is given by the sum of the average number of tokens in place c_i and in place s_i.

Blocking After Service (BAS). This blocking mechanism works as follows: if a customer attempts to enter a full capacity node j upon completion of service at node i, it is forced to wait in node i, until it is allowed to enter destination node j. The server node i stops servicing customers (it is blocked) until destination node j releases a customer. The node i service will be resumed as soon as a departure occurs from node j. At that time the customer waiting in node i immediately moves to node j.

GSPN Subnet of a BAS Node. Figure 1(c) shows a BAS node having one possible destination node. The place c_i represents the queue while transition T_i represents the server of node i. A customer receives its service and reaches place v_i, if the buffer of the destination (place d_{i+1}) is full the customer waits in place v_i. In this case the transition T_i is blocked (inhibitor arc from place v_i to transition T_i). When a position in the buffer of the node $i+1$ is available the customer moves immediately towards its destination and the service of transition T_i is resumed. The place d_i records the free positions in the buffer of node i. The capacity of node i is given by $\boldsymbol{m}[c_i] + \boldsymbol{m}[v_i] + \boldsymbol{m}[d_i]$.

4 Structural Petri Net Results for Product Form Analysis

In this section we present the main contribution of this paper: we show that the structural analysis can be useful for the study of the product form solution of FC-QNs. In the following we first give an interpretation of a known result for FC-QNs and then we discuss some new product form cases using the same arguments.

Theorem 1. (From [3]) *A homogeneous closed cyclic queueing network with exponential service centers, load independent service rates, and with*

$$N \geq \sum_{i=1}^{M} b_i - \min_{j=1}^{M} b_j \tag{1}$$

has product form equilibrium distribution under BBS-SO blocking mechanism.

The proof of the previous theorem has been obtained using the concept of *holes* that has been introduced by Gordon and Newell [8]. Since the capacity of node i is b_i, let us assume that this node consists of b_i cells. If there are n_i customer at node i, then n_i cells are occupied and $b_i - n_i$ are empty. We may say that these empty cells are occupied by holes. Then the total number of holes in the network is equal to $\sum_{i=1}^{M} b_i - N$. As the customers move sequentially through the cyclic network, the holes execute a counter sequential motion since each movement of customer from the i-th node to the $(i+1)$-th node corresponds to the movement of a hole in the opposite direction (from the $(i+1)$-th node to the the i-th node). It is then shown that these two networks are *dual*. That is, if a customer (hole) at node i is blocked in one system, then node $i+1$ has no holes (customers) in its dual. Let (b_i, μ_i) be the capacity and the service rate of node i and $\{(b_1, \mu_1), \ldots, (b_M, \mu_M)\}$ be a cyclic network with N customers. Then its dual is $\{(b_1, \mu_M), (b_M, \mu_{M-1}), \ldots, (b_2, \mu_1)\}$ with $\sum_{i=1}^{M} b_i - N$ customers. Let $\pi(\boldsymbol{n})$ and $\pi^D(\boldsymbol{n})$ be the steady state equilibrium probabilities of the cyclic network and its dual, respectively, where $\boldsymbol{n} = [n_1, n_2, \ldots, n_M]$ is the state of the network with n_i being the number of customers at node i. Then for all the feasible states, we have $\pi([n_1, n_2, \ldots, n_M]) = \pi^D([b_1 - n_1, b_M - n_M, \ldots, b_2 - n_2])$. We note that if the number of customers in the network is such that no node can be empty, then the dual network is a non-blocking network, i.e., the number of holes is less than or equal to the minimum node capacity, and hence the network has product form equilibrium distribution. Inequality (1) ensures exactly this condition, i.e., in a cyclic FC-QN satisfying this inequality no node can be empty.
If we consider the GSPN representation of a closed cyclic queueing network with BBS-SO blocking mechanisms, we can say that Inequality (1) ensures the places corresponding to the queues (the c_i-s of the GSPN of Figure 2) are always marked and hence they cannot block the movement of the holes, i.e., these places never restrict the firing of their output transitions and then they can be removed without affecting the behaviour of the GSPN. In the Petri net literature places that behave in this manner are called *implicit places.*

Definition 1. (From [18]) *Let $\mathcal{S} = \langle \mathcal{P}, \mathcal{T}, W, \boldsymbol{m_0} \rangle$ a Petri net and $\mathcal{S}' = \langle \mathcal{P}', \mathcal{T}, W' \boldsymbol{m_0'} \rangle$ the net resulting from removing place p from $\mathcal{S}$. The place p is an* implicit place *if the removing of p preserves all the firing sequences of the original Petri net. The Petri net $\mathcal{S}'$ is obtained from $\mathcal{S}$ by removing place p.*

A place is implicit depending on the initial marking of the Petri net. Places which can be implicit for any initial marking are said to be *structurally implicit places.*

Definition 2. (From [7]) *Given a Petri net $\mathcal{N}_p = \langle \mathcal{P}, \mathcal{T}, W \rangle$, the place p (with $p \in \mathcal{P}$) is structurally implicit iff $\forall\, \boldsymbol{m_0'}$ of $\mathcal{N}'$ (the net without place p), there exists an $\boldsymbol{m_0}[p]$ such that p is an implicit place in $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$.*

A structurally implicit place p may become implicit for any initial marking of the places $\mathcal{P} \setminus \{p\}$ if we have the freedom to select an adequate initial marking for it.
The following result allows to recognise whether a place is structurally implicit.

