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# Parametric parameter passing λ-calculus

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**Abstract**

A λ-calculus is defined, which is parametric with respect to a set $\mathsf{V}$ of input values and subsumes all the different λ-calculi given in the literature, in particular the classical one and the call-by-value λ-calculus of Plotkin. It is proved that it enjoy the confluence property, and a necessary and sufficient condition is given, under which it enjoys the standardization property. Its operational semantics is given through a reduction machine, parametric with respect to both $\mathsf{V}$ and a set $\mathsf{V}_o$ of output values.


*Keywords:* λ-Calculus; Confluence; Standardization

## 1. Introduction

The λ-calculus, in its different variants, has been used as paradigmatic language for studying various properties of programming languages. In particular, classical $\lambda\beta$-calculus of Curry [2,10] and $\lambda\beta_v$-calculus of Plotkin [18] are paradigms for two different parameter passing policies, the call-by-name and the call-by-value, respectively. Although the lexicon of both languages is the same, the reduction rule of $\lambda\beta_v$-calculus is obtained as a restriction of the classical $\beta$-rule. Thus, these two λ-calculi appear different both from syntactic and semantic point of view: in fact they have been studied using different tools [2,9,17].

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In this paper we propose a new λ-calculus, the λV-calculus, which is parametric with respect to a subset V of terms that we call *input values*. The λV-calculus is a call-by-value calculus, in the sense that the reduction rule is a kind of conditioned $\beta$-rule, firing just in case the argument belong to V. Informally, input values represent partially evaluated terms, that can be passed as parameters. The only conditions we ask on the set V is to be closed under substitution and reduction: these conditions are quite natural, in order to preserve the status of an input, during the computation.

The λV-calculus subsumes a plethora of different variants of calculi, including both $\lambda\beta$ and $\lambda\beta_v$ calculi. The $\lambda\beta$-calculus is obtained by putting $\mathsf{V} = \Lambda$, while $\lambda\beta_v$-calculus by putting $\mathsf{V} = \mathsf{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$, i.e., variables and abstractions. Moreover it can suggest new kinds of calculi: in particular, we can easily prove that calculi already studied, as the calculus obtained by choosing as input values the set $\mathsf{Var} \cup \{M \mid M$ is a closed $\beta$-normal-form$\}$ [8,11], enjoy good properties.

The interest of such a new λ-calculus is that it is a setting where different λ-calculi can be studied in an uniform manner. As an example, we explore the conditions on the set of input values that guarantee confluence property and standardization property, which are two basic properties we expect for a sequential programming language.

Confluence assures us that, when the result of a computation exists, it is unique; we prove that, for every choice of input values, the λV-calculus enjoys this property.

The standardization property says that every reduction sequence can be "sequentialized" in a given order. At a first sight it's difficult to deal with the standardization in a uniform manner. Both $\lambda\beta$ and $\lambda\beta_v$ calculi enjoy standardization, but in the first calculus a reduction when redexes are reduced from left to right is always standard, while in the second one the order is very tricky, see [15,18].

For example, let us consider the term $M \equiv (\lambda x.xx)(II)$, where $I \equiv \lambda x.x$. Clearly $M$ reduces to $I$ in both $\lambda\beta$ and $\lambda\beta_v$ calculi, but in $\lambda\beta$-calculus the standard reduction sequence is: $(\lambda x.xx)(II) \to_\beta II(II) \to_\beta I(II) \to_\beta II \to_\beta I$, while in $\lambda\beta_v$-calculus the standard reduction sequence is: $(\lambda x.xx)(II) \to_{\beta_v} (\lambda x.xx)I \to_{\beta_v} II \to_{\beta_v} I$.

We give a notion of "sequentialization" that subsumes both cases, and we state a necessary and sufficient condition on the set of input values that assures the standardization property.

In the literature about λ-calculus, two notions of standardization has been defined, the classical one [2], and a "strong" one, [12]. According to the former, a given reduction sequence can be standardized in more than one way, while, according to the latter, there is just one standard reduction sequence corresponding to a given one. We choose this second approach.

In fact, in this case, the standardization implies the existence of a principal reduction strategy (reducing always the first redex in the "sequentialization"), which is normalizing. Thus, we show as various operational semantics can be defined in a uniform, parametric way. Namely we define a reduction machine, parametric with respect to both the set of input values V and a set of *output values* $\mathsf{V}_o$, that implements such a strategy, and that can be seen as a "universal λ-machine."

The machine is described in a logical form. Standard reduction machines, as that one performing the head reduction for the λ-calculus or the S.E.C.D. machine of Landin [13] can be obtained from it just instantiating V and $\mathsf{V}_o$ in suitable ways.

A discussion about the motivations for our choice of strong standardization is the topic of the last section of this paper, together with the comparisons with other definitions of standardization given in the literature.

Not all the key properties of $\lambda\beta$-calculus can be studied in an uniform manner using as tool the λV-calculus: in [16] we proved that the notion of solvability is quite different in $\lambda\beta$ and in $\lambda\beta_v$

settings. The definition is uniform, i.e., a term is solvable if and only if it can reduce to the identity, when applied to suitable arguments. But in $\lambda\beta$ this notion corresponds to an operational property of terms, being solvable all and only the terms that reduce to head normal form, while in $\lambda\beta_v$ the solvability cannot be expressed through the reduction rule.

The $\lambda\mathsf{V}$-calculus has been already introduced in [20], and has been used for defining a new parametric notion of extensionality, related to operational semantics. The proof of confluence we develop in this paper has been already suggested there.

We plan to extend the study of the $\lambda\mathsf{V}$-calculus, by studying its semantics, both operational and denotational, hoping to solve some characterization problems of the models of $\lambda\beta_v$ stated in [19].

An interesting uniform approach to call-by-name and call-by-value computations, in a typed setting, has been presented in [5], using a language derived from Gentzen's sequence calculus *LK*.

The paper is organized as follows. In Section 2 we define the basic notions of the $\mathsf{V}$-calculus, Section 3 and 4 contain, respectively, the proofs of the confluence property and of the standardization property, Section 5 contains a discussion on the constraints we imposed on sets of input values, and in Section 6 the operational machine is defined. Section 7 contains the comparison with other approaches to the standardization present in the literature.

