

Call-By-Value Lambda Calculus

Solvability and Separability

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Outline

Parametric

Valuability

Solvability

Separability

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Parametric Parameter Passing Lambda-Calculus

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Parametric Lambda-Calculus

- The set Λ is produced by

$$M ::= x \mid MM \mid \lambda x.M$$

- Let $\Delta \subseteq \Lambda$.

The Δ -reduction (\rightarrow_{Δ}) is the **contextual** closure of the following rule:

$$(\lambda x.M)N \rightarrow_{\Delta} M[N/x] \quad \text{whenever } N \in \Delta$$

- 👉 When a set Δ induces an interesting calculus ?

It is reasonable to respect next remarks.

1. **Variables** can be considered as placeholder for values.
2. The status of being an **input value** should be preserved during the evaluation process.

Definition 1

$\Delta \subseteq \Lambda$ is a **set of input values**, when:

- $\text{Var} \subseteq \Delta$ (variable closure)
- $P, Q \in \Delta$ implies $P[Q/x] \in \Delta$, (substitution closure)
- $M \in \Delta$ and $M \rightarrow_{\Delta} N$ imply $N \in \Delta$ (reduction closure)

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Instances of Values

- Λ is a set of input values
(the $\lambda\Lambda$ -calculus is the classical λ -calculus)
- $\Gamma = \text{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$ is a set of input values
(the $\lambda\Gamma$ -calculus is the Plotkin's λ_v -calculus)
- $\Xi = \text{Var} \cup \{M \mid M \text{ is a closed } \beta \text{ normal form}\}$ is a set of input value

On the other hand,

- Λ -normal forms are not input values
- Λ -head normal forms are not input values
- Γ -normal forms are not input values

Confluence

Theorem 2

$M \rightarrow_{\Delta}^* N_1$ and $M \rightarrow_{\Delta}^* N_2$ imply that

there is N_3 such that both $N_1 \rightarrow_{\Delta}^* N_3$ and $N_2 \rightarrow_{\Delta}^* N_3$.

The closure conditions on the set Δ of input values assure that the corresponding Δ -calculus enjoys the confluence property.

Corollary 3

The Δ -normal form of a term, if it exists, is unique.

Standard Strategies

Assume $M \rightarrow_{\Delta}^* N$, and assume that there is more than one Δ -reduction sequence from M to N . The standardization theorem says that, in case the set of input values enjoys a particular property, there is a “standard” reduction sequence from M to N , reducing the redexes in a given order.

- A zero-step reduction sequence is always standard
- A one-stp reduction sequence is always standard.
- Standardization is not Normalization.
- A standard reduction sequence does not need to reduce the lefmost redex at every step.

In case of Λ -calculus a standard reduction sequence is a strategy choosing redexes from left to right. For instance,

$$(\lambda x.x(KI))(II) \rightarrow_{\Lambda} II(KI) \rightarrow_{\Lambda} I(KI) \rightarrow_{\Lambda} I(\lambda y.I)$$

is a standard reduction sequence in the Λ -calculus.

The reduction sequence reducing the same term from left to right in the Γ -calculus:

$$(\lambda x.x(KI))(II) \rightarrow_{\Gamma} (\lambda x.x(\lambda y.I))(II) \rightarrow_{\Gamma} (\lambda x.x(\lambda y.I))I \rightarrow_{\Gamma} I(\lambda y.I)$$

is a standard reduction sequence?

It is not easy to formalize such a strategy for other sets of input values.

