Call-By-Value Lambda Calculus

Solvability and Separability

Luca Paolini
Università di Torino
Dipartimento di Informatica

May 31, 2009

Outline

Parametric

Valuability

Solvability

Separability

Parametric Parameter Passing Lambda-Calculus

Parametric Lambda-Calculus

- The set Λ is produced by

$$M ::= x \mid MM \mid \lambda x.M$$

- Let $\Delta \subseteq \Lambda$.
  The Δ-reduction ($\rightarrow_\Delta$) is the contextual closure of the following rule:

$$(\lambda x.M)N \rightarrow_\Delta M[N/x] \quad \text{whenever} \quad N \in \Delta$$

☞ When a set Δ induces an interesting calculus ?

It is reasonable to respect next remarks.

1. Variables can be considered as placeholder for values.
2. The status of being an input value should be preserved during the evalution process.

Definition 1

$\Delta \subseteq \Lambda$ is a set of input values, when:

- $\mathrm{Var} \subseteq \Delta$ (variable closure)
- $P, Q \in \Delta$ implies $P[Q/x] \in \Delta$, (substitution closure)
- $M \in \Delta$ and $M \rightarrow_\Delta N$ imply $N \in \Delta$ (reduction closure)

Instances of Values

- Λ is a set of input values

  (the $\lambda\Lambda$-calculus is the classical λ-calculus)

- $\Gamma = \text{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$ is a set of input values

  (the $\lambda\Gamma$-calculus is the Plotkin's λ_v-calculus)

- $\Xi = \text{Var} \cup \{M \mid M \text{ is a closed } \beta \text{ normal form }\}$ is a set of input value

On the othe hand,

- Λ-normal forms are not input values
- Λ-head normal forms are not input values
- Γ-normal forms are not input values

Confluence

Theorem 2

$M \to_\Delta^* N_1$ and $M \to_\Delta^* N_2$ imply that

there is N_3 such that both $N_1 \to_\Delta^* N_3$ and $N_2 \to_\Delta^* N_3$.

The closure conditions on the set Δ of input values assure that the corresponding Δ-calculus enjoys the confluence property.

Corollary 3

The Δ-normal form of a term, if it exists, is unique.

Standard Strategies

Assume $M \to_\Delta^* N$, and assume that there is more than one Δ-reduction sequence from M to N. The standardization theorem says that, in case the set of input values enjoys a particular property, there is a "standard" reduction sequence from M to N, reducing the redexes in a given order.

- A zero-step reduction sequence is always standard
- A one-stp reduction sequence is always standard.
- Standardization is not Normalization.
- A standard reduction sequence does not need to reduce the lefmost redex at every step.

In case of Λ-calculus a standard reduction sequence is a strategy choosing redexes from left to right. For instance,

$$(\lambda x.x(KI))(II) \to_\Lambda II(KI) \to_\Lambda I(KI) \to_\Lambda I(\lambda y.I)$$

is a standard reduction sequence in the Λ-calculus.

The reduction sequence reducing the same term from left to right in the Γ-calculus:

$$(\lambda x.x(KI))(II) \to_\Gamma (\lambda x.x(\lambda y.I))(II) \to_\Gamma (\lambda x.x(\lambda y.I))I \to_\Gamma I(\lambda y.I)$$

is a standard reduction sequence?
It is not easy to formalize such a strategy for other sets of input values.