## 2. $\Lambda_{\mathsf{V}}$-language and $\mathsf{V}$-reduction

The $\lambda\mathsf{V}$-calculus is the language $\Lambda$ equipped with a set $\mathsf{V} \subseteq \Lambda$ of input values, satisfying some closure conditions. Informally, input values represent partially evaluated terms, that can be passed as parameters. Call-value and call-by-name parameter passing can be seen as the two most radical choices: parameters are not evaluated in the former policy, while in the latter they are evaluated until an output result is reached.

Most of the known variants of $\lambda$-calculus can be obtained from this parametric calculus by instantiating $\mathsf{V}$ in a suitable way. The set $\mathsf{V}$ of input values and the reduction $\rightarrow_{\mathsf{V}}$, induced by it, are defined in the next definition.

**Definition 1.** Let $\mathsf{V} \subseteq \Lambda$.

(i) The $\mathsf{V}$-*reduction* ($\rightarrow_{\mathsf{V}}$) is the contextual closure of the following rule:

$(\lambda x.M)N \rightarrow M[N/x]$ if $N \in \mathsf{V}$.

$(\lambda x.M)N$ is called a $\mathsf{V}$-*redex* (or simply redex) and $M[N/x]$ is called its $\mathsf{V}$-*contractum* (or simply contractum).

(ii) $\rightarrow^*_{\mathsf{V}}$ and $=_{\mathsf{V}}$ are, respectively, the reflexive and transitive closure of $\rightarrow_{\mathsf{V}}$ and the symmetric, reflexive and transitive closure of $\rightarrow_{\mathsf{V}}$.

(iii) A set $\mathsf{V} \subseteq \Lambda$ is said a *set of input values*, when the following conditions are satisfied:
- $\mathsf{Var} \subseteq \mathsf{V}$ ($\mathsf{Var}$-*closure*);
- $P, Q \in \mathsf{V}$ implies $P[Q/x] \in \mathsf{V}$, for every $x \in \mathsf{Var}$ (*substitution closure*);
- $M \in \mathsf{V}$ and $M \rightarrow_{\mathsf{V}} N$ imply $N \in \mathsf{V}$ (*reduction closure*).

(iv) A term *is in* $\mathsf{V}$-*normal form* ($\mathsf{V}$-nf) if it has no $\mathsf{V}$-redexes and it *has a* $\mathsf{V}$-*normal form*, or it is $\mathsf{V}$-*normalizing* if it $\mathsf{V}$-reduces to a $\mathsf{V}$-normal form; the set of $\mathsf{V}$-nf is denoted by $\mathsf{V}\text{-}\mathsf{NF}$.

The closure conditions on the set of input values are quite natural. Since, as already said, input values represent partially evaluated terms, we ask that this partial evaluation is preserved by reduction, which is the rule on which is based the evaluation process. The substitution closure comes from the fact that variables always belong to the set of input values, and so semantically a variable denotes a generic input value.

The classical $\lambda\beta$-calculus can be obtained by choosing $\mathsf{V} = \Lambda$.

In all the paper the symbol $\mathsf{V}$ will denote a generic set of input values. We will omit the prefix $\mathsf{V}$ in case it will be clear from the context. Moreover we will use the symbol $\equiv$ for denoting the $\alpha$-equivalence and, if $\Delta \subseteq \Lambda$ then we will denote by $\Delta^0$ the set of closed terms (without free variables) belonging to $\Delta$.

Now some possible sets of input values will be defined.

**Definition 2.**
(i) $\Lambda_v = \mathsf{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$;
(ii) $\Lambda_I$ is the language obtained from the following grammar:
- $x \in \Lambda_I$
- $M, N \in \Lambda_I$ imply $MN \in \Lambda_I$
- $M \in \Lambda_I$ and $x \in FV(M)$ imply $\lambda x.M \in \Lambda_I$.

The $\lambda\Lambda_v$-calculus is the $\lambda\beta_v$-calculus defined by Plotkin in [18], also called the call-by-value $\lambda$-calculus. $\Lambda_I$ is the language first defined by Church [4].

**Property 3.**
(1) $\Lambda$ *is a set of input values*;
(2) $\Lambda_v$ *is a set of input values*;
(3) $\Lambda_I$ *is a set of input values*;
(4) $\Lambda$-$\mathsf{NF}$ *is not a set of input values*;
(5) $\mathsf{Var} \cup \Lambda$-$\mathsf{NF}^0$ *is a set of input values*;
(6) $\Upsilon = \mathsf{Var} \cup \{\lambda z.P \mid z \in \mathsf{FV}(P)\}$ *is not a set of input values*.

**Proof.** The first case is obvious. In cases 2, 3, and 5 it is easy to check that the closure properties of Definition 1 are satisfied. $\Lambda$-NF is not closed under substitution.

It is easy to see that $\Upsilon$ is closed under substitution. But it is not closed under reduction. In fact $\lambda x.KIx \in \Upsilon$, while $\lambda x.KIx \rightarrow_\Upsilon \lambda z.I \notin \Upsilon$. $\square$

It is easy to check that every term $M$ has the following shape:

$$\lambda x_1 \ldots x_n.\zeta M_1 \ldots M_m \quad (n, m \geqslant 0),$$

where $M_i \in \Lambda$ are the *arguments* of $M$ ($1 \leqslant i \leqslant m$) and $\zeta$ is the *head* of $M$. $\zeta$ is either a variable (*head variable*) or an application of the shape $(\lambda z.P)Q$, which can be either a redex (*head redex*) or not (*head block*), depending on $Q$.

The proofs both of confluence and of standardization are based on the notion of parallel reduction.