Sequentialization

- The Δ -sequentialization $(M)^{\circ}$ of a term M is a function from Λ to Λ defined as follows:
 - $(xM_1 \dots M_m)^{\circ} = x(M_1)^{\circ} \dots (M_m)^{\circ}$;
 - $((\lambda x.P)QM_1 \dots M_m)^{\circ} = (\lambda x.P)^{\circ}(Q)^{\circ}(M_1)^{\circ} \dots (M_m)^{\circ}$ if $Q \in \Delta$;
 - $((\lambda x.P)QM_1 \dots M_m)^{\circ} = (Q)^{\circ}(\lambda x.P)^{\circ}(M_1)^{\circ} \dots (M_m)^{\circ}$ if $Q \notin \Delta$;
 - $(\lambda x.P)^{\circ} = \lambda x.(P)^{\circ}$.
- A symbol λ in a term M is **active** if and only if it is the first symbol of a Δ -redex of M .
- The **degree** of a redex R in M is the numbers of λ 's which both are active in M and occur on the left of $(R)^{\circ}$ in $(M)^{\circ}$.
- A sequence $M \equiv P_0 \rightarrow_{\Delta} P_1 \rightarrow_{\Delta} \dots \rightarrow_{\Delta} P_n \rightarrow_{\Delta} N$ is **standard** if and only if the degree of the redex contracted in P_i is less than or equal to the degree of the redex contracted in P_{i+1} , for every $i < n$.
We denote by $M \rightarrow_{\Delta}^{\circ} N$ a standard reduction sequence from M to N .
- The **principal redex** of M , if it exists, is the redex of M with minimum degree. The **principal reduction** $M \rightarrow_{\Delta}^p N$ denotes that N is obtained from M by reducing the principal redex of M . Moreover, $\rightarrow_{\Delta}^{*p}$ is the reflexive and transitive closure of \rightarrow_{Δ}^p .

Standard Values

$(\lambda x.x(KI))(II) \rightarrow_{\Gamma} (\lambda x.x(KI))I \rightarrow_{\Gamma} I(KI) \rightarrow_{\Gamma} I(\lambda y.I)$ is a standard reduction sequence in the Γ -calculus.

Definition 4

A set Δ of input values is **standard** if and only if $M \notin \Delta$ and $M \rightarrow_{\Delta}^* N$ by reducing at every step a not principal redex imply $N \notin \Delta$.

- The set of input values Λ, Γ are standard.
- The set of input values Λ_I is a not standard set of input value.

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Standardization

Theorem 5

Let Δ be standard. $M \rightarrow_{\Delta}^* N$ implies there is a standard reduction sequence from M to N .

Corollary 6

Let Δ be standard.

If $M \rightarrow_{\Delta}^* N$ and N is a normal form then $M \rightarrow_{\Delta}^{*p} N$.

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Δ -Normal Forms

In the classical λ -calculus,
 β -normal forms are crucial ingredients for semantical analysis.

Instead in Plotkin's cbv λ -calculus,
 λ_v -normal forms are essentially meaningless, while **potentially valuable terms** are relevant. Let $\Delta \subseteq \Lambda$ be a set of terms.

- The **lazy Δ -reduction** ($\rightarrow_{\Delta\ell}$) is the **applicative** closure of the following rule:

$$(\lambda x.M)N \rightarrow M[N/x] \text{ if and only if } N \in \Delta$$

- Let $I \equiv \lambda x.x$; then $\lambda x.II \rightarrow_{\Delta} \lambda x.I$
while $\lambda x.II \not\rightarrow_{\Delta\ell} \lambda x.I$

- Let $\spadesuit \in \{\Delta, \Delta\ell\}$.
 - $\rightarrow_{\spadesuit}^+$ is the **transitive** closure of \rightarrow_{\spadesuit}
 - $\rightarrow_{\spadesuit}^*$ is the **reflexive and transitive** closure of \rightarrow_{\spadesuit}
 - $=_{\spadesuit}$ is the **symmetric, reflexive and transitive** closure of \rightarrow_{\spadesuit}

Let $\spadesuit \in \{\Delta, \Delta\ell\}$ where $\Delta \subseteq \Lambda$ is a set of input values.

- A term M is in **\spadesuit -normal form** iff
it has no occurrences of \spadesuit -redexes.
- A term M **has \spadesuit -normal form** iff
 $\exists N$ in \spadesuit -normal form s.t. $M \rightarrow_{\spadesuit}^* N$.
- A term M is **\spadesuit -strongly normalizing** iff
all sequences of \spadesuit -reduction starting from M are finite.