Sequentialization

- The Δ-sequentialization $(M)^\circ$ of a term M is a function from Λ to Λ defined as follows:
  - $(xM_1...M_m)^\circ = x(M_1)^\circ...(M_m)^\circ$;
  - $((\lambda x.P)QM_1...M_m)^\circ = (\lambda x.P)^\circ(Q)^\circ(M_1)^\circ...(M_m)^\circ$ if $Q \in \Delta$;
  - $((\lambda x.P)QM_1...M_m)^\circ = (Q)^\circ(\lambda x.P)^\circ(M_1)^\circ...(M_m)^\circ$ if $Q \notin \Delta$;
  - $(\lambda x.P)^\circ = \lambda x.(P)^\circ$.
- A symbol λ in a term M is active if and only if it is the first symbol of a Δ-redex of M.
- The degree of a redex R in M is the numbers of λ's which both are active in M and occur on the left of $(R)^\circ$ in $(M)^\circ$.
- A sequence $M \equiv P_0 \to_\Delta P_1 \to_\Delta ... \to_\Delta P_n \to_\Delta N$ is standard if and only if the degree of the redex contracted in P_i is less than or equal to the degree of the redex contracted in P_{i+1}, for every $i < n$.
  We denote by $M \to_\Delta^\circ N$ a standard reduction sequence from M to N.
- The principal redex of M, if it exists, is the redex of M with minimum degree. The principal reduction $M \to_\Delta^p N$ denotes that N is obtained from M by reducing the principal redex of M. Moreover, $\to_\Delta^{*p}$ is the reflexive and transitive closure of $\to_\Delta^p$.

Standard Values

$(\lambda x.x(KI))(II) \to_\Gamma (\lambda x.x(KI))I \to_\Gamma I(KI) \to_\Gamma I(\lambda y.I)$ is a standard reduction sequence in the Γ-calculus.

Definition 4

A set Δ of input values is standard if and only if $M \notin \Delta$ and $M \to_\Delta^* N$ by reducing at every step a not principal redex imply $N \notin \Delta$.

- The set of input values Λ, Γ are standard.
- The set of input values Λ_I is a not standard set of input value.

Standardization

Theorem 5

Let Δ be standard. $M \to_\Delta^* N$ implies there is a standard reduction sequence from M to N.

Corollary 6

Let Δ be standard.
If $M \to_\Delta^* N$ and N is a normal form then $M \to_\Delta^{*p} N$.

Δ-Normal Forms

In the classical λ-calculus,
β-normal forms are crucial ingredients for semantical analysis.

Instead in Plotkin's cbv λ-calulus,
λ_v-normal forms are essentially meaningless, while potentially valuable terms are relevants. Let $\Delta \subseteq \Lambda$ be a set of terms.

- The lazy Δ-reduction ($\rightarrow_{\Delta\ell}$) is the applicative closure of the following rule:

$$(\lambda x.M)N \rightarrow M[N/x] \text{ if and only if } N \in \Delta$$

- Let $I \equiv \lambda x.x$; then $\lambda x.II \rightarrow_{\Lambda} \lambda x.I$
  while $\lambda x.II \not\rightarrow_{\Lambda\ell} \lambda x.I$

- Let $\spadesuit \in \{\Delta, \Delta\ell\}$.
  $\rightarrow^{+}_{\spadesuit}$ is the transitive closure of $\rightarrow_{\spadesuit}$
  $\rightarrow^{*}_{\spadesuit}$ is the reflexive and transitive closure of $\rightarrow_{\spadesuit}$
  $=_{\spadesuit}$ is the symmetric, reflexive and transitive closure of $\rightarrow_{\spadesuit}$

Let $\spadesuit \in \{\Delta, \Delta\ell\}$ where $\Delta \subseteq \Lambda$ is a set of input values.

- A term M is in $\spadesuit$-normal form iff
  it has no occurrences of $\spadesuit$-redexes.

- A term M has $\spadesuit$-normal form iff
  $\exists N$ in $\spadesuit$-normal form s.t. $M \rightarrow^{*}_{\spadesuit} N$.

- A term M is $\spadesuit$-strongly normalizing iff
  all sequences of $\spadesuit$-reduction starting from M are finite.

  So, in particular, M has $\spadesuit$-normal form.

Blocked Δ-Normal Forms

If M is a Λ-NF then it has the following shape:

$$\lambda x_1 ... x_n . x M_1 ... M_m \qquad (n, m \geq 0)$$

where $M_i \in \Lambda$-NF, for all $i \leq m$.