Theorem 2. (From [7]) *Let $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, W \rangle$ be a Petri net. A place $p \in \mathcal{P}$ is structurally implicit iff there exists a subset $\mathcal{I}_p \subseteq \mathcal{P} \setminus \{p\}$ such that $\boldsymbol{C}[p, \cdot] \geq \sum_{q \in \mathcal{I}_p} y_q \cdot \boldsymbol{C}[q, \cdot]$, where y_q is a nonnegative rational number (i.e., $\exists\, \boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{y}[p] = 0$ such that $\boldsymbol{y} \cdot \boldsymbol{C} \leq \boldsymbol{C}[p, \cdot]$ and $\mathcal{I}_p = \{q \ : q \in \mathcal{P}$ such that $\boldsymbol{y}[q] > 0\}$).*

Next result allows to compute the initial marking such that the structurally implicit place becomes implicit. This result is obtained using the Linear Programming technique.

Theorem 3. (From [7]) *Let $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, W \rangle$ with initial marking $\boldsymbol{m_0}$. A structurally implicit place p of $\mathcal{N}$, with initial marking $\boldsymbol{m_0}[p]$, is an implicit place if $\boldsymbol{m_0}[p] \geq z$, where z is the optimal value of the following linear programming problem:*

$$
\begin{aligned}
z = \min\ & \boldsymbol{y} \cdot \boldsymbol{m_0} + h \\
s.\ t.\ & \boldsymbol{y} \cdot \boldsymbol{C} \leq \boldsymbol{C}[p, \cdot] \\
& \boldsymbol{y} \cdot \boldsymbol{Pre}[\cdot, t] + h \geq \boldsymbol{Pre}[p, t] \quad \forall\, t \in p^{\bullet} \\
& \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{y}[p] = 0.
\end{aligned} \tag{2}
$$

Now we illustrate the previous results concerning implicit places in the case of GSPNs representing closed cyclic queueing networks with BBS-SO blocking mechanisms.

Let us consider a closed cyclic queueing network with M nodes and BBS-SO blocking mechanisms, Figure 2 shows its GSPN representation.

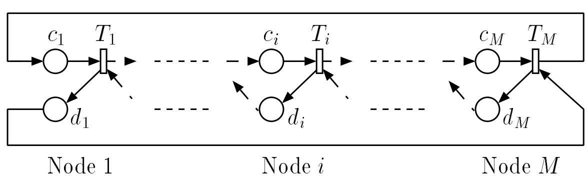

Fig. 2. The GSPN modeling a cyclic FC-QN with M BBS-SO nodes.

The incidence and the preconditions matrices, for this Petri net, are:

$$
\boldsymbol{C} = \begin{array}{c|cccccccc}
 & T_1 & \cdots & T_{i-1} & T_i & \cdots & T_{M-1} & T_M \\
\hline
c_1 & -1 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
d_1 & 1 & \cdots & 0 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_i & 0 & \cdots & 1 & -1 & \cdots & 0 & 0 \\
d_1 & 0 & \cdots & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_M & 0 & \cdots & 0 & 0 & \cdots & 1 & -1 \\
d_M & 0 & \cdots & 0 & 0 & \cdots & -1 & 1
\end{array}
\qquad
\boldsymbol{Pre} = \begin{array}{c|cccccccc}
 & T_1 & \cdots & T_{i-1} & T_i & \cdots & T_{M-1} & T_M \\
\hline
c_1 & -1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_i & 0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\
d_1 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_M & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\
d_M & 0 & \cdots & 0 & 0 & \cdots & -1 & 0
\end{array}
$$

From the incidence matrix $\boldsymbol{C}$ we can observe that for any place c_i (with $1 \leq i \leq M$) $\boldsymbol{C}[c_i, \cdot] = \sum_{j\,=\,1,\ j \neq i}^{M} \boldsymbol{C}[d_j, \cdot]$. It follows that any place c_i is structurally

implicit and the corresponding vector $\boldsymbol{y}$ has the components equal to 1 in the positions corresponding to places d_j (with $j \neq i$) and all the others are equal to 0.
The initial marking of place c_i such that it becomes implicit can be computed using Theorem 3. We can see that from vectors $\boldsymbol{y}$-s and from the structure of the GSPN the linear programming problem (2) has a simple interpretation.
The second inequality of (2), that is, $\boldsymbol{y} \cdot \boldsymbol{C} \leq \boldsymbol{C}[c_i, \cdot]$, derives from Theorem 2 and defines the vector $\boldsymbol{y}$.
Furthermore, let be $T_i \in c_i^\bullet$, we have that $\boldsymbol{y} \cdot \boldsymbol{Pre}[\cdot, T_i] = \boldsymbol{Pre}[d_{i+1}, T_i] = -1$, but also $\boldsymbol{Pre}[c_i, T_i] = -1$, hence the third inequality of (2) becomes an equality when $h = 0$. In this case the initial marking of place c_i such that it becomes implicit has to satisfy the following inequality

$$\boldsymbol{m_0}[c_i] \geq \sum_{j=1,\ j \neq i}^{M} \boldsymbol{m_0}[d_j]. \tag{3}$$