So, in particular, M has \spadesuit -normal form.

Blocked Δ -Normal Forms

If M is a Λ -NF then it has the following shape:

$$\lambda x_1 \dots x_n. x M_1 \dots M_m \quad (n, m \geq 0)$$

where $M_i \in \Lambda$ -NF, for all $i \leq m$.

Namely, the set Λ -NF of Λ -normal forms is

$$\Lambda\text{-NF} = \text{Var} \cup \{x M_1 \dots M_n \mid M_k \in \Lambda\text{-NF} (1 \leq k \leq n)\} \\ \cup \{\lambda x_1 \dots x_n. M \mid M \in \Lambda\text{-NF}\}.$$

If M is a Δ -NF then it has the following shape:

$$\lambda x_1 \dots x_n. \zeta M_1 \dots M_m \quad (n, m \geq 0)$$

where $M_i \in \Delta$ -NF, for all $i \leq m$ and

$$\text{either } \zeta = \begin{cases} x \in \text{Var} \\ (\lambda x. P) Q \end{cases} \quad \text{or,} \\ \text{where } P, Q \in \Delta\text{-NF}, Q \notin \Delta.$$

Namely, the set Δ -NF of Δ -normal forms is

$$\Delta\text{-NF} = \text{Var} \cup \{x M_1 \dots M_n \mid M_k \in \Delta\text{-NF} (1 \leq k \leq n)\} \\ \cup \{\lambda x_1 \dots x_n. M \mid M \in \Delta\text{-NF}\} \\ \cup \{(\lambda x. P) Q M_1 \dots M_n \mid P, Q, M_i \in \Delta\text{-NF}, Q \notin \Delta\}.$$

As an example, $\lambda x. I(xx) \in \Lambda\Gamma$ -NF.

A Λ -theory equating two extensionally different Λ -NF is trivial

(Böhm Theorem).

On the other hand, we can equate different Δ -NF.

As an example, in the Γ -calculus just consider:

$$(\lambda xy. xy)(uu)(vI) \\ (\lambda xy. yx)(vI)(uu)$$

Γ -Liar Normal Forms

- Let $D \equiv \lambda x.xx$, since $(xI) \notin \Gamma$ and $(xI) \in \Gamma\text{-NF}$

$(\lambda y.D)(xI)D$ is a $\Gamma\text{-NF}$.

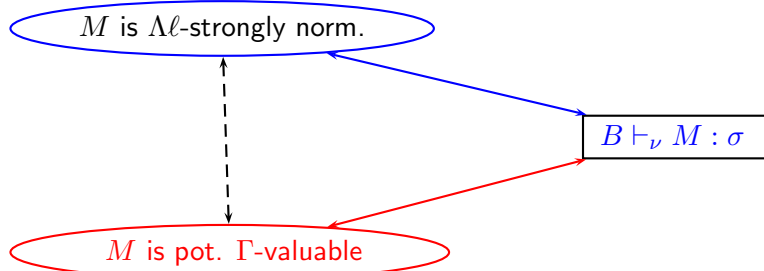
- Let $C[\cdot]$ be a context:

$$C[(\lambda y.D)(xI)D] =_{\Delta} I \quad \text{iff} \quad C[DD] =_{\Delta} I$$

Definition 7

- A term M is Δ -valuable if and only if there is $N \in \Delta$ such that $M \rightarrow_{\Delta}^* N$.
- A term M is **potentially Δ -valuable** if and only if there is a (capture free) substitution s , replacing variables by terms belonging to Δ , such that $s(M)$ is Δ -valuable.
- M is a **Δ -liar normal forms** iff M is a Δ -normal forms non potentially Δ -valuable.

A Relating Result



Intersection Types

- Let C be a countable set of **type-constants** (ranging over α, β, \dots) containing at least the type constant ν .
- The set $T(C)$ of **types**, ranging over by $\sigma, \tau, \pi, \rho, \dots$ is defined by:

$$\sigma ::= \alpha \mid (\sigma \rightarrow \tau) \mid (\sigma \wedge \tau)$$
- A **basis** is a partial function from Var to $T(C)$ having a finite domain of definition.