Namely, the set Λ-NF of Λ-normal forms is

$$\begin{aligned}\Lambda\text{-NF} = \ \mathrm{Var} \ & \cup \{x M_1 ... M_n \mid M_k \in \Lambda\text{-NF } (1 \leq k \leq n)\} \\ & \cup \{\lambda x_1 ... x_n . M \mid M \in \Lambda\text{-NF}\}.\end{aligned}$$

If M is a Δ-NF then it has the following shape:

$$\lambda x_1 ... x_n . \zeta M_1 ... M_m \qquad (n, m \geq 0)$$

where $M_i \in \Delta$-NF, for all $i \leq m$ and

$$\text{either } \zeta = \begin{cases} x \in \mathrm{Var} & \text{or,} \\ (\lambda x.P)Q & \text{where} P, Q \in \Delta\text{-NF}, \ Q \notin \Delta. \end{cases}$$

Namely, the set Δ-NF of Δ-normal forms is

$$\begin{aligned}\Delta\text{-NF} = \mathrm{Var} \ & \cup \{x M_1 ... M_n \mid M_k \in \Delta\text{-NF } (1 \leq k \leq n)\} \\ & \cup \{\lambda x_1 ... x_n . M \mid M \in \Delta\text{-NF}\} \\ & \cup \{(\lambda x.P) Q M_1 ... M_n \mid P, Q, M_i \in \Delta\text{-NF}, Q \notin \Delta\}.\end{aligned}$$

As an example, $\lambda x.I(xx) \in \lambda\Gamma$-NF.
A Λ-theory equating two extensionally different Λ-NF is trivial

(Böhm Theorem).

On the other hand, we can equates different Δ-NF.
As an example, in the Γ-calculus just consider:

$$(\lambda xy.xy)(uu)(vI)$$
$$(\lambda xy.yx)(vI)(uu)$$

Γ-Liar Normal Forms

- Let $D \equiv \lambda x.xx$, since $(xI) \notin \Gamma$ and $(xI) \in \Gamma$-NF

$$(\lambda y.D)(xI)D \qquad \text{is a } \Gamma\text{-NF}.$$

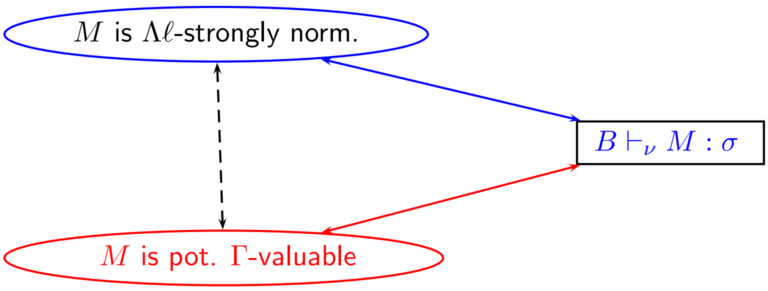
- Let $C[.]$ be a context:

$$C[(\lambda y.D)(xI)D] =_\Delta I \quad \text{iff} \quad C[DD] =_\Delta I$$

Definition 7

- A term M is Δ-valuable if and only if there is $N \in \Delta$ such that $M \to^*_\Delta N$.
- A term M is potentially Δ-valuable if and only if there is a (capture free) substitution $\mathbf{s}$, replacing variables by terms belonging to Δ, such that $\mathbf{s}(M)$ is Δ-valuable.
- M is a Δ-liar normal forms iff
  M is a Δ-normal forms non potentially Δ-valuable.

A Relating Result



Intersection Types

- Let C be a countable set of type-constants (ranging over $\alpha, \beta, ..$) containing at least the type constant ν.
- The set $T(\mathsf{C})$ of types, ranging over by $\sigma, \tau, \pi, \rho, ..$ is defined by:
  $\sigma ::= \alpha \mid (\sigma \to \tau) \mid (\sigma \wedge \tau)$
- A basis is a partial function from Var to $T(\mathsf{C})$ having a finite domain of definition.

Types will be considered modulo associativity, commutativity and idempotency of $\wedge$.