The previous results can be interpreted in terms of the parameters of the FC-QN with BBS-SO nodes represented by the GSPN. We have to remember that the tokens in all the places c_i represent the customers circulating within the FC-QN, i. e., for any reachable marking $\boldsymbol{m}$ we have that $\sum_{i=1}^{M} \boldsymbol{m}[c_i] = N$ (the set of places c_i is the support set of a minimal P-semiflow). The tokens in places d_i represent the available positions in the buffer of the i-th node. For any reachable marking $\boldsymbol{m}$ we have that $\boldsymbol{m}[c_i] + \boldsymbol{m}[d_i] = b_i$ $(i = 1, \ldots, M)$, where b_i is the size of the buffer of the i-th node. From this it follows that Inequality (3) can be rewritten as

$$\begin{aligned} \boldsymbol{m_0}[c_i] &\geq \sum_{j=1,\ j \neq i}^{M} b_j - \boldsymbol{m_0}[c_j] \\ N &\geq \sum_{j=1,\ j \neq i}^{M} b_j. \end{aligned} \tag{4}$$

If Inequality (1) of Theorem 1 is satisfied then, for any $i = 1, \ldots, M$, we have that

$$N \geq \sum_{j=1,\ j \neq i}^{M} b_j.$$

Hence all the places c_i are implicit.
On the other hand, if all places c_i are implicit, Inequalities (4) are satisfied for any $1 \leq i \leq M$ and this implies that Inequality (1) holds. The previous results can be summarised with the following theorem that represent the GSPN interpretation of Theorem 1.

Theorem 4. *Let $\langle \mathcal{P}, \mathcal{T}, W, Q, \boldsymbol{m_0} \rangle$ be a GSPN modeling a closed cyclic queueing network with BBS-SO blocking mechanisms. The places of the GSPN represent-*

ing the queues are implicit iff

$$N \geq \sum_{i=1}^{M} b_i \; - \; \min_{j=1}^{M} \; b_j .$$

4.1 Implicit Places for Deriving New Product Form Results

Let us consider a closed cyclic queueing network with $M-1$ BBS-SO nodes, and one BBS-SNO node. Without loss of generality we assume that l is the index of the BBS-SNO node. Figure 3 shows the GSPN representation of a such FC-QN.

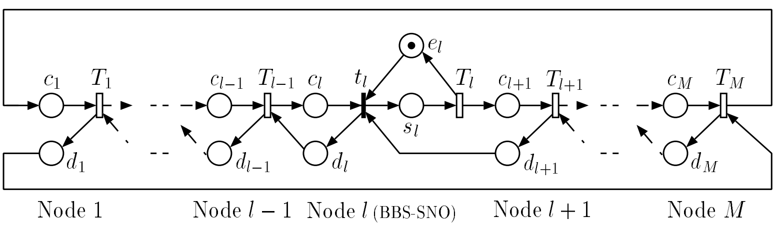

Fig. 3. The GSPN modeling a cyclic FC-QN with one BBS-SNO node and $M-1$ BBS-SO nodes.

The incidence and the preconditions matrices, for this Petri net, are:

$$
\boldsymbol{C} = \begin{array}{r|cccccccc}
 & T_1 & \cdots & T_{l-1} & t_l & T_l & T_{l+1} & \cdots & T_M \\ \hline
c_1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\
d_1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{l-1} & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 \\
d_{l-1} & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
c_l & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 \\
d_l & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 \\
s_l & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \\
e_l & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\
c_{l+1} & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 0 \\
d_{l+1} & 0 & \cdots & 0 & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_M & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 \\
d_M & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}
\qquad
\boldsymbol{Pre} = \begin{array}{r|cccccccc}
 & T_1 & \cdots & T_{l-1} & t_l & T_l & T_{l+1} & \cdots & T_M \\ \hline
c_1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
d_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{l-1} & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 \\
d_{l-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
c_l & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
d_l & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 \\
s_l & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
e_l & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
c_{l+1} & 0 & \cdots & 0 & 0 & 0 & -1 & \cdots & 0 \\
d_{l+1} & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_M & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 \\
d_M & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}
$$

We can split the analysis of this FC-QN into three parts: identification of the implicit places, product form analysis, and computation of the analytical solution.

Identification of the Implicit Places. From the incidence matrix $\boldsymbol{C}$ we can observe that for any place c_i, with $i = 1, \ldots, M$, and $i \neq l+1$ (the place corresponding to the BBS-SO node that follows the BBS-SNO node) $\boldsymbol{C}[c_i, \cdot] = \sum_{j=1,\; j \neq i}^{M} \boldsymbol{C}[d_j, \cdot]$. For the place c_{l+1}, i. e., the node that immediately follows the