**Definition 4.** Let $\mathsf{V}$ be a set of input values.
(i) The *deterministic parallel reduction* $\hookrightarrow_{\mathsf{V}}$ is inductively defined as follows:
(1) $x \hookrightarrow_{\mathsf{V}} x$;
(2) $M \hookrightarrow_{\mathsf{V}} N$ implies $\lambda x.M \hookrightarrow_{\mathsf{V}} \lambda x.N$;
(3) $M \hookrightarrow_{\mathsf{V}} M'$,$N \hookrightarrow_{\mathsf{V}} N'$ and $N \in \mathsf{V}$ imply $(\lambda x.M)N \hookrightarrow_{\mathsf{V}} M'[N'/x]$;
(4) $M \hookrightarrow_{\mathsf{V}} M'$,$N \hookrightarrow_{\mathsf{V}} N'$ and $MN$ is not a redex imply $MN \hookrightarrow_{\mathsf{V}} M'N'$.
(ii) The *non-deterministic parallel reduction* $\Rightarrow_{\mathsf{V}}$ is inductively defined as follows:
(1) $x \Rightarrow_{\mathsf{V}} x$;
(2) $M \Rightarrow_{\mathsf{V}} N$ implies $\lambda x.M \Rightarrow_{\mathsf{V}} \lambda x.N$;
(3) $M \Rightarrow_{\mathsf{V}} M'$,$N \Rightarrow_{\mathsf{V}} N'$ and $N \in \mathsf{V}$ imply $(\lambda x.M)N \Rightarrow_{\mathsf{V}} M'[N'/x]$;
(4) $M \Rightarrow_{\mathsf{V}} M'$,$N \Rightarrow_{\mathsf{V}} N'$ imply $MN \Rightarrow_{\mathsf{V}} M'N'$.

Roughly speaking, the deterministic parallel reduction reduces in one step all and only the redexes present in a term, while the non-deterministic one reduces a subset of them.

**Example 5.** Let $M \equiv I(II)$, where $I \equiv \lambda x.x$. If $\mathsf{V} \equiv \Lambda$ then $M \hookrightarrow_{\mathsf{V}} I$, while $M \Rightarrow_{\mathsf{V}} M$, $M \Rightarrow_{\mathsf{V}} II$ and $M \Rightarrow_{\mathsf{V}} I$. If $\mathsf{V} \equiv \Lambda_v$ then $M \hookrightarrow_{\mathsf{V}} II$ while $M \Rightarrow_{\mathsf{V}} M$ and $M \Rightarrow_{\mathsf{V}} II$.

The following lemma shows the relation between $\Rightarrow_{\mathsf{V}}$ and $\rightarrow_{\mathsf{V}}$ reduction.

**Lemma 6.** *Let* $\mathsf{V}$ *be a set of input values.*
(i) $M \rightarrow_{\mathsf{V}} N$ *implies* $M \Rightarrow_{\mathsf{V}} N$;
(ii) $M \Rightarrow_{\mathsf{V}} N$ *implies* $M \rightarrow^*_{\mathsf{V}} N$;
(iii) $\rightarrow^*_{\mathsf{V}}$ *is the transitive closure of* $\Rightarrow_{\mathsf{V}}$.

**Proof.** Easy. □

$\Rightarrow_{\mathsf{V}}$ enjoys a useful substitution property.

**Lemma 7.** $M \Rightarrow_{\mathsf{V}} M'$, $N \Rightarrow_{\mathsf{V}} N'$ *and* $N \in \mathsf{V}$ *imply* $M[N/x] \Rightarrow_{\mathsf{V}} M'[N'/x]$.

**Proof.** By induction on $M$. Let us prove just the most difficult case, i.e., the term $M$ is a $\mathsf{V}$-redex. Let $M \equiv (\lambda z.P)Q$, $Q \in \mathsf{V}$, $P \Rightarrow_{\mathsf{V}} P'$, $Q \Rightarrow_{\mathsf{V}} Q'$ and $M' \equiv P'[Q'/z]$. By induction $P[N/x] \Rightarrow_{\mathsf{V}} P'[N'/x]$ and $Q[N/x] \Rightarrow_{\mathsf{V}} Q'[N'/x]$, where $Q'[N'/x] \in \mathsf{V}$ for the closure conditions on $\mathsf{V}$. Thus

$$((\lambda z.P)Q)[N/x] \equiv (\lambda z.P[N/x])Q[N/x] \Rightarrow_{\mathsf{V}} P'[N'/x][Q'[N'/x]/z] \equiv (P'[Q'/z])[N'/x],$$

by point 3 of $\Rightarrow_{\mathsf{V}}$ definition. □

The next property, whose proof is obvious, states that, for every term $M$, there is a unique term $N$ such that $M \hookrightarrow_{\mathsf{V}} N$.

**Property 8.** *$M \hookrightarrow_{\mathsf{V}} P$ and $M \hookrightarrow_{\mathsf{V}} Q$ imply $P \equiv Q$.*

**Proof.** Trivial. □

Let $[M]_{\mathsf{V}}$ be the unique term such $M \hookrightarrow_{\mathsf{V}} [M]_{\mathsf{V}}$. $[M]_{\mathsf{V}}$ is called in the literature the *complete development* of $M$ (see [21]). The following lemma holds.

**Lemma 9.** *$M \Rightarrow_{\mathsf{V}} N$ implies $N \Rightarrow_{\mathsf{V}} [M]_{\mathsf{V}}$.*

**Proof.** By induction on $M$.
- If $M \equiv x$, then $N \equiv x$ and $[M]_{\mathsf{V}} \equiv x$.
- If $M \equiv \lambda x.P$ then $N \equiv \lambda x.Q$, for some $Q$ such that $P \Rightarrow_{\mathsf{V}} Q$. By induction $Q \Rightarrow_{\mathsf{V}} [P]_{\mathsf{V}}$, and so $N \Rightarrow_{\mathsf{V}} \lambda x.[P]_{\mathsf{V}} \equiv [M]_{\mathsf{V}}$.
- If $M \equiv P_1P_2$ and it is not a $\mathsf{V}$-redex, then $N \equiv Q_1Q_2$ for some $Q_1$ and $Q_2$ such that $P_1 \Rightarrow_{\mathsf{V}} Q_1$ and $P_2 \Rightarrow_{\mathsf{V}} Q_2$. So, by induction, $Q_1 \Rightarrow_{\mathsf{V}} [P_1]_{\mathsf{V}}$ and $Q_2 \Rightarrow_{\mathsf{V}} [P_2]_{\mathsf{V}}$, which implies $N \Rightarrow_{\mathsf{V}} [P_1]_{\mathsf{V}}[P_2]_{\mathsf{V}} \equiv [M]_{\mathsf{V}}$.
- If $M \equiv (\lambda x.P_1)P_2$ is a redex (i.e. $P_2 \in \mathsf{V}$) then either $N \equiv (\lambda x.Q_1)Q_2$ or $N \equiv Q_1[Q_2/x]$, for some $Q_i$ such that $P_i \Rightarrow_{\mathsf{V}} Q_i$ $(1 \leqslant i \leqslant 2)$.
  By induction, $Q_i \Rightarrow_{\mathsf{V}} [P_i]_{\mathsf{V}}$ $(1 \leqslant i \leqslant 2)$. In both cases, $N \Rightarrow_{\mathsf{V}} [P_1]_{\mathsf{V}}[[P_2]_{\mathsf{V}}/x] \equiv [M]_{\mathsf{V}}$, in the former case simply by induction, in the latter by both induction and Lemma 7. □

## 3. Confluence

The proof of confluence follows the Takahashi pattern [21], which is a simplification of the original proof made by Taït and Martin Löf for classical call-by-name λ-calculus. It is based on the property that a reduction which is the transitive closure of another one enjoying the Diamond Property is confluent.