$[\sigma_1/x_1, ..., \sigma_n/x_n]$ will denote the basis B
s.t. $dom(B) = \{x_1, ..., x_n\}$ and $B(x_i) = \sigma_i$.

An Intersection Type Assignment System

Typing judgments $B \vdash_\nu M : \sigma$ are proved by the following rules:

$$\frac{}{B[\sigma/x] \vdash_\nu x : \sigma}\ (var) \qquad \frac{}{B \vdash_\nu \lambda x.M : \nu}\ (\nu)$$

$$\frac{B[\sigma/x] \vdash_\nu M : \tau}{B \vdash_\nu \lambda x.M : \sigma \to \tau}\ (\to I) \qquad \frac{B \vdash_\nu M : \sigma \to \tau \quad B \vdash_\nu N : \sigma}{B \vdash_\nu MN : \tau}\ (\to E)$$

$$\frac{B \vdash_\nu M : \sigma \quad B \vdash_\nu M : \tau}{B \vdash_\nu M : \sigma \wedge \tau}\ (\wedge I)$$

$$\frac{B \vdash_\nu M : \sigma \wedge \tau}{B \vdash_\nu M : \sigma}\ (\wedge E_l) \qquad \frac{B \vdash_\nu M : \sigma \wedge \tau}{B \vdash_\nu M : \tau}\ (\wedge E_r)$$

System Properties

Proposition 8

If d is a derivation of $B \vdash_\nu M : \sigma$ then every subterm of M, which is not under the scope of a λ-abstraction, is typed by a subderivation of d.

Proposition 9 **Subject-reduction**

If $B \vdash_\nu M : \sigma$ and $M \to_\Lambda N$ then $B \vdash_\nu N : \sigma$.

Proposition 10 **Typed subject-expansion**

Let $C[.]$ be a context. Then $B \vdash_\nu C[P[Q/x]] : \sigma$ and $B \vdash_\nu Q : \tau$ imply $B \vdash_\nu C[(\lambda x.P)Q] : \sigma$.

Characterizations

Theorem 11

$M \in \Lambda\ell$-SN implies M is typable in $\vdash_\nu$.

Theorem 12

$M \in \Gamma$-PV implies M is typable in $\vdash_\nu$.

Theorem 13

1. $B \vdash_\nu M : \sigma$ implies $M \in \Gamma$-PV.
2. $B \vdash_\nu M : \sigma$ implies $M \in \Lambda\ell$-SN.

Δ-Solvability

- A term M is Δ-solvable if and only if there is a head Δ-context $C[.] \equiv (\lambda\vec{x}.[.])\vec{N}$ such that:

$$\vec{x} \subseteq \mathrm{FV}(M)$$

$$\vec{N} \subseteq \Delta \qquad \text{and} \qquad C[M] =_\Delta I$$

- A term is Λ-solvable if and only if it has a Λ-head normal form.

Solvable of zero-degree

Definition 14

1. M is of Δ-order 0 if and only if there is no P such that $M \to^*_{\Delta\ell} \lambda x.P$;

2. M is of Δ-order $n \geq 1$ if and only if n is the maximum integer such that $M \to^*_{\Delta\ell} \lambda x_1.M_1$, $M_i \to^*_{\Delta\ell} \lambda x_{i+1}.M_{i+1}$ $(1 \leq i \leq n)$ and M_n is Δ-unsolvable of order 0. If such an n does not exists M is of Δ-order ∞.

Lemma 15

The class of Γ-solvable terms is properly included in the class of potentially Γ-valuable terms.