unique BBS-SNO node, we have that $\boldsymbol{C}[c_{l+1},\cdot] = \boldsymbol{C}[e_l,\cdot] + \sum_{j=1,\ j\neq l+1}^{M} \boldsymbol{C}[d_j,\cdot]$.
From these equations it follows that all the places c_i (included places c_l and c_{l+1}) are structurally implicit. For each place c_i we identify the vector $\boldsymbol{y}$ involved in the LPP (2). To distinguish these vectors we denote by $\boldsymbol{y}^{(i)}$ the vector corresponding to place c_i. For any place c_i, with $i \neq l+1$ we have that $\boldsymbol{y}^{(i)}$ has the components equal to 1 in the positions corresponding to places d_j (with $j \neq i$) and all the other components are equal to 0. For place c_{l+1}, the vector $\boldsymbol{y}^{(l+1)}$ has the components equal to 1 in the positions corresponding to places d_j (with $j \neq l+1$), and in the one corresponding to place e_l, and all the other components are equal to 0.
Given the structure of the GSPN that models this FC-QN, and the form of vectors $\boldsymbol{y}^{(i)}$ (for $i = 1,\ldots,M$) we see that LPP (2) has a simple interpretation. Let us consider the following two cases: $i = l+1$, and $1 \leq i \leq M$ with $i \neq l+1$. In each one of these possible cases, if $t \in c_i^\bullet$ then the vector $\boldsymbol{y}^{(i)}$ has the entry corresponding to place c_i equal to 0 and the entry corresponding to place d_{i+1} equal to 1. From this it follows that $\boldsymbol{y}^{(i)} \cdot \boldsymbol{Pre}[\cdot,t] = \boldsymbol{Pre}[d_{i+1},t] = -1$, but also $\boldsymbol{Pre}[c_i,t] = -1$. This implies that the third inequality of the LPP (2) becomes an equality when $h = 0$. To compute the initial marking of place c_i such that it becomes implicit we must take into account the different structures of the vector $\boldsymbol{y}^{(i)}$-s. In particular, for any $i = 1,\ldots,M$, the only non zero entries are those corresponding to places d_j, with $j = 1,\ldots,M$, and $j \neq i$. When $i = l+1$ we must consider that also the entry corresponding to place e_l is equal to 1. It follows that LLP (2), which gives the initial marking of c_i such that it becomes implicit, assumes the following form:

$$\boldsymbol{m_0}[c_i] \geq \begin{cases} \displaystyle\sum_{j=1,\ j\neq i}^{M} \boldsymbol{m_0}[d_j] & \text{if } i \neq l+1 \\ \boldsymbol{m_0}[e_l] + \displaystyle\sum_{j=1,\ j\neq i}^{M} \boldsymbol{m_0}[d_j] & \text{if } i = l+1. \end{cases} \tag{5}$$

Next step is the interpretation of the previous inequalities in terms of the parameters of the FC-QN. To this aim we need to know the structure of the P-semiflows of the GSPN of Figure 3. In this GSPN we have that $N = \sum_{i=1}^{M} \boldsymbol{m}[c_i] + \boldsymbol{m}[s_l]$, $b_i = \boldsymbol{m}[c_i] + \boldsymbol{m}[d_i]$ (with $i \neq l, i \neq l+1$), $b_l - 1 = \boldsymbol{m}[c_l] + \boldsymbol{m}[d_l]$, $b_{l+1} = \boldsymbol{m}[c_{l+1}] + \boldsymbol{m}[d_{l+1}] + \boldsymbol{m}[s_l]$, and $1 = \boldsymbol{m}[s_l] + \boldsymbol{m}[e_l]$. We split the analysis of Inequalities (5) into three cases:

Node i, with $i \neq l$ and $i \neq l+1$

$$\boldsymbol{m_0}[c_i] \geq \sum_{\substack{j=1 \\ j\neq i, j\neq l, j\neq l+1}}^{M} \boldsymbol{m_0}[d_j] + \boldsymbol{m_0}[d_l] + \boldsymbol{m_0}[d_{l+1}]$$

$$\boldsymbol{m_0}[c_i] \geq \sum_{\substack{j=1\\ j\neq i, j\neq l, j\neq l+1}}^{M} (b_j - \boldsymbol{m_0}[c_j]) + b_l - 1 - \boldsymbol{m_0}[c_l] + b_{l+1} - \boldsymbol{m_0}[c_{l+1}] - \boldsymbol{m_0}[s_l]$$

$$N \geq \sum_{j=1,\ \ j\neq i}^{M} b_j - 1.$$

Node l

$$\boldsymbol{m_0}[c_l] \geq \sum_{\substack{j=1\\ j\neq l, j\neq l+1}}^{M} \boldsymbol{m_0}[d_j] + \boldsymbol{m_0}[d_{l+1}]$$

$$\boldsymbol{m_0}[c_l] \geq \sum_{\substack{j=1\\ j\neq l, j\neq l+1}}^{M} (b_j - \boldsymbol{m_0}[c_j]) + b_{l+1} - \boldsymbol{m_0}[c_{l+1}] - \boldsymbol{m_0}[s_l]$$

$$N \geq \sum_{j=1,\ \ j\neq l}^{M} b_j.$$

Node $l+1$

$$\boldsymbol{m_0}[c_{l+1}] \geq \sum_{\substack{j=1\\ j\neq l+1, j\neq l}}^{M} \boldsymbol{m_0}[d_j] + \boldsymbol{m_0}[d_l] + \boldsymbol{m}[e_l]$$

$$\boldsymbol{m_0}[c_{l+1}] \geq \sum_{\substack{j=1\\ j\neq l+1, j\neq l}}^{M} (b_j - \boldsymbol{m_0}[c_j]) + b_l - 1 - \boldsymbol{m_0}[c_l] + \boldsymbol{m}[e_l]$$

$$\sum_{j=1}^{M} \boldsymbol{m_0}[c_j] \geq \sum_{j=1,\ \ j\neq l+1}^{M} b_j - \boldsymbol{m}[s_l]$$

$$N \geq \sum_{j=1,\ \ j\neq l+1}^{M} b_j.$$

The previous inequalities are satisfied if

$$N \geq \sum_{i=1}^{M} b_i - \min_{j=1}^{M}\ b_j.$$

We summarise all the previous reasoning in the following lemma.

