Session Types = Intersection Types + Union Types

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Behaviors versus Types

$$c : P$$

$$\overline{a}.\overline{a}.b$$

$$\overline{a} \oplus \overline{b}$$

$$a + b$$

Behaviors versus Types

$c : P$

$\overline{a}.\overline{a}.b$

$\overline{a} \oplus \overline{b}$ $\qquad$ $\overline{a} \wedge \overline{b}$

$a + b$ $\qquad$ $a \vee b$

Outline

1. Crash course on session types
2. Behaviors
3. Types
4. Conclusion

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Processes

$$P \quad ::= \quad \mathsf{end} \quad | \quad \alpha.P \quad | \quad P \oplus P \quad | \quad P + P$$

$$\alpha.P \xrightarrow{\alpha} P \qquad P \oplus Q \longrightarrow P \qquad \frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$$

Systems

$P \mid Q$

System evolution

$\overline{a} \oplus \overline{c} \mid b$	$\longrightarrow$	$\overline{a} \mid b$	internal choice
$\overline{a}.P \mid a.Q + b$	$\longrightarrow$	$P \mid Q$	communication

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P  Q

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Well-formed Systems & Subtyping

$$wf(P \mid Q) \stackrel{\text{def}}{\iff} P \mid Q \longrightarrow \cdots \longrightarrow P' \mid Q' \not\longrightarrow \text{ implies } P' = Q' = \mathsf{end}$$

Examples

$$wf(\overline{a} \oplus \overline{b} \mid a + b) \qquad \neg wf(\overline{a} \oplus \overline{b} \mid a)$$

(Semantic) Subtyping

$$[\![P]\!] = \{Q \mid wf(P \mid Q)\} \qquad P \preceq Q \stackrel{\text{def}}{\iff} [\![P]\!] \subseteq [\![Q]\!]$$

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Interpretation of choices

Internal choice = intersection

$$[\![P \oplus Q]\!] = [\![P]\!] \cap [\![Q]\!]$$

External choice $\neq$ union

$$\alpha.P + \alpha.Q \approx \alpha.(P \oplus Q)$$

$\preceq$ is *not* a pre-congruence

$$a.P \preceq a.P + b.Q$$

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Syntax

$$T \quad ::= \quad \textsf{end} \quad | \quad \alpha.T \quad | \quad T \wedge T \quad | \quad T \vee T$$

$$P \qquad \overline{a} \wedge \overline{b} \qquad Q$$

$$P \qquad a \vee b \qquad Q$$

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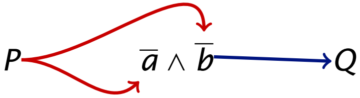

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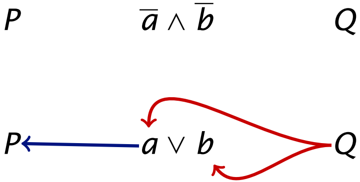
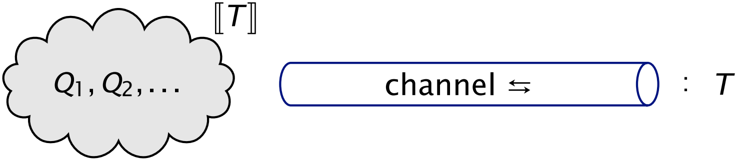


(Tentative) type semantics



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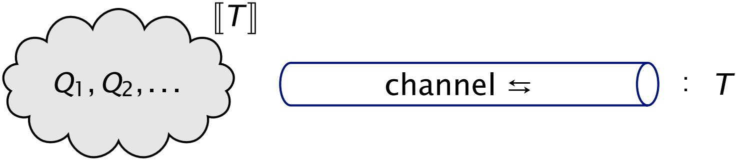

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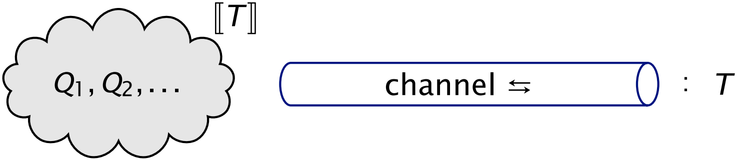

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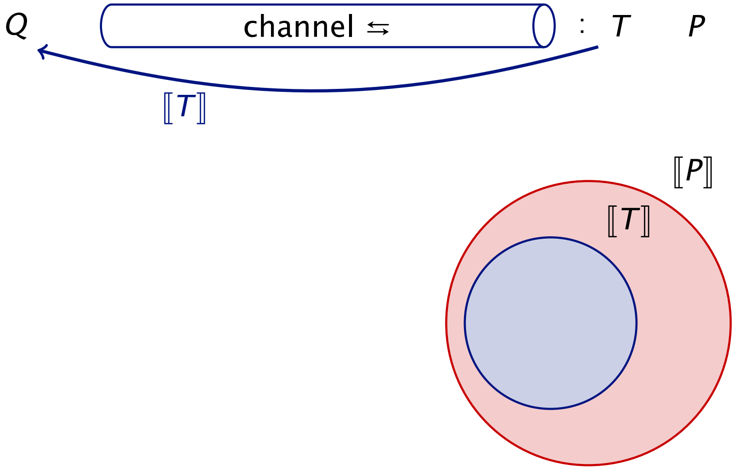
Orthogonal set