Proof. Let us first prove the inclusion. Let M be Γ-solvable, so there is a head context $(\lambda\vec{x}.[.])\vec{N}$ such that $(\lambda\vec{x}.M)\vec{N} \to^*_\Gamma I$ (since I is in normal form). Assume $\|\vec{x}\| \leq \|\vec{N}\|$ (otherwise, we can consider the context $(\lambda\vec{x}.[.])\vec{N}\underbrace{I.....I}_{p}$, where $p = \|\vec{x}\| - \|\vec{N}\|$)

and $\vec{N} \equiv \vec{N}_1\vec{N}_2$ such that $\|\vec{x}\| = \|\vec{N}_1\|$. So $M[\vec{N}_1/\vec{x}]\vec{N}_2 \to^*_\Gamma I$.
But $M[\vec{N}_1/\vec{x}]\vec{N}_2$ Γ-reduces to an abstraction implies that $M[\vec{N}_1/\vec{x}]$ reduces to an abstraction too, i.e it is Γ-valuable. So M is Γ-potentially valuable.
The inclusion is proper, since $\lambda x.DD$ is valuable, and so potentially valuable, but clearly Γ-unsolvable.

Γ-Head Normal Forms

Definition 16

1. The relation $\searrow \subseteq \Lambda \times \Lambda$ is defined inductively in the following way:
- $\lambda x.P \searrow \lambda x.Q$ if and only if $P \searrow Q$,
- $xM_1 \ldots M_m \searrow xN_1 \ldots N_m$ if and only if $M_i \rightarrow_{\Delta\ell} N_i \in \Lambda\ell$-NF $(1 \leq i \leq m)$,
- $(\lambda x.P)QM_1 \ldots M_m \searrow R$ if and only if $Q \rightarrow_{\Delta\ell} \bar{Q} \in \Lambda\ell$-NF and $P[\bar{Q}/x]M_1 \ldots M_m \searrow R$.

2. M is in Γ-head normal form (Γ-hnf) if and only if $M \equiv \lambda\vec{x}.xM_1 \ldots M_m$, and for all $1 \leq i \leq m$, $M_i \in \Lambda\ell$-NF;
   Γ-HNF denotes the set of all Γ-head normal forms.
3. M has Γ-head-normal form if and only if $M \searrow \lambda\vec{x}.xM_1 \ldots M_m$ and $M_i \in \Xi$, for all $1 \leq i \leq m$. $\|\vec{x}\|$ is the Γ-order and m is the Γ-degree of M.

Note that Γ-HNF is a proper subclass of Λ-HNF.
In fact, $\lambda x.x(DD) \in \Lambda$-HNF, but $\lambda x.x(DD) \notin \Gamma$-HNF since $DD \notin \Xi$.

Γ-Solvabilty Characterization

Theorem 17 **Γ-Solvability**

A term is Γ-solvable if and only if it has Γ-head-normal form.

A strange result: the theory is semi-sensible.

A call-by-value recursion operator is a term Z such that $ZM =_\Gamma M(\lambda z.ZMz)$, for all Γ-valuable terms M. Thus $\lambda x.(\lambda y.x(\lambda z.yyz))(\lambda y.x(\lambda z.yyz))$ is a such operator.

Theorem 18

Let Z be a call-by-value recursion operator.
If $B \equiv \lambda xyz.x(yz)$ then $I \sim_{\mathcal{V}} ZB$.

Separability

- Two terms M, N are Δ-separable if and only if there exists a context $C[.]$,
  such that $C[M] =_{\Delta} x$ and $C[N] =_{\Delta} y$.

- Böhm Theorem:
  2 different $\beta\eta$-normal forms are Λ-separable.

- The BT implies that if one were ever to postulate, as an extra axiom, the equality of two distinct normal forms, the resulting Λ-theory would be inconsistent.

  (A theory of the Δ-calculus is a congruence relation containing the relation $=_{\Delta}$.)

Different $\beta_v\eta_v$-normal forms are Γ-separable?

NO!

Theorem 19

Different $\beta\eta$-normal forms are Γ-separable.

The result is strongly based on characterization of potentially valuable terms.
In fact $\beta\eta$-normal forms are potentially valuable terms.