Lemma 1. *Given a cyclic queueing network with $M-1$ BBS-SO nodes and one BBS-SNO node, in the GSPN representation of this FC-QN, the places corresponding to the queues of the network are implicit if*

$$N \geq \sum_{i=1}^{M} b_i - \min_{j=1}^{M}\ b_j. \tag{6}$$

There is an interesting relation between Inequality (6) and the condition to ensure that the FC-QN is deadlock free. Deadlock prevention for some types of blocking mechanisms has been discussed in [9,12,14]. In these papers it is stated that a cyclic queueing network with $M-1$ BBS-SO nodes and one BBS-SNO node is deadlock free if the number of customers is

$$N < \sum_{i=1}^{M} b_i - 1. \tag{7}$$

If the minimum among the buffer capacities is equal to 1 then the value of N satisfying Inequality (6) implies that the network is deadlocked. To satisfy both Inequality (6) and Inequality (7) the minimum among the buffer capacities must be

$$\min_{j=1}^{M} \; b_j \geq 2.$$

Product Form Analysis. Figure 4 shows the GSPN of Figure 3 without the implicit places.

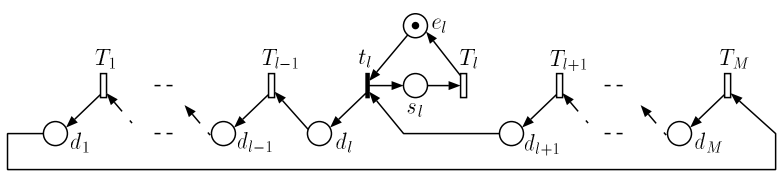

Fig. 4. The GSPN of Figure 3 without the implicit places.

In this GSPN the tokens in the places d_i represent the empty positions circulating within the FC-QN. If $N = \sum_{i=1}^{M} b_i - 1$, from Inequality (7), it follows that the GSPN of Figure 3 is deadlocked because for any node i of the network there is no available room in the destination node $i+1$ and hence all nodes are blocked. For the GSPN of Figure 4 the sum of the number of tokens in places d_i is zero. If $\min_{j=1}^{M} \; b_j \geq 2$ then $N = \sum_{i=1}^{M} b_i - 2$, satisfies Inequality (6) and Inequality (7). For this value of N the number of tokens in the places d_i is equal to 1. We show that, in this case, the model has a closed form expression for its steady state probability distribution, and that there exists a SPN having the same average number of tokens in the places d_i (for $i = 1, \ldots, M$), and the same throughput of the transitions T_i of the GSPN of Figure 4. Figure 5 shows this SPN.

The SPN of Figure 5 represents a cyclic queueing network with M stations and only one customer. It is easy to see that the equilibrium distribution of this SPN is product form

$$\pi(\boldsymbol{m}) = \frac{1}{G} \left(\frac{1}{\overline{\mu_l}} \right)^{m_{l+1}} \cdot \prod_{\substack{i=1 \\ i \neq l+1}}^{M} \left(\frac{1}{\mu_{i-1}} \right)^{m_i}, \tag{8}$$

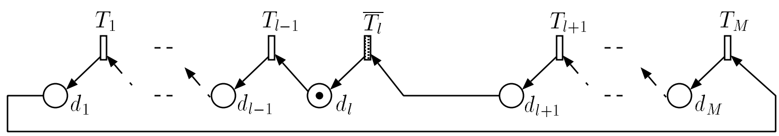

Fig. 5. The SPN with the same performance indices of the GSPN of Figure 4.

where G is a normalisation constant, μ_0 is the rate of transition T_M, and $\overline{\mu_l}$ is the rate of transition $\overline{T_l}$.
Now we derive the rate of $\overline{T_l}$ such that the models of Figure 4 and of Figure 5 have the same equilibrium distributions of the number of tokens in the places d_i (for $i = 1, \ldots, M$), the same average waiting time of the token in these places, and the same throughput of the transitions. Since all transitions T_i have the same throughput, we denote this measure by χ. We can write $\chi = 1/\left(\sum_{i=j}^{M} \overline{w_j}\right)$, where $\overline{w_j}$ is the average waiting time time spent by the token in place d_j. In the SPN of Figure 5 there is only one token circulating in the net and hence for any $j \neq l+1$ we have that $\overline{w_j} = 1/\mu_{j-1}$. For $j = 1$ we have that $\overline{w_1} = \frac{1}{\mu_M}$. The computation of the average waiting time spent by the token in place d_{l+1} requires a more complex analysis on the GSPN of Figure 4. We can compute $\overline{w_{l+1}}$ using this expression:

$$\overline{w_{l+1}} = \frac{P^a}{\mu_l}\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\big\} + P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#e_l > 0\big\} \cdot 0,$$

where $P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\big\}$ (resp. $P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#e_l > 0\big\}$) is the probability that the arriving token in place d_{l+1} finds place s_l marked (resp. the probability that the arriving token in place d_{l+1} finds place e_l marked). The arrival-instant probabilities used in the previous equation can be expressed in terms of steady state solution of the model of Figure 4 as follows: $P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\big\}$ is the ratio between the frequency of arrivals in d_{l+1} when s_l is marked and the frequency of arrivals in d_{l+1}, that is,

$$P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\big\} = \frac{\pi(\{d_{l+2}, s_l\})\mu_{l+1}}{\chi},$$

where $\pi(\{d_{l+2}, s_l\})$ is the equilibrium probability of the state with one token in d_{l+2} and one in s_l, and χ is the frequency of arrivals in d_{l+1}, that is the throughput of T_{l+1}. Since all the service rates of the timed transitions are marking independent we can compute the throughput of T_{l+1} as $\chi = \pi(\{d_{l+2}\})\mu_{l+1}$, where $\pi(\{d_{l+2}\})$ is the equilibrium probability of the state with one token in d_{l+2}. It follows that we can write the previous arrival-instant probability in the following manner

$$P^a\big\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\big\} = \frac{\pi(\{d_{l+2}, s_l\})}{\pi(\{d_{l+2}\})}. \tag{9}$$