**Lemma 10** (Diamond Property of $\Rightarrow_{\mathsf{V}}$). *If $M \Rightarrow_{\mathsf{V}} N_0$ and $M \Rightarrow_{\mathsf{V}} N_1$ then there is $N_2$ such that both $N_0 \Rightarrow_{\mathsf{V}} N_2$ and $N_1 \Rightarrow_{\mathsf{V}} N_2$.*

**Proof.** By Lemma 9, $M \Rightarrow_{\mathsf{V}} N$ implies $N \Rightarrow_{\mathsf{V}} [M]_{\mathsf{V}}$. So, if $M \Rightarrow_{\mathsf{V}} M_1$ and $M \Rightarrow_{\mathsf{V}} M_2$, then both $M_1 \Rightarrow_{\mathsf{V}} [M]_{\mathsf{V}}$ and $M_2 \Rightarrow_{\mathsf{V}} [M]_{\mathsf{V}}$. See Fig. 1. □

$$
\begin{tikzcd}
 & M \arrow[dl, Rightarrow, "\mathsf{V}"'] \arrow[dr, Rightarrow, "\mathsf{V}"] & \\
N_0 \arrow[dr, Rightarrow, "\mathsf{V}"'] & & N_1 \arrow[dl, Rightarrow, "\mathsf{V}"] \\
 & N_2 &
\end{tikzcd}
$$

Fig. 1. Diamond property.

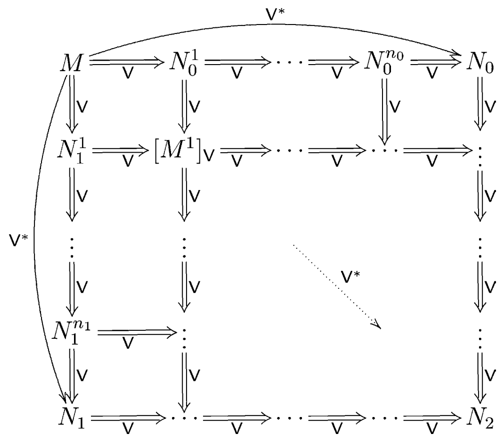

Fig. 2. Diamond closure.

**Theorem 11** (Confluence). *If $M \to^*_{\mathsf{V}} N_0$ and $M \to^*_{\mathsf{V}} N_1$ then there is $N_2$ such that both $N_0 \to^*_{\mathsf{V}} N_2$ and $N_1 \to^*_{\mathsf{V}} N_2$.*

**Proof.** By Property 6, $\to^*_{\mathsf{V}}$ is the transitive closure of $\Rightarrow_{\mathsf{V}}$. This means that there are $N_0^1, \ldots, N_0^{n_0}$, $N_1^1, \ldots, N_1^{n_1}$ $(n_0, n_1 \leqslant 0)$ such that $M \Rightarrow_{\mathsf{V}} N_0^1 \ldots \Rightarrow_{\mathsf{V}} N_0^{n_0} \Rightarrow_{\mathsf{V}} N_0$ and $M \Rightarrow_{\mathsf{V}} N_1^1 \ldots \Rightarrow_{\mathsf{V}} N_m^{n_1} \Rightarrow_{\mathsf{V}} N_1$. Then the proof follows by applying repeatedly the diamond property of $\Rightarrow_{\mathsf{V}}$ (diamond closure), as shown in Fig. 2. □

Theorem 11 has as immediate corollary the *uniqueness of the normal form* of a term, if this exists.

**Corollary 12.** *The $\mathsf{V}$-normal form of a term $M$ is unique.*

**Proof.** Assume by absurdum that a term $M$ has two different normal forms $M_1$ and $M_2$. Then, by the Confluence Theorem, there is a term $N$ such that both $M_1$ and $M_2$ $\mathsf{V}$-reduce to $N$, against the hypothesis that both are normal forms. □

## 4. Standardization

In this section the notion of standardization is formalized, and a necessary and sufficient condition under which the $\lambda\mathsf{V}$-calculus enjoy this property is given.

The notion of standard reduction sequence is given in the next definition, and it is based on a measure of the redexes called *degree*. As in the case of classical $\lambda$-calculus, a reduction sequence is standard if it reduces the redexes in increasing order of their degree. But here the computation of the degree of a redex is parametric with respect to the set $\mathsf{V}$ of input values.

**Definition 13.**

(i) A symbol $\lambda$ in a term $M$ is active if and only if it is the first symbol of a $\mathsf{V}$-redex of $M$.

(ii) The V-sequentialization $(M)^\circ$ of a term $M$ is a function in $\Lambda \to \Lambda$ defined as follows:
- $(xM_1 \dots M_m)^\circ = x(M_1)^\circ \dots (M_m)^\circ$;
- $((\lambda x.P)QM_1 \dots M_m)^\circ = (\lambda x.P)^\circ (Q)^\circ (M_1)^\circ \dots (M_m)^\circ$, if $Q \in \mathsf{V}$;
- $((\lambda x.P)QM_1 \dots M_m)^\circ = (Q)^\circ (\lambda x.P)^\circ (M_1)^\circ \dots (M_m)^\circ$, if $Q \notin \mathsf{V}$;
- $(\lambda x.P)^\circ = \lambda x.(P)^\circ$.

(iii) The degree of a redex $R$ in $M$ is the numbers of $\lambda$'s which both are active in $M$ and occur on the left of $(R)^\circ$ in $(M)^\circ$.

(iv) A sequence $M \equiv P_0 \to_{\mathsf{V}} P_1 \to_{\mathsf{V}} \dots \to_{\mathsf{V}} P_n \to_{\mathsf{V}} P_{n+1} \equiv N$ is standard if and only if the degree of the redex contracted in $P_i$ is less than or equal to the degree of the redex contracted in $P_{i+1}$, for every $i < n$.
We denote by $M \to^\circ_{\mathsf{V}} N$ a standard reduction sequence from $M$ to $N$.