Completeness of Γ-Separability

$\forall M \in \Lambda$ without "Ω in its Böhm Tree" $\exists N \in \beta\eta$-normal form such that,

$$\text{for all } C[.] \begin{cases} C[M] \to_{\Gamma} x \quad \text{implies} \quad C[N] \to_{\Gamma} x \\ \\ C[N] \to_{\Gamma} x \quad \text{implies} \quad \begin{cases} C[M] \to_{\Gamma} x \\ \text{or} \\ C[M] \not\to_{\Gamma} \Gamma \end{cases} \end{cases}$$

Preliminaries

Let $B^n \equiv \lambda x_1 ... x_{n+1}.x_{n+1} x_1 ... x_n$,

$O^n \equiv \lambda x_1 ... x_{n+1}.x_{n+1}$ and

$U_i^n \equiv \lambda x_1 ... x_n.x_i$ $(i \leq n,\ n \in \mathbb{N})$.

Furthermore, if $S \subseteq \mathrm{Var}$ then let

$$X_{x,S}^n \equiv \begin{cases} \lambda z_1 \ldots z_n. \quad x_i z_1 \ldots z_n & \text{if } x \notin S, \\ x & \text{otherwise.} \end{cases}$$

Definition 20

Let c be a sequence of $n \geq 0$ natural numbers (ϵ denotes the empty sequence) and M, N be Λ-normal forms.

$M \not\simeq_c N$ if and only if one of following cases arises:

1. if $c \equiv \epsilon$ then either $|p - m| \neq |q - n|$ or $x \not\equiv y$;
2. if $c \equiv i, c'$ then $M =_\eta \lambda x_1 \ldots x_p.x M_1 \ldots M_m$ and $N =_\eta \lambda x_1 \ldots x_p.y N_1 \ldots N_m$ where $M_i \not\simeq_{c'} N_i$ $(1 \leq i \leq m)$.

Definition 21

Let $M \in \Lambda$-NF; $\mathsf{args}(M) \in \mathbb{N}$ is defined inductively as:

- $\mathsf{args}(x M_1 ... M_m) = \max\{m, \mathsf{args}(M_1), ..., \mathsf{args}(M_n)\}$;
- $\mathsf{args}(\lambda x.M) = \mathsf{args}(M)$.

Separability Algorithm

The proof will be given in a constructive way, by showing a separability algorithm. The algorithm is defined as a formal system, proving statements of the shape

$$M, N \Rrightarrow_\Gamma C[.],$$

where M, N are Λ-normal forms such that $M \not=_{\Lambda\eta} N$.

Let $M, N \in \Lambda$-NF, $M \not\simeq_c N$, $r \geq \max\{\mathsf{args}(M), \mathsf{args}(N)\}$ and $\tilde{x}, \tilde{y}$ be fresh variables such that $\tilde{x} \not\equiv \tilde{y}$.

$$\frac{p \leq q \qquad C^r_k[.] \equiv (\lambda x_1 ... x_k.[.])X^r_{x_1,\{x,y\}} ... X^r_{x_k,\{x,y\}} \quad (k \in \{p,q\}) \qquad \left.\begin{array}{r} x\mathsf{nf}_\Lambda(C^r_p[M_1])...\mathsf{nf}_\Lambda(C^r_p[M_m])X^r_{x_{p+1},\{x,y\}}...X^r_{x_q,\{x,y\}}, \\ y\mathsf{nf}_\Lambda(C^r_q[N_1])...\mathsf{nf}_\Lambda(C^r_q[N_n]) \end{array}\right\} \Rrightarrow_\Gamma C[.]}{\lambda x_1 \ldots x_p.xM_1 \ldots M_m, \lambda x_1 \ldots x_q.yN_1 \ldots N_n \Rrightarrow_\Gamma C[[.]X^r_{x_1,\{x,y\}} ... X^r_{x_q,\{x,y\}}]} \; (\Gamma 1)$$