If we set the rate of transition $\overline{T_l}$, i. e. , $\overline{\mu_l}$ as $\overline{\mu_l} = 1/\overline{w_{l+1}}$, we obtain that the throughput of the transitions in the model of Figure 5 is the same as the one of the model Figure 4. This is the same for the average waiting time of the token in the places d_i (for $i = 1, \ldots, M$), and hence it follows from the Little law that also the number of tokens in these places is equal. Since there is only one token circulating in the net, the average number of tokens also give the marginal distributions of the tokens in these places.

Analytical Solution. Now we derive a method for the computation of the arrival-instant probability of Equation (9).

The GSPN of Figure 4 shows the GSPN representation of a FC-QN with $M-1$ BBS-SO nodes and one BBS-SNO node without the implicit places. If we consider the model of Figure 4 we can see that this system (with only one token circulating in places d_i) is equivalent to the $M/H_{M-1}/1/2$ queueing system. The $M/H_{M-1}/1/2$ is a queueing system where the customers arrivals form a Poisson process with rate λ. The service times of customers are independent identically distributed random variables, the common distribution being $M-1$ stages hypo-exponential where ν_i, for $1 \leq i \leq M-1$ is the rate of the i-th stage. The third parameter of the notation $M/H_{M-1}/1/2$ means single server queue, while the last parameter is maximum number of customers, that is, in the queueing system there can be up to 2 customers. For equivalence we mean that continuous time Markov chain of the GSPN of Figure 4 is equal to that one of the $M/H_{M-1}/1/2$ queueing system.

From this it follows that we can compute the arrival-instant probability of Equation (9) on the $M/H_{M-1}/1/2$ queueing system.

The arrival rate λ of the queueing system corresponds to the service rate of the BBS-SNO node (the one with index l). The rate of the first stage of the hypo-exponential distribution is equal to the service rate of the BBS-SO node with index $l-1$, the rate of the second stage is equal to the service rate of the BBS-SO node with index $l-2$, and so on up to the service rate of the last stage that is equal to the service rate of the BBS-SO node with index $l+1$.

Table 1 shows the mapping between the states (markings) of the GSPN of Figure 4 and states of the $M/H_{M-1}/1/2$. In this table each state of the $M/H_{M-1}/1/2$ is denoted by a pair (n, s), where n represents the number of customers in the system and s is the stage of the customer currently in service. Each marking of the GSPN is represented by a list of the marked places. From the previous table we can see that the arrival-instant probability of Equation (9) can be computed as

$$P^a\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\} = \frac{P\{(1, M-1)\}}{P\{(1, M-1)\} + P\{(2, M-1)\}}, \quad (10)$$

where $P\{(1, M-1)\}$ and $P\{(2, M-1)\}$ are the equilibrium probabilities of the states $(1, M-1)$ and $(2, M-1)$ of the $M/H_{M-1}/1/2$ queueing system. These equilibrium probabilities can be obtained using standard queueing techniques (see [11] for details). From Equation (10) we can derive that:

$$\overline{\mu_l} = \frac{1}{\overline{w_{l+1}}}$$

State of the $M/H_{M-1}/1/2$	Marking of the GSPN
$(0,0)$	$\{s_l, d_{l+1}\}$
$(1,1)$	$\{s_l, d_l\}$
$\vdots$	$\vdots$
$(1, M-1)$	$\{s_l, d_{l+2}\}$
$(2,1)$	$\{e_l, d_l\}$
$\vdots$	$\vdots$
$(2, M-1)$	$\{e_l, d_{l+2}\}$

Table 1. Mapping between states of the queueing system and markings of the GSPN of Figure 4

$$= \frac{\mu_l}{P^a\{\text{the token arr. in } d_{l+1} \text{ finds } \#s_l > 0\}}$$
$$= \frac{\mu_l \cdot (P\{(1, M-1)\} + P\{(2, M-1)\})}{P\{(1, M-1)\}}. \tag{11}$$

We summarise the previous derivations in the following lemma.

Lemma 2. *Given a cyclic queueing network with $M-1$ BBS-SO nodes and one BBS-SNO node. If the number of customers N circulating within the network is such that*

$$N = \sum_{i=1}^{M} b_i - 2, \tag{12}$$

with $\min_{j=1}^{M} b_j \geq 2$, then the queueing network has a closed form expression for the steady state probability distribution. Moreover there exists a product form solution model having the same performance measures of the cyclic queueing network. The product form model has M stations. The service rate of station i (for $i = 1, \ldots, M$, and $i \neq l$) is the same of the corresponding station of the FC-QN. The service rate of station l is $\overline{\mu_l}$ and it is obtained using Equation (11).