**Example 14.**
(1) Let $\mathsf{V} = \Lambda$, and let $M \equiv (\lambda x.x(KI))(II)$, where $K \equiv \lambda xy.x$. Then $M$ has degree 0, $KI$ has degree 1 and $II$ has degree 2. The following reduction sequence is standard: $(\lambda x.x(KI))(II) \to_\Lambda (II)(KI) \to_\Lambda I(KI) \to_\Lambda I(\lambda y.I)$.
(2) Let $M$ be as before, and let $\mathsf{V} = \Lambda_v$. Then $II$ has degree 0, and $KI$ has degree 1. Note that now $M$ is no more a redex. The following reduction sequence is standard: $(\lambda x.x(KI))(II) \to_{\Lambda_v} (\lambda x.x(KI))I \to_{\Lambda_v} I(KI) \to_{\Lambda_v} I(\lambda y.I) \to_{\Lambda_v} \lambda y.I$.

It is important to notice that the degree of a redex can change during the reduction, in particular the redex of minimum degree has always degree zero. Moreover note that the reduction sequences of length 0 and 1 are always standard.

It is easy to check that, for every $M$, the $\Lambda$-sequentialization $(M)^\circ \equiv M$; thus in this case the redex of degree 0 is always the leftmost one.

$M \Rightarrow^\circ_{\mathsf{V}} N$ will stand for "$M \to^\circ_{\mathsf{V}} N$ and $M \Rightarrow_{\mathsf{V}} N$".

Let $\vec{N}$ be an abbreviation for a sequence of terms $N_1, \dots, N_l$, having length $\|\vec{N}\| = l$, and let $M[\vec{N}/\vec{x}]$ denote $M[N_1/x_1] \dots [N_l/x_l]$.

The following lemma, at the point (ii), shows that a nondeterministic parallel reduction can always be transformed into a standard reduction sequence.

**Lemma 15.** *Let $\vec{P}, \vec{Q}$ be two sequences of terms, such that $\|\vec{P}\| = \|\vec{Q}\|$ and $\forall i \leqslant \|\vec{P}\|$ $P_i \in \mathsf{V}$ and $P_i \Rightarrow^\circ_{\mathsf{V}} Q_i$.*
(i) *If $M \Rightarrow^\circ_{\mathsf{V}} N$ then $M[\vec{P}/\vec{x}] \Rightarrow^\circ_{\mathsf{V}} N[\vec{Q}/\vec{x}]$.*
(ii) *If $M \Rightarrow_{\mathsf{V}} N$ then $M \Rightarrow^\circ_{\mathsf{V}} N$.*

**Proof.** (i) and (ii) by mutual induction on $M$.
(i) By Lemma 7, $M[\vec{P}/\vec{x}] \Rightarrow_{\mathsf{V}} N[\vec{Q}/\vec{x}]$, so it suffices to show that $M[\vec{P}/\vec{x}] \to^\circ_{\mathsf{V}} N[\vec{Q}/\vec{x}]$. Let $M \equiv \lambda y_1 \dots y_n.\zeta M_1 \dots M_m$, where $\zeta \in \mathsf{Var}$ or $\zeta \equiv (\lambda z.T)U$.
If $n > 0$, then the proof follows by induction on $(i)$.
Let $n = 0$, thus $N \equiv \xi N_1 \dots N_m$ such that $\zeta \Rightarrow^\circ_{\mathsf{V}} \xi$ and $M_i \Rightarrow^\circ_{\mathsf{V}} N_i$ $(1 \leqslant i \leqslant m)$. Further, let $M'_i \equiv M_i[\vec{P}/\vec{x}]$ and $N'_i \equiv N_i[\vec{Q}/\vec{x}]$ $(1 \leqslant i \leqslant m)$.

The proof is organized according to the possible shapes of $\zeta$.

(1) Let $\zeta$ be a variable. If $m = 0$ then the proof is trivial, so let $m > 0$. There are two cases to be considered.

(1.1) $\zeta \notin \vec{x}$, so $\xi[\vec{Q}/\vec{x}] \equiv \zeta$. Thus $M_i[\vec{P}/\vec{x}] \to^{\circ}_{\mathsf{V}} N_i[\vec{Q}/\vec{x}]$, by induction. The standard reduction sequence is

$$\zeta M'_1 \dots M'_m \to^{\circ}_{\mathsf{V}} \zeta N'_1 M'_2 \dots M'_m \to^{\circ}_{\mathsf{V}} \cdots\cdots \to^{\circ}_{\mathsf{V}} \zeta N'_1 \dots N'_m.$$

(1.2) $\zeta \equiv x_j \in \vec{x}$ $(1 \leqslant j \leqslant l)$, so $\xi[\vec{Q}/\vec{x}] \equiv Q_j$. But $P_j \Rightarrow^{\circ}_{\mathsf{V}} Q_j$ means that there is a standard sequence $P_j \equiv S_0 \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_n \equiv Q_j$ $(n \in \mathbb{N})$.
Two cases can arise.

(1.2.1) $\forall i \leqslant n, S_i \not\equiv \lambda z.S'$. Then the following reduction sequence:

$$\sigma : S_0 M'_1 \dots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_n M'_1 \dots M'_m,$$

is standard. Since by induction $M_i[\vec{P}/\vec{x}] \to^{\circ}_{\mathsf{V}} N_i[\vec{Q}/\vec{x}]$, there is a standard reduction sequence

$$\tau : S_n M'_1 \dots M'_m \to^{\circ}_{\mathsf{V}} S_n N'_1 M'_2 \dots M'_m \to^{\circ}_{\mathsf{V}} \cdots\cdots \to^{\circ}_{\mathsf{V}} S_n N'_1 \dots N'_m.$$

Note that $S_0 M'_1 \dots M'_m \equiv M[\vec{P}/\vec{x}]$ and $S_n N'_1 \dots N'_m \equiv N[\vec{Q}/\vec{x}]$, so $\sigma$ followed by $\tau$ is the desired standard reduction sequence.

