# Session Types = Intersection Types + Union Types

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# Behaviors versus Types

$$c : P$$

$$\overline{a}.\overline{a}.b$$

$$\overline{a} \oplus \overline{b}$$

$$a + b$$

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$c : P$

$\overline{a}.\overline{a}.b$

$\overline{a} \oplus \overline{b}$ $\qquad$ $\overline{a} \wedge \overline{b}$

$a + b$ $\qquad$ $a \vee b$

# Outline

1. Crash course on session types
2. Behaviors
3. Types
4. Conclusion

# Outline

# Processes

$$P \quad ::= \quad \mathsf{end} \quad | \quad \alpha.P \quad | \quad P \oplus P \quad | \quad P + P$$

$$\alpha.P \xrightarrow{\alpha} P \qquad P \oplus Q \longrightarrow P \qquad \frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$$

# Systems

$P$


$Q$

$$P \mid Q$$

## System evolution

$$\overline{a} \oplus \overline{c} \mid b \longrightarrow \overline{a} \mid b \qquad \text{internal choice}$$

$$\overline{a}.P \mid a.Q + b \longrightarrow P \mid Q \qquad \text{communication}$$

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# Well-formed Systems & Subtyping

$$wf(P \mid Q) \stackrel{\text{def}}{\iff} P \mid Q \longrightarrow \cdots \longrightarrow P' \mid Q' \not\longrightarrow \text{ implies } P' = Q' = \mathsf{end}$$

Examples

$$wf(\overline{a} \oplus \overline{b} \mid a + b) \qquad \neg wf(\overline{a} \oplus \overline{b} \mid a)$$

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$$[\![P]\!] = \{Q \mid wf(P \mid Q)\} \qquad P \preceq Q \stackrel{\text{def}}{\iff} [\![P]\!] \subseteq [\![Q]\!]$$

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## Internal choice = intersection

$$[\![P \oplus Q]\!] = [\![P]\!] \cap [\![Q]\!]$$

## External choice $\neq$ union

$$\alpha.P + \alpha.Q \approx \alpha.(P \oplus Q)$$

## $\preceq$ is *not* a pre-congruence

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# Syntax

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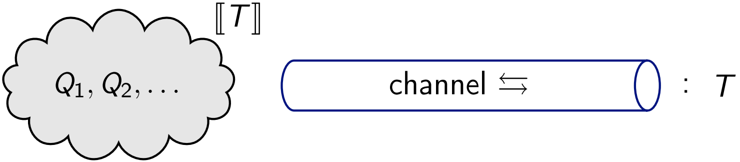
# (Tentative) type semantics

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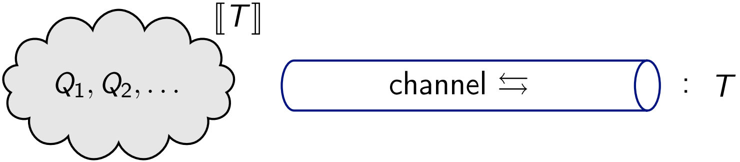


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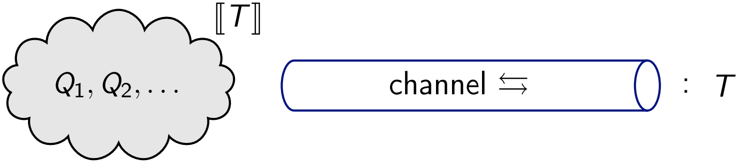


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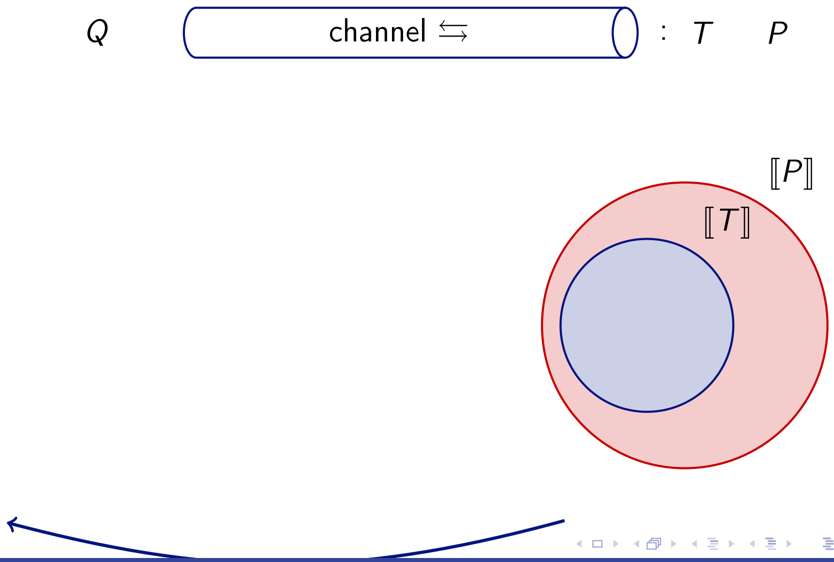
# Orthogonal set