Types will be considered modulo associativity, commutativity and idempotency of \wedge .

$[\sigma_1/x_1, \dots, \sigma_n/x_n]$ will denote the basis B

s.t. $\text{dom}(B) = \{x_1, \dots, x_n\}$ and $B(x_i) = \sigma_i$.

An Intersection Type Assignment System

Typing judgments $B \vdash_\nu M : \sigma$ are proved by the following rules:

$$\frac{}{B[\sigma/x] \vdash_\nu x : \sigma} \text{ (var)}$$

$$\frac{}{B \vdash_\nu \lambda x.M : \nu} \text{ } (\nu)$$

$$\frac{B[\sigma/x] \vdash_\nu M : \tau}{B \vdash_\nu \lambda x.M : \sigma \rightarrow \tau} \text{ } (\rightarrow I) \quad \frac{B \vdash_\nu M : \sigma \rightarrow \tau \quad B \vdash_\nu N : \sigma}{B \vdash_\nu MN : \tau} \text{ } (\rightarrow E)$$

$$\frac{B \vdash_\nu M : \sigma \quad B \vdash_\nu M : \tau}{B \vdash_\nu M : \sigma \wedge \tau} \text{ } (\wedge I)$$

$$\frac{B \vdash_\nu M : \sigma \wedge \tau}{B \vdash_\nu M : \sigma} \text{ } (\wedge E_l) \quad \frac{B \vdash_\nu M : \sigma \wedge \tau}{B \vdash_\nu M : \tau} \text{ } (\wedge E_r)$$

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System Properties

Proposition 8

If d is a derivation of $B \vdash_\nu M : \sigma$ then every subterm of M , which is not under the scope of a λ -abstraction, is typed by a subderivation of d .

Proposition 9 Subject-reduction

If $B \vdash_\nu M : \sigma$ and $M \rightarrow_\Delta N$ then $B \vdash_\nu N : \sigma$.

Proposition 10 Typed subject-expansion

Let $C[\cdot]$ be a context. Then $B \vdash_\nu C[P[Q/x]] : \sigma$ and $B \vdash_\nu Q : \tau$ imply $B \vdash_\nu C[(\lambda x.P)Q] : \sigma$.

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Characterizations

Theorem 11

$M \in \Lambda\ell\text{-SN}$ implies M is typable in \vdash_ν .

Theorem 12

$M \in \Gamma\text{-PV}$ implies M is typable in \vdash_ν .

Theorem 13

1. $B \vdash_\nu M : \sigma$ implies $M \in \Gamma\text{-PV}$.
2. $B \vdash_\nu M : \sigma$ implies $M \in \Lambda\ell\text{-SN}$.

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Δ -Solvability

- A term M is **Δ -solvable** if and only if there is a head **Δ -context** $C[\cdot] \equiv (\lambda \vec{x}. [\cdot]) \vec{N}$ such that:

$$\vec{x} \subseteq \text{FV}(M)$$

$$\vec{N} \subseteq \Delta \quad \text{and} \quad C[M] =_{\Delta} I$$

- A term is **Λ -solvable** if and only if

it has a **Λ -head normal form**.

Solvable of zero-degree

Definition 14

1. M is of Δ -order 0 if and only if there is no P such that $M \rightarrow_{\Delta \ell}^* \lambda x.P$;
2. M is of Δ -order $n \geq 1$ if and only if n is the maximum integer such that $M \rightarrow_{\Delta \ell}^* \lambda x_1.M_1$, $M_i \rightarrow_{\Delta \ell}^* \lambda x_{i+1}.M_{i+1}$ ($1 \leq i \leq n$) and M_n is Δ -unsolvable of order 0. If such an n does not exist M is of Δ -order ∞ .


Lemma 15

The class of Γ -solvable terms is properly included in the class of potentially Γ -valuable terms.