Using the product form equilibrium distribution (8) with the well known computational algorithms (for instance the normalisation constant algorithm [6], or the mean value analysis [16]) we can derive the performance measures for the SPN of Figure 5, in particular we can compute the average number of tokens in place d_i and the throughput of the transitions. From these indices we can derive the measures of the FC-QN represented by the GSPN of Figure 3. Let us denote with $\overline{n_i}$ the average number of customers in the node i and with $\overline{d_i}$ the average number of tokens in place d_i. For any node i, with $i = 1, \ldots, M$, and $i \neq l$, $i \neq l+1$, the average number of customers in the node $\overline{n_i}$ is given by $\overline{n_i} = b_i - \overline{d_i}$. For $i = l$ we have that $\overline{n_l} = b_l - 1 - \overline{d_l}$ gives the average number of customers queued at node l and does not take into account the position in front of the server (represented by place s_l). To derive the average number of tokens in place s_l we can use the following relation: let χ be the throughput of the transitions. Since the service rates are marking independent, we have that on the GSPN of

Figure 4, $\chi = P\{\#s_l > 0\} \cdot \mu_l$, where $P\{\#s_l > 0\}$ is the probability that place s_l is marked. From the knowledge of χ we can derive $P\{\#s_l > 0\}$. Since place s_l can contain at most one token, we have that $\overline{s_l} = P\{\#s_l > 0\}$. In the GSPN of Figure 3 we can observe that places s_l, c_{l+1}, and d_{l+1} are covered by a minimal P-semiflow. From this we can compute the average number of customers at node $l+1$ as follows: $\overline{n_{l+1}} = b_{l+1} - \overline{s_l} - \overline{d_{l+1}}$. In this manner from the performance measures of the SPN of Figure 5 we have derived the measures for the FC-QN represented by the GSPN of Figure 3.

Remarks. We must point out that the GSPN of Figure 4 does not have product form solution. However, we are claiming that the SPN of Figure 5 has product form equilibrium distribution and that the performance measures (average number of tokens in the places, average waiting times, throughput of the transitions) of this SPN are the same as those of the GSPN of Figure 4.
In other words, with the help of the implicit places we can compute the performance measures of a non-product form model (the GSPN of Figure 4) using a product form one (the SPN of Figure 5).
In principle we could derive the analytical form of the steady state probabilities for the GSPN of Figure 4 without using any product form analysis and without the help of the SPN of Figure 5. Since the GSPN of Figure 4 is equivalent to the $M/H_{M-1}/1/2$ queueing system, we can use the closed form for the equilibrium distribution of this system for deriving the steady state probabilities for the GSPN. Nevertheless we present the product form analysis because product form results can also be used to investigate models that do not have this nice property.
In the literature there are several proposals of this type of studies. One possible technique would be the derivation of Mean Value Analysis equations [16] for the product form case and then use these equations as a basis for developing approximate techniques similar to those proposed in [17]. Other examples of use of the product form as a basis for approximate techniques are described in [5]. In [19] the product form is the basis for developing bounding techniques.

Another issue that we have to point out is that unfortunately the product form result is valid only under very special circumstances: the total number of customers circulating within the queueing network must be equal to the sum of all buffer capacities minus 2, and this result can not be easily generalised. However, the availability of the product form can be used for the exploitation of approximate and bounding techniques for non-product form models.
As can be seen in [3,14], the product form solution exists only for a limited class of queueing network with finite capacity, and the results of this paper represent an extension of this class.

4.2 Cyclic Networks with More Than One BBS-SNO Node

In principle, the previous method can be generalised to cases of FC-QNs with more than one BBS-SNO node. We illustrate this idea by means of the following example.

Example 1. Figure 6(a) shows a GSPN modeling a cyclic FC-QN with 4 BBS-SNO and one BBS-SO node.

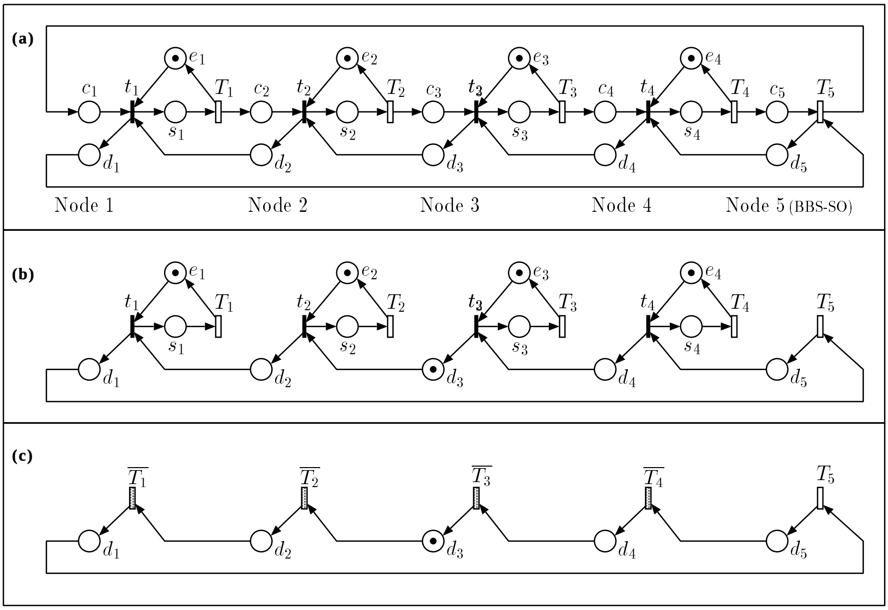

Fig. 6. The GSPN modeling a cyclic FC-QN with 4 BBS-SNO and one BBS-SO node (a), the same GSPN without the implicit places (b), the SPN with the same performance indices of the GSPN without the implicit places (c).