$$\frac{p > q \qquad C^r_k[.] \equiv (\lambda x_1 ... x_k.[.])X^r_{x_1,\{x,y\}} ... X^r_{x_k,\{x,y\}} \quad (k \in \{p,q\}) \qquad \left.\begin{array}{r} x\mathsf{nf}_\Lambda(C^r_p[M_1])...\mathsf{nf}_\Lambda(C^r_p[M_m]), \\ y\mathsf{nf}_\Lambda(C^r_q[N_1])...\mathsf{nf}_\Lambda(C^r_q[N_n])X^r_{x_{q+1},\{x,y\}}...X^r_{x_p,\{x,y\}} \end{array}\right\} \Rrightarrow_\Gamma C[.]}{\lambda x_1 \ldots x_p.xM_1 \ldots M_m, \lambda x_1 \ldots x_q.yN_1 \ldots N_n \Rrightarrow_\Gamma C[[.]X^r_{x_1,\{x,y\}} ... X^r_{x_p,\{x,y\}}]} \; (\Gamma 2)$$

$$\frac{n < m}{xM_1 \ldots M_m, xN_1 \ldots N_n \Rrightarrow_\Gamma (\lambda x.[.])O^{r+n} \underbrace{B^r .....B^r}_{r+n-m} (\lambda x_1 \ldots x_{m-n}.\tilde{x}) \underbrace{\tilde{y} .....\tilde{y}}_{m-n}} \; (\Gamma 3)$$

$$\frac{m < n}{xM_1 \ldots M_m, xN_1 \ldots N_n \Rrightarrow_\Gamma (\lambda x.[.])O^{r+m} \underbrace{B^r .....B^r}_{r+m-n} (\lambda x_1 \ldots x_{n-m}.\tilde{y}) \underbrace{\tilde{x} .....\tilde{x}}_{n-m}} \; (\Gamma 4)$$

$$\frac{x \not\equiv y}{xM_1 \ldots M_m, yN_1 \ldots N_n \Rrightarrow_\Gamma (\lambda xy.[.])(\lambda x_1 \ldots x_{r+m}.\tilde{x})(\lambda x_1 \ldots x_{r+n}.\tilde{y}) \underbrace{B^r .....B^r}_{r}} \; (\Gamma 5)$$

$$\frac{M_k \not=_{\Lambda\eta} N_k \qquad x \notin \mathrm{FV}(M_k) \cup \mathrm{FV}(N_k) \quad M_k, N_k \Rrightarrow_\Gamma C[.]}{xM_1 \ldots M_m, xN_1 \ldots N_m \Rrightarrow_\Gamma C[(\lambda x.[.])U^r_k \underbrace{B^r .....B^r}_{r-m}]} \; (\Gamma 6)$$

$$\frac{M_k \not=_{\Lambda\eta} N_k \qquad x \in \mathrm{FV}(M_k) \cup \mathrm{FV}(N_k) \qquad C^r_x[.] \equiv (\lambda x.[.])B^r \quad \mathsf{nf}_\Lambda(C^r_x[M_k]), \mathsf{nf}_\Lambda(C^r_x[N_k]) \Rrightarrow_\Gamma C[.]}{xM_1 \ldots M_m, xN_1 \ldots N_m \Rrightarrow_\Gamma C[C^r_x[.] \underbrace{B^r .....B^r}_{r-m} U^r_k]} \; (\Gamma 7)$$

Bibliography

- L. Paolini, E. Pimentel and S. Ronchi Della Rocca. Strong Normalization from an unusual point of view, Theoretical Computer Science, ??? 2009.
- L. Paolini, E. Pimentel and S. Ronchi Della Rocca. Lazy strong normalization, Electronic Notes in Theoretical Computer Science, volume 136C, pages 103-116, 2005.
- S. Ronchi Della Rocca and L. Paolini. The Parametric -Calculus: a Metamodel for Computation, Springer-Verlag, Texts in Theoretical Computer Science: An EATCS Series, 2004, Berlin.
- L. Paolini. Call-by-Value Separability and Computability. Lecture Notes in Computer Science volume 2202, pages 74–89, 2002, Springer-Verlag.
- L. Paolini and S. Ronchi Della Rocca. Call-by-value Solvability, Theoretical Informatics and Applications, 33(6):507-534, 1999.
- L. Paolini. La chiamata per Valore e La valutazione pigra nel -calcolo, Tesi di Laurea, Torino 1998.

Thank you !