Proof. Let us first prove the inclusion. Let M be Γ -solvable, so there is a head context $(\lambda \vec{x}. [\cdot]) \vec{N}$ such that $(\lambda \vec{x}. M) \vec{N} \rightarrow_{\Gamma}^* I$ (since I is in normal form). Assume $\|\vec{x}\| \leq \|\vec{N}\|$ (otherwise, we can consider the context $(\lambda \vec{x}. [\cdot]) \vec{N} \underbrace{I, \dots, I}_p$, where $p = \|\vec{x}\| - \|\vec{N}\|$)

and $\vec{N} \equiv \vec{N}_1 \vec{N}_2$ such that $\|\vec{x}\| = \|\vec{N}_1\|$. So $M[\vec{N}_1/\vec{x}] \vec{N}_2 \rightarrow_{\Gamma}^* I$.

But $M[\vec{N}_1/\vec{x}] \vec{N}_2$ Γ -reduces to an abstraction implies that $M[\vec{N}_1/\vec{x}]$ reduces to an abstraction too, i.e it is Γ -valuable. So M is Γ -potentially valuable.

The inclusion is proper, since $\lambda x.DD$ is valuable, and so potentially valuable, but clearly Γ -unsolvable. 

Γ -Head Normal Forms

Definition 16

1. The relation $\searrow \subseteq \Lambda \times \Lambda$ is defined inductively in the following way:
 - $\lambda x.P \searrow \lambda x.Q$ if and only if $P \searrow Q$,
 - $xM_1 \dots M_m \searrow xN_1 \dots N_m$ if and only if $M_i \rightarrow_{\Delta\ell} N_i \in \Lambda\ell\text{-NF}$ ($1 \leq i \leq m$),
 - $(\lambda x.P)QM_1 \dots M_m \searrow R$ if and only if $Q \rightarrow_{\Delta\ell} \bar{Q} \in \Lambda\ell\text{-NF}$ and $P[\bar{Q}/x]M_1 \dots M_m \searrow R$.
2. M is in Γ -head normal form (Γ -hnf) if and only if $M \equiv \lambda \vec{x}.xM_1 \dots M_m$, and for all $1 \leq i \leq m$, $M_i \in \Lambda\ell\text{-NF}$;
 $\Gamma\text{-HNF}$ denotes the set of all Γ -head normal forms.
3. M has Γ -head-normal form if and only if $M \searrow \lambda \vec{x}.xM_1 \dots M_m$ and $M_i \in \Xi$, for all $1 \leq i \leq m$.
 $\|\vec{x}\|$ is the Γ -order and m is the Γ -degree of M .

Note that $\Gamma\text{-HNF}$ is a proper subclass of $\Lambda\text{-HNF}$.

In fact, $\lambda x.x(DD) \in \Lambda\text{-HNF}$, but $\lambda x.x(DD) \notin \Gamma\text{-HNF}$ since $DD \notin \Xi$.

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Γ -Solvability Characterization

Theorem 17 Γ -Solvability

A term is Γ -solvable if and only if it has Γ -head-normal form.

A strange result: the theory is **semi-sensible**.

A call-by-value recursion operator is a term Z such that $ZM =_{\Gamma} M(\lambda z.ZMz)$, for all Γ -valuable terms M . Thus $\lambda x.(\lambda y.x(\lambda z.yyz))(\lambda y.x(\lambda z.yyz))$ is a such operator.

Theorem 18

Let Z be a call-by-value recursion operator.
 If $B \equiv \lambda xyz.x(yz)$ then $I \sim_{\gamma} ZB$.

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Separability

- Two terms M, N are Δ -separable if and only if there exists a context $C[.]$, such that $C[M] =_{\Delta} x$ and $C[N] =_{\Delta} y$.
- Böhm Theorem: 2 different $\beta\eta$ -normal forms are Δ -separable.
- The BT implies that if one were ever to postulate, as an extra axiom, the equality of two distinct normal forms, the resulting Δ -theory would be **inconsistent**.