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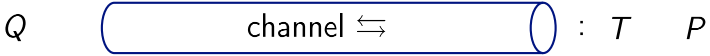
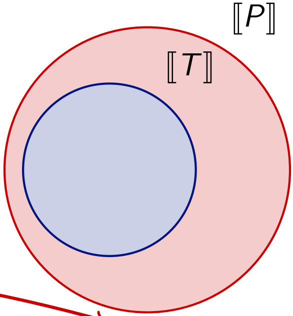
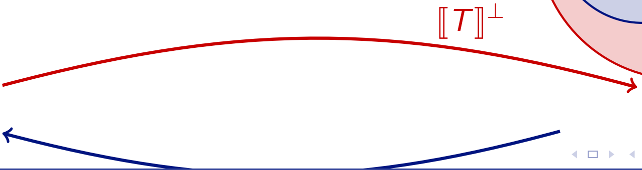



$$X^{\perp} \stackrel{\text{def}}{=} \{P \mid X \subseteq [\![P]\!]\}$$



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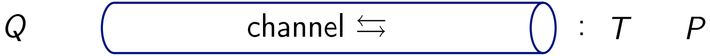


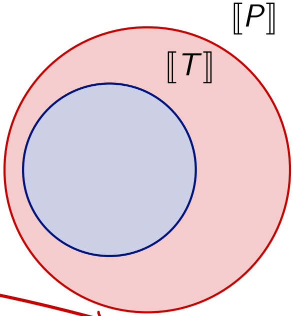


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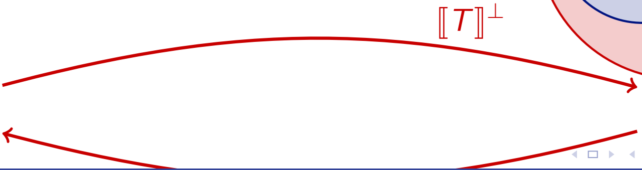

# Closure = bi-orthogonal

**Proposition ($(\cdot)^{\perp\perp}$ is a closure)**

1. $X \subseteq X^{\perp\perp}$
2. $X \subseteq Y$ *implies* $X^{\perp\perp} \subseteq Y^{\perp\perp}$
3. $X^{\perp\perp\perp\perp} = X^{\perp\perp}$

$$\begin{array}{rcccl} \{a\}^{\perp\perp} & = & \{\overline{a}\}^{\perp} & = & \{a, a + b, \dots\} \\ \{\overline{a}, \overline{b}\}^{\perp\perp} & = & \{a + b, a + b + c, \dots\}^{\perp} & = & \{\overline{a} \oplus \overline{b}, \overline{a}, \overline{b}\} \end{array}$$

# Type semantics

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# Behaviors versus types: subtyping

$$\alpha.P + \alpha.Q \approx \alpha.(P \oplus Q) \qquad \alpha.T \vee \alpha.S \approx \alpha.(T \vee S)$$

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# On combining incompatible behaviors

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$$
T ::= \mathbb{0} \mid \mathbb{1} \mid \cdots
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- $\mathbb{0}$ can be implemented, cannot be consumed
- $\mathbb{1}$ cannot be implemented, can be consumed

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# Theory validation

**Proposition (correctness)**

*For every* $T \neq \mathbb{0}, \mathbb{1}$ *there exists* $P$ *such that* $[\![T]\!] = [\![P]\!]$

**Proposition (completeness)**

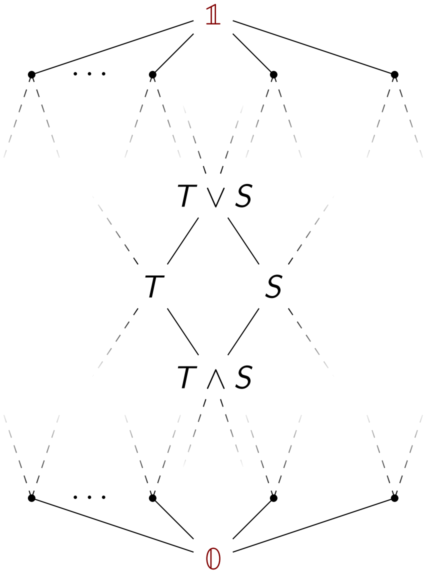
*For every* $P$ *there exists* $T$ *such that* $[\![P]\!] = [\![T]\!]$

$$\alpha.P + \alpha.Q \qquad \alpha.(T \wedge S)$$

# Outline

# The lattice of session types

- intersection types and union types for branching
- advantages over behavioral interpretation
- the bounds are "misbehaving processes"



# Future work

- infinite behaviors

$$\texttt{rec}\ x.T$$

- refined actions

$$!\texttt{int} \wedge !\texttt{bool}$$

$$!T \wedge !S \approx !(T \vee S) \qquad !T \vee !S \approx !(T \wedge S)$$