Using Theorem 2 we can prove that places c_1, c_2, c_3, c_4, and c_5 are structurally implicit. Theorem 3 allows us to compute the the initial marking such that these places become implicit. Let us assume that the initial marking is such that in the GSPN obtained by removing the implicit places there is only one token circulating within places d_i ($i = 1, \ldots, 5$). In Figure 6(b) it is depicted a such GSPN. We can apply the same technique used in the case of a cyclic network with only one BBS-SNO node (arrival instant probabilities). We can build a product form solution model that has the same performance measures of the GSPN of Figure 6(b). Figure 6(c) shows this measure equivalent model. This SPN has product form solution:

$$\pi(\boldsymbol{m}) = \frac{1}{G}\left(\frac{1}{\mu_5}\right)^{m_1}\left(\frac{1}{\overline{\mu_1}}\right)^{m_2}\left(\frac{1}{\overline{\mu_2}}\right)^{m_3}\left(\frac{1}{\overline{\mu_3}}\right)^{m_4}\left(\frac{1}{\overline{\mu_4}}\right)^{m_5} \quad \forall\ \boldsymbol{m} \in RS,$$

where G is a normalisation constant, m_i (for $i = 1, \ldots, 5$) is the marking of place d_i, and $\overline{\mu_i}$ is the rate of transition $\overline{T_i}$ (for $i = 1, \ldots, 4$).

However, we must point out that, in this case, we do not have an auxiliary model (the $M/H_{M-1}/1/2$ queueing system) that allows to compute in a closed form the rates of the transitions $\overline{T_i}$ (the shadow transitions). The example only shows that there is a class of FC-QNs for which there exist equivalent models with product form solution and this equivalence can be found by using the technique based on the structural implicit places.

4.3 Another New Case of Product Form Solution

Let us consider a closed cyclic queueing network with $M-1$ BBS-SO nodes, and one BAS node. Figure 7(a) shows the GSPN representation of a such FC-QN. Here, with respect to the representation of a BAS node presented in Figure 1(c), we have removed the inhibitor arc by using the complementary place e_l. We can

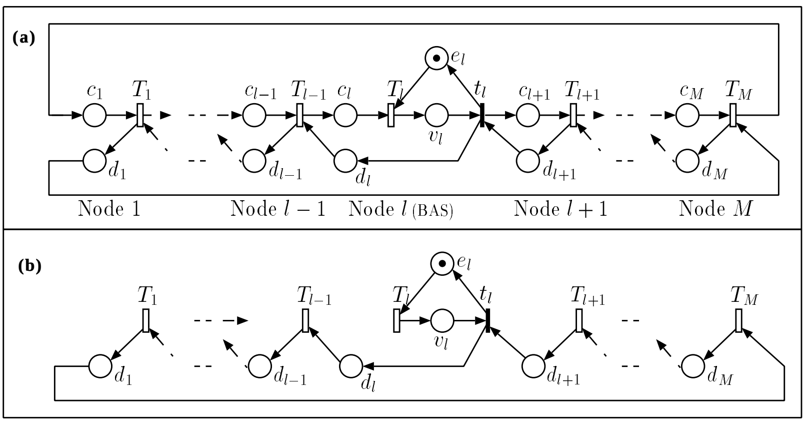

Fig. 7. The GSPN modeling a cyclic FC-QN with one BAS node and $M-1$ BBS-SO nodes (a), the same GSPN witout the implicit places (b).

see that the GSPN representation of a such FC-QN is similar to that one of the case with only one BBS-SNO node. We can repeat the same reasoning used for that cyclic FC-QN. We can summarise the result in the following lemmas.

Lemma 3. *Given a cyclic queueing network with $M-1$ BBS-SO nodes and one BAS node in the GSPN representation of this FC-QN the places corresponding to the queues of the network are implicit if*

$$N-1 \geq \sum_{i=1}^{M} b_i - \min_{j=1}^{M} \; b_j. \tag{13}$$

Figure 7(b) shows the GSPN without the implicit places.

Lemma 4. *Given a cyclic queueing network with $M-1$ BBS-SO nodes and one BAS node. If the number of customers N circulating within the network is such that*

$$N = \sum_{i=1}^{M} b_i - 1, \tag{14}$$

then we can build a product form solution model having the same performance measures of the cyclic queueing network. The product form model has M stations. For any $i = 1, \ldots, M$, with $i \neq l$, the service rate of station i is the same of the corresponding station of the FC-QN. The service rate of station l is $\overline{\mu_l}$ and it is obtained using Equation (11).

Please note the difference between the case of FC-QN with one BBS-SNO node and several BBS-SO nodes and the one with one BAS node and several BBS-SO nodes. In both cases there exist the measure equivalent PF models only when the sum of tokens in places d_i $(i = 1, \ldots, M)$ is equal to 1. For the case of one BAS node and several BBS-SO nodes this is obtained with $N = \sum_{i=1}^{M} b_i - 1$. The measure equivalent model is the same for both cases (SPN of Figure 5).

5 Conclusions

In this paper we have proposed an approach that allows to discover new quantitative results for queueing networks with blocking using structural Petri net properties. The studies are based on the representation of FC-QNs by means of GSPNs. We have used the notion of *implicit places* for studying product form equilibrium distributions of FC-QNs. In particular using this notion we have characterised a class of FC-QNs such that for each element of this class there exists a model with the same performance measures and with product form equilibrium distributions.

Future plans include the application of the described technique for deriving other new product form results. Another direction of future research could be the derivation of other Petri net driven techniques that allows to transform a GSPN representation of a non-product form FC-QN into a product form one.

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