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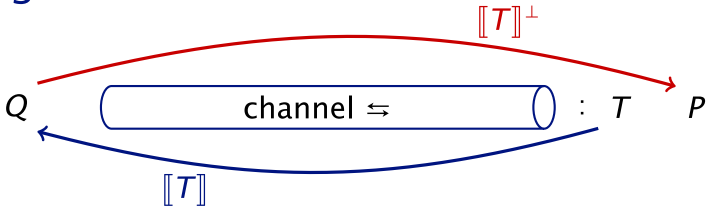
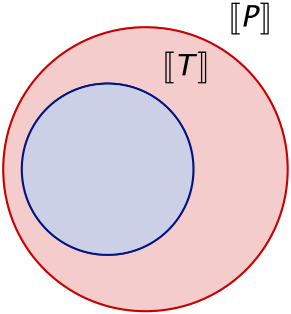


$$X^\perp \stackrel{\text{def}}{=} \{P \mid X \subseteq [\![P]\!]\}$$



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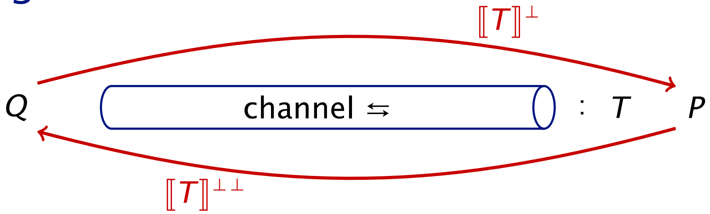

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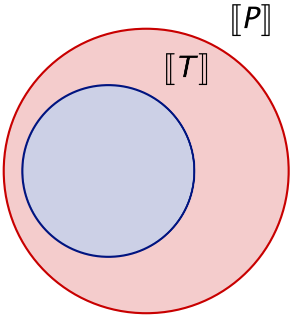

Closure = bi-orthogonal

Proposition ($(\cdot)^{\perp\perp}$ is a closure)

1. $X \subseteq X^{\perp\perp}$
2. $X \subseteq Y$ *implies* $X^{\perp\perp} \subseteq Y^{\perp\perp}$
3. $X^{\perp\perp\perp\perp} = X^{\perp\perp}$

$$
\begin{aligned}
\{a\}^{\perp\perp} &= \{\overline{a}\}^{\perp} &&= \{a, a+b, \dots\} \\
\{\overline{a}, \overline{b}\}^{\perp\perp} &= \{a+b, a+b+c, \dots\}^{\perp} &&= \{\overline{a} \oplus \overline{b}, \overline{a}, \overline{b}\}
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Type semantics

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Behaviors versus types: subtyping

$$\alpha.P + \alpha.Q \quad \approx \quad \alpha.(P \oplus Q) \qquad\qquad \alpha.T \vee \alpha.S \quad \approx \quad \alpha.(T \vee S)$$

$\preceq$ is not a pre-congruence

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On combining incompatible behaviors

$$
\begin{aligned}
[\![a \wedge \overline{b}]\!] &= ([\![a]\!] \cap [\![\overline{b}]\!])^{\perp\perp} = \emptyset^{\perp\perp} = \mathcal{P}^{\perp} = \emptyset \\
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$$
T ::= \mathbb{0} \mid \mathbb{1} \mid \cdots
$$

- $\mathbb{0}$ can be implemented, cannot be consumed
- $\mathbb{1}$ cannot be implemented, can be consumed

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Theory validation

Proposition (correctness)

For every $T \neq \mathbb{0}, \mathbb{1}$ *there exists* P *such that* $[\![T]\!] = [\![P]\!]$

Proposition (completeness)

For every P *there exists* T *such that* $[\![P]\!] = [\![T]\!]$

$$\alpha.P + \alpha.Q \qquad \alpha.(T \wedge S)$$

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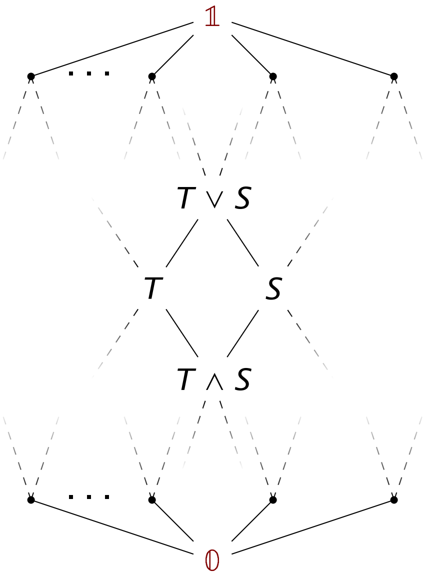
The lattice of session types

- intersection types and union types for branching
- advantages over behavioral interpretation
- the bounds are "misbehaving processes"



Future work

- infinite behaviors

$$\mathtt{rec}\ x.T$$

- refined actions

$$!\mathtt{int} \wedge !\mathtt{bool}$$

$$!T \wedge !S \approx !(T \vee S) \qquad !T \vee !S \approx !(T \wedge S)$$