(A theory of the Δ -calculus is a congruence relation containing the relation $=_{\Delta}$.)

Different $\beta_v\eta_v$ -normal forms are Γ -separable?

NO!

Theorem 19

Different $\beta\eta$ -normal forms are Γ -separable.

The result is strongly based on characterization of potentially valuable terms.
In fact $\beta\eta$ -normal forms are potentially valuable terms.

Completeness of Γ -Separability

$\forall M \in \Lambda$ without “ Ω in its Böhm Tree” $\exists N \in \beta\eta$ -normal form such that,

$$\text{for all } C[.] \left\{ \begin{array}{l} C[M] \rightarrow_{\Gamma} x \text{ implies } C[N] \rightarrow_{\Gamma} x \\ C[N] \rightarrow_{\Gamma} x \text{ implies } \left\{ \begin{array}{l} C[M] \rightarrow_{\Gamma} x \\ \text{or} \\ C[M] \not\rightarrow_{\Gamma} \Gamma \end{array} \right. \end{array} \right.$$

Preliminaries

Let $B^n \equiv \lambda x_1 \dots x_{n+1}. x_{n+1} x_1 \dots x_n$,

$O^n \equiv \lambda x_1 \dots x_{n+1}. x_{n+1}$ and

$U_i^n \equiv \lambda x_1 \dots x_n. x_i$ ($i \leq n, n \in \mathbb{N}$).

Furthermore, if $S \subseteq \text{Var}$ then let

$$X_{x,S}^n \equiv \begin{cases} \lambda z_1 \dots z_n. x_i z_1 \dots z_n & \text{if } x \notin S, \\ x & \text{otherwise.} \end{cases}$$

Definition 20

Let c be a sequence of $n \geq 0$ natural numbers (ϵ denotes the empty sequence) and M, N be Λ -normal forms.

$M \not\equiv_c N$ if and only if one of following cases arises:

1. if $c \equiv \epsilon$ then either $|p - m| \neq |q - n|$ or $x \not\equiv y$;
2. if $c \equiv i, c' \equiv i'$ then $M =_{\eta} \lambda x_1 \dots x_p. x M_1 \dots M_m$ and $N =_{\eta} \lambda x_1 \dots x_p. y N_1 \dots N_m$ where $M_i \not\equiv_{c'} N_i$ ($1 \leq i \leq m$).

Definition 21

Let $M \in \Lambda\text{-NF}$; $\text{args}(M) \in \mathbb{N}$ is defined inductively as:

- $\text{args}(x M_1 \dots M_m) = \max\{m, \text{args}(M_1), \dots, \text{args}(M_m)\}$;
- $\text{args}(\lambda x. M) = \text{args}(M)$.

Separability Algorithm

The proof will be given in a constructive way, by showing a **separability algorithm**. The algorithm is defined as a formal system, proving statements of the shape

$$M, N \Rightarrow_{\Gamma} C[.],$$

where M, N are Λ -normal forms such that $M \neq_{\Lambda\eta} N$.

Let $M, N \in \Lambda\text{-NF}$, $M \not\approx_c N$, $r \geq \max\{\text{args}(M), \text{args}(N)\}$ and \tilde{x}, \tilde{y} be fresh variables such that $\tilde{x} \neq \tilde{y}$.

$$\frac{\begin{array}{l} p \leq q \quad C_k^r[.] \equiv (\lambda x_1 \dots x_k \cdot [.] X_{x_1, \{x, y\}}^r \dots X_{x_k, \{x, y\}}^r \quad (k \in \{p, q\}) \\ \left. \begin{array}{l} \text{nf}_{\lambda}(C_p^r[M_1]) \dots \text{nf}_{\lambda}(C_p^r[M_m]) X_{x_{p+1}, \{x, y\}}^r \dots X_{x_q, \{x, y\}}^r, \\ \text{ynf}_{\lambda}(C_q^r[N_1]) \dots \text{nf}_{\lambda}(C_q^r[N_n]) \end{array} \right\} \Rightarrow_{\Gamma} C[.] \end{array}}{\lambda x_1 \dots x_p \cdot x M_1 \dots M_m, \lambda x_1 \dots x_q \cdot y N_1 \dots N_n \Rightarrow_{\Gamma} C[[.] X_{x_1, \{x, y\}}^r \dots X_{x_q, \{x, y\}}^r]} \quad (\Gamma 1)$$