(1.2.2) There is a minimum $k \leqslant n$ such that $S_k \equiv \lambda z.S'$.
By induction on (ii), $M_1 \Rightarrow^{\circ}_{\mathsf{V}} N_1$. So, by induction $M_1[\vec{P}/\vec{x}] \Rightarrow^{\circ}_{\mathsf{V}} N_1[\vec{Q}/\vec{x}]$, where $M_1[\vec{P}/\vec{x}] \to^{\circ}_{\mathsf{V}} N_1[\vec{Q}/\vec{x}]$ is $M_1[\vec{P}/\vec{x}] \equiv R_0 \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} R_p \equiv N_1[\vec{Q}/\vec{x}]$ $(p \in \mathbb{N})$.
There are two subcases:

(1.2.2.1) $\forall i \leqslant p$ $R_i \notin \mathsf{V}$. Then the following reduction sequence:

$$\sigma' : M[\vec{P}/\vec{x}] \equiv S_0 R_0 M'_2 \dots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_k R_0 M'_2 \dots M'_m \to_{\mathsf{V}} \cdots\cdots$$
$$\to_{\mathsf{V}} S_k R_p M'_2 \dots M'_m \to_{\mathsf{V}} S_{k+1} R_p M'_2 \dots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_n R_p M'_2 \dots M'_m,$$

is standard too. Moreover, since $M_i[\vec{P}/\vec{x}] \to^{\circ}_{\mathsf{V}} N_i[\vec{P}/\vec{x}]$, also the following reduction sequence:

$$\tau' : S_n R_p M'_2 \dots M'_m \to^{\circ}_{\mathsf{V}} S_n R_p N'_2 M'_3 \dots M'_m \to^{\circ}_{\mathsf{V}} \cdots\cdots \to^{\circ}_{\mathsf{V}} S_n R_p N'_2 \dots N'_m,$$

is standard. Clearly $\sigma'$ followed by $\tau'$ is the desired standard reduction sequence.

(1.2.2.2) There is a minimum $q \leqslant p$ such that $R_q \in \mathsf{V}$. So

$$\sigma'' : M[\vec{P}/\vec{x}] \equiv S_0 R_0 M'_2 \dots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_k R_0 M'_2 \dots M'_m$$
$$\to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_k R_q M'_2 \dots M'_m \to_{\mathsf{V}} S_{k+1} R_q M'_2 \dots M'_m$$
$$\to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_n R_q M'_2 \dots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} S_n R_p M'_2 \dots M'_m,$$

is a standard reduction sequence. The desired standard reduction sequence is $\sigma''$ followed by $\tau'$.

(2) Let $\zeta \equiv (\lambda z.T)U$. So, either $N \equiv (\lambda z.\bar{T})\bar{U} N_1 \dots N_m$ or $N \equiv \bar{T}[\bar{U}/z] N_1 \dots N_m$, where $T \Rightarrow_{\mathsf{V}} \bar{T}$, $U \Rightarrow_{\mathsf{V}} \bar{U}$ and $M_i \Rightarrow_{\mathsf{V}} N_i$ $(1 \leqslant i \leqslant m)$.
By induction, $U' \equiv U[\vec{P}/\vec{x}] \Rightarrow^{\circ}_{\mathsf{V}} \bar{U}[\vec{Q}/\vec{x}] \equiv U''$, $T' \equiv T[\vec{P}/\vec{x}] \Rightarrow^{\circ}_{\mathsf{V}} \bar{T}[\vec{Q}/\vec{x}] \equiv T''$ and $M'_i \equiv M_i[\vec{P}/\vec{x}] \Rightarrow^{\circ}_{\mathsf{V}} N_i[\vec{Q}/\vec{x}] \equiv N'_i$ $(1 \leqslant i \leqslant m)$.
Let $U' \equiv R_0 \to_{\mathsf{V}} \cdots \to_{\mathsf{V}} R_p \equiv U''$ $(p \in \mathbb{N})$ be the standard sequence $U' \to^{\circ}_{\mathsf{V}} U''$. Without loss of generality assume $z \notin \vec{x}$.

(2.1) Let $N \equiv (\lambda z.\bar{T})\bar{U}N_1 \ldots N_m$. There are two cases.

(2.1.1) $\forall i \leqslant p \;\; R_i \notin \mathsf{V}$.

Then the standard reduction sequence $M[\vec{P}/\vec{x}] \to^{\circ}_{\mathsf{V}} N[\vec{Q}/\vec{x}]$ is

$$(\lambda z.T')R_0M'_1 \ldots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} (\lambda z.T')R_pM'_1 \ldots M'_m$$
$$\to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_pM'_1 \ldots M'_m \to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_pN'_1M'_2 \ldots M'_m$$
$$\to^{\circ}_{\mathsf{V}} \cdots\cdots \to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_pN'_1 \ldots N'_m.$$

(2.1.2) There is a minimum $q \leqslant p$ such that $R_q \in \mathsf{V}$. Thus the desired standard reduction sequence is:

$$(\lambda z.T')R_0M'_1 \ldots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} (\lambda z.T')R_qM'_1 \ldots M'_m$$
$$\to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_qM'_1 \ldots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} (\lambda z.T'')R_pM'_1 \ldots M'_m$$
$$\to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_pN'_1M'_2 \ldots M'_m \to^{\circ}_{\mathsf{V}} \cdots\cdots \to^{\circ}_{\mathsf{V}} (\lambda z.T'')R_pN'_1 \ldots N'_m.$$

(2.2) Let $N \equiv \bar{T}[\bar{U}/z]N_1 \ldots N_m$. So, there is a minimum $q \leqslant p$ such that $R_q \in \mathsf{V}$; let $\mu$ be the standard reduction sequence:

$$M[\vec{P}/\vec{x}] \equiv (\lambda z.T')R_0M'_1 \ldots M'_m \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} (\lambda z.T')R_qM'_1 \ldots M'_m$$
$$\to_{\mathsf{V}} T'[R_q/z]M'_1 \ldots M'_m.$$

By induction on (ii), it follows $T \Rightarrow^{\circ}_{\mathsf{V}} \bar{T}$. Furthermore, since $R_q \Rightarrow^{\circ}_{\mathsf{V}} U''$, it follows by induction that $T[\vec{P}/\vec{x}][R_q/z] \Rightarrow^{\circ}_{\mathsf{V}} \bar{T}[\vec{Q}/\vec{x}][U''/z]$.
Let $T[\vec{P}/\vec{x}][R_q/z] \equiv T_0 \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} T_t \equiv \bar{T}[\vec{Q}/\vec{x}][U''/z]$ be the corresponding standard reduction sequence. Two subcases can arise:

(2.2.1) $\forall i \leqslant t,\; T_i \not\equiv \lambda z.S'$. The desired standard reduction sequence is $\mu$ followed by:

$$T'[R_p/z]M'_1 \ldots M'_m \equiv T[\vec{P}/\vec{x}][R_p/z]M'_1 \ldots M'_m \to_{\mathsf{V}} T_1M'_1 \ldots M'_m$$
$$\to_{\mathsf{V}} \ldots\ldots \to_{\mathsf{V}} T_tM'_1 \ldots M'_m \to^{\circ}_{\mathsf{V}} \ldots\ldots \to^{\circ}_{\mathsf{V}} T_tN'_1 \ldots N'_m \equiv [\vec{Q}/\vec{x}]$$

(2.2.2) Let $k \leqslant t$ be the minimum index such that $T_k \equiv \lambda y.T'_k$. The construction of the standard reduction sequence depends on the fact that $M_2$ become or not an input values, but, in every case, it can be easily build as in the previous cases.

(ii) By induction on $M$. The cases $M \equiv x$ and $M \equiv \lambda z.M'$ are trivial.

(1) Let $M \equiv PQ \Rightarrow_{\mathsf{V}} P'Q' \equiv N$, $P \Rightarrow_{\mathsf{V}} P'$ and $Q \Rightarrow_{\mathsf{V}} Q'$.
By induction, there are standard sequences $P \equiv P_0 \to_{\mathsf{V}} \cdots \to_{\mathsf{V}} P_p \equiv P'$ and $Q \equiv Q_0 \to_{\mathsf{V}} \cdots \to_{\mathsf{V}} Q_q \equiv Q'$.
If $\forall i \leqslant p \;\; P_i \not\equiv \lambda z.P'_i$, then $M \to^{\circ}_{\mathsf{V}} N$ is $P_0Q_0 \to^{\circ}_{\mathsf{V}} P_pQ_0 \to^{\circ}_{\mathsf{V}} P_pQ_q$.
Otherwise, let $k$ the minimum index such that $P_k \equiv \lambda z.P'_k$.
If $\forall j \leqslant q \;\; Q_j \notin \mathsf{V}$, then $M \to^{\circ}_{\mathsf{V}} N$ is

$$P_0Q_0 \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} P_kQ_0 \to^{\circ}_{\mathsf{V}} P_kQ_q \to_{\mathsf{V}} P_{k+1}Q_q \to_{\mathsf{V}} \cdots\cdots \to_{\mathsf{V}} P_pQ_q.$$

If there is a minimum $h$ such that $Q_h \in \mathsf{V}$ the standard sequence is $P_0Q_0 \to^{\circ}_{\mathsf{V}} P_kQ_0 \to^{\circ}_{\mathsf{V}} P_kQ_h \to_{\mathsf{V}} P_{k+1}Q_h \to^{\circ}_{\mathsf{V}} P_pQ_h \to^{\circ}_{\mathsf{V}} P_pQ_q$.

(2) Let $M \equiv (\lambda x.P)Q \Rightarrow_{\mathsf{V}} P'[Q'/x] \equiv N$, $P \Rightarrow_{\mathsf{V}} P'$, $Q \Rightarrow_{\mathsf{V}} Q'$ and $Q \in \mathsf{V}$.
$P \Rightarrow^{\circ}_{\mathsf{V}} P'$ and $Q \Rightarrow^{\circ}_{\mathsf{V}} Q'$ follow by induction on (ii), so $P[Q/x] \Rightarrow^{\circ}_{\mathsf{V}} P'[Q'/x]$, by induction on (i). So, the desired standard reduction sequence is $(\lambda x.P)Q \to_{\mathsf{V}} P[Q/x] \to^{\circ}_{\mathsf{V}} P'[Q'/x]$. $\square$

In order to prove the standardization some auxiliary definitions are necessary.

**Definition 16.** Let $M, N \in \Lambda$.

(i) The principal redex of $M$, if it exists, is the redex of $M$ with minimum degree.
(ii) $M \rightarrow^{p}_{\mathsf{V}} N$ denotes that $N$ is obtained from $M$ by reducing the principal redex of $M$; $\rightarrow^{*p}_{\mathsf{V}}$ denotes the reflexive and transitive closure of $\rightarrow^{p}_{\mathsf{V}}$.
(iii) $M \rightarrow^{i}_{\mathsf{V}} N$ denotes that $N$ is obtained from $M$ by reducing a redex which is not the principal redex.
(iv) $M \Rightarrow^{i}_{\mathsf{V}} N$ denotes $M \Rightarrow_{\mathsf{V}} N$ and $M \rightarrow^{*i}_{\mathsf{V}} N$.

The notion of a standard set of input values, which will be given in the next definition, is the key one for having the standardization property.

**Definition 17** (*Standard Input Values*)**.** A set $\mathsf{V}$ of input values is *standard* if and only if $M \notin \mathsf{V}$ and $M \rightarrow^{i}_{\mathsf{V}} N$ imply $N \notin \mathsf{V}$.

Not all the set of input values we defined are standard, as proved in the next property.

**Property 18.**

(i) $\Lambda$ *and* $\Lambda_v$ *are standard;*
(ii) *For every* $\mathsf{V}$, $\mathsf{Var} \cup \mathsf{V}\text{-}\mathsf{NF}^0$ *is standard;*
(iii) $\Lambda_I$ *is not standard.*