$$\frac{\begin{array}{l} p > q \quad C_k^r[.] \equiv (\lambda x_1 \dots x_k \cdot [.] X_{x_1, \{x, y\}}^r \dots X_{x_k, \{x, y\}}^r \quad (k \in \{p, q\}) \\ \left. \begin{array}{l} \text{nf}_{\lambda}(C_p^r[M_1]) \dots \text{nf}_{\lambda}(C_p^r[M_m]), \\ \text{ynf}_{\lambda}(C_q^r[N_1]) \dots \text{nf}_{\lambda}(C_q^r[N_n]) X_{x_{q+1}, \{x, y\}}^r \dots X_{x_p, \{x, y\}}^r \end{array} \right\} \Rightarrow_{\Gamma} C[.] \end{array}}{\lambda x_1 \dots x_p \cdot x M_1 \dots M_m, \lambda x_1 \dots x_q \cdot y N_1 \dots N_n \Rightarrow_{\Gamma} C[[.] X_{x_1, \{x, y\}}^r \dots X_{x_p, \{x, y\}}^r]} \quad (\Gamma 2)$$

$$\frac{n < m}{x M_1 \dots M_m, x N_1 \dots N_n \Rightarrow_{\Gamma} (\lambda x \cdot [.] O^{r+n} \underbrace{B^r \dots B^r}_{r+n-m} (\lambda x_1 \dots x_{m-n} \cdot \tilde{x}) \underbrace{\tilde{y} \dots \tilde{y}}_{m-n})} \quad (\Gamma 3)$$

$$\frac{m < n}{x M_1 \dots M_m, x N_1 \dots N_n \Rightarrow_{\Gamma} (\lambda x \cdot [.] O^{r+m} \underbrace{B^r \dots B^r}_{r+m-n} (\lambda x_1 \dots x_{n-m} \cdot \tilde{y}) \underbrace{\tilde{x} \dots \tilde{x}}_{n-m})} \quad (\Gamma 4)$$

$$\frac{x \neq y}{x M_1 \dots M_m, y N_1 \dots N_n \Rightarrow_{\Gamma} (\lambda x y \cdot [.] (\lambda x_1 \dots x_{r+m} \cdot \tilde{x}) (\lambda x_1 \dots x_{r+n} \cdot \tilde{y}) \underbrace{B^r \dots B^r}_r)} \quad (\Gamma 5)$$

$$\frac{M_k \neq_{\Lambda\eta} N_k \quad x \notin \text{FV}(M_k) \cup \text{FV}(N_k) \quad M_k, N_k \Rightarrow_{\Gamma} C[.]}{x M_1 \dots M_m, x N_1 \dots N_m \Rightarrow_{\Gamma} C[(\lambda x \cdot [.] U_k^r \underbrace{B^r \dots B^r}_{r-m})]} \quad (\Gamma 6)$$

$$\frac{\begin{array}{l} M_k \neq_{\Lambda\eta} N_k \quad x \in \text{FV}(M_k) \cup \text{FV}(N_k) \\ C_x^r[.] \equiv (\lambda x \cdot [.] B^r \quad \text{nf}_{\lambda}(C_x^r[M_k]), \text{nf}_{\lambda}(C_x^r[N_k]) \Rightarrow_{\Gamma} C[.] \end{array}}{x M_1 \dots M_m, x N_1 \dots N_m \Rightarrow_{\Gamma} C[C_x^r[.] \underbrace{B^r \dots B^r}_{r-m} U_k^r]} \quad (\Gamma 7)$$

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Thank you !

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