**Proof.**

(i) $\Lambda$ is trivially standard. Let us consider $\Lambda_v$. It is sufficient to prove that, if $M \notin \Lambda_v$, and $M \rightarrow^{*}_{\Lambda_v} N$ through a not principal reduction, then $N \notin \Lambda_v$. $M \notin \Lambda_v$ implies that $M$ has one of the following shapes:
(1) $yM_1 \ldots M_m$ $(m > 1)$;
(2) $(\lambda x.M_1)M_2 \ldots M_m$ $(m \geqslant 2)$ and either $(\lambda x.M_1)M_2$ is a redex or it is a head block.
Case (1) is trivial, since $M$ can never be reduced to a $\Lambda_v$-value. The only non trivial case is $m = 2$ and $M_2 \in \Lambda_v$. Every reduction sequence not reducing the principal redex is such that $(\lambda x.M_1)M_2 \rightarrow^{*}_{\Lambda_v} (\lambda x.M_1')M_2'$, where $M_1 \rightarrow^{*}_{\Lambda_v} M_1'$ and $M_2 \rightarrow^{*}_{\Lambda_v} M_2'$; thus an input value cannot be reached.
(ii) $\mathsf{Var} \cup \Lambda\text{-}\mathsf{NF}^0$ is standard since not principal reductions preserve the presence of the redex of minimum degree.
(iii) Just consider the term: $M \equiv (DD)((\lambda z.I)I)$, where $D \equiv \lambda x.xx$. In fact $M \rightarrow_{\Lambda_I} (DD)I \in \Lambda_I$ and in this reduction the reduced redex is not principal, while for every sequence of $\rightarrow^{*p}_{\Lambda_I}$ reductions: $M \rightarrow^{*p}_{\Lambda_I} M \notin \Lambda_I$. □

**Lemma 19.** *$M \Rightarrow_{\mathsf{V}} N$ implies there is $P$ such that $M \rightarrow^{*p}_{\mathsf{V}} P \Rightarrow^{i}_{\mathsf{V}} N$.*

**Proof.** Trivial, by Lemma (15(ii)). Notice that it can be $M \equiv P$, by definition of $\rightarrow^{*p}_{\mathsf{V}}$. □

**Example 20.** We have $M \equiv (\lambda xy.I(\lambda z.IK(II)))I \Rightarrow_{\Lambda_v} \lambda yz.IKI$. Clearly $M \rightarrow^{p}_{\Lambda_v} \lambda y.I(\lambda z.IK(II)) \rightarrow^{p}_{\Lambda_v} \lambda yz.IK(II) \Rightarrow^{i}_{\Lambda_v} \lambda yz.IKI$ and $\lambda yz.IK(II) \in \Lambda_v$.

Note that, if $\mathsf{V}$ is standard and $R$ is the principal redex of $M$ and $M \rightarrow^{*i}_{\mathsf{V}} N$, then $R$ is the principal redex of $N$.

**Lemma 21.** *Let* $\mathsf{V}$ *be standard*; $M \Rightarrow^{i}_{\mathsf{V}} P \rightarrow^{p}_{\mathsf{V}} N$ *implies* $M \rightarrow^{*p}_{\mathsf{V}} Q \Rightarrow^{i}_{\mathsf{V}} N$, *for some* $Q$.

**Proof.** By induction on $M$. If either $M \equiv \lambda x.M'$, or the head of $M$ is a variable, then the proof follows by induction. Otherwise, let $M \equiv (\lambda y.M_0)M_1 \dots M_m$; thus it must be $P \equiv (\lambda y.P_0)P_1 \dots P_m$. Note that $M \Rightarrow^{i}_{\mathsf{V}} P$ implies $M_i \Rightarrow_{\mathsf{V}} P_i$ $(1 \leqslant i \leqslant m)$. Now there are two cases, according to $P_1 \in \mathsf{V}$ or not.

Let $P_1 \in \mathsf{V}$; it follows that $P_1$ is the argument of the principal redex of $P$, thus $N \equiv P_0[P_1/y]P_2 \dots P_m$.

Let $M_1 \in \mathsf{V}$. Then we can build the following reduction sequence:

$M \equiv (\lambda y.M_0)M_1 \dots M_m \rightarrow^{p}_{\mathsf{V}} M_0[M_1/y] \dots M_m \Rightarrow_{\mathsf{V}} P_0[P_1/y]P_2 \dots P_m$, which can be transformed into a standard one, by Lemma 19.

Let $M_1 \notin \mathsf{V}$ and $P_1 \in \mathsf{V}$; since the set $\mathsf{V}$ is standard, $M_1 \Rightarrow_{\mathsf{V}} P_1 \in \mathsf{V}$ if and only if $M_1 \rightarrow^{*p}_{\mathsf{V}} P'_1 \Rightarrow^{i}_{\mathsf{V}} P_1$, where $P'_1 \in \mathsf{V}$. But this would imply that, in the reduction $M \Rightarrow^{i}_{\mathsf{V}} P$ the principal redex of $M_1$ has been reduced; but by definition the principal redex of $M_1$ coincides with the principal redex of $M$, against the hypothesis that $M \Rightarrow^{i}_{\mathsf{V}} P$. So this case is not possible.

Let $P_1 \notin \mathsf{V}$. Then there is $j \geqslant 0$ such that the principal redex of $P_j$ is the principal redex of $P$. Let $j \geqslant 2$; so $\forall k \leqslant j$ $P_k$ is a normal form. So $N \equiv (\lambda y.P_0)P_1 \dots P'_j..P_m$, where $P_j \rightarrow^{p}_{\mathsf{V}} P'_j$. From the hypothesis $M \Rightarrow^{i}_{\mathsf{V}} P$, it follows that $M_i \equiv P_i$ $(1 \leqslant i \leqslant j-1)$, and $M_i \Rightarrow_{\mathsf{V}} P_i$ $(j < i \leqslant m)$. Then by induction there is $P^*_j$ such that $M_j \rightarrow^{*p}_{\mathsf{V}} P^*_j \Rightarrow^{i}_{\mathsf{V}} P'_j$, and we can build the following reduction sequence:

$$(\lambda y.M_0)M_1 \dots M_m \rightarrow^{*p}_{\mathsf{V}} (\lambda y.M_0)M_1 \dots P^*_j P_{j+1} \dots P_m \Rightarrow_{\mathsf{V}} (\lambda y.M_0)M_1 \dots P'_j \dots P_m,$$

which can be transformed into a standard one, by Lemma 19.

The cases $j < 2$ are similar. □

This Lemma has a key corollary:

**Corollary 22.** *Let* $\mathsf{V}$ *be standard.*

*If* $M \rightarrow^{*}_{\mathsf{V}} N$ *then* $M \rightarrow^{*p}_{\mathsf{V}} Q \underbrace{\Rightarrow^{i}_{\mathsf{V}} \dots \Rightarrow^{i}_{\mathsf{V}}}_{k} N$, *for some* $Q$ *and some* $k$.

**Proof.** Note that, if $P \rightarrow_{\mathsf{V}} P'$ then $P \Rightarrow_{\mathsf{V}} P'$. So $M \rightarrow^{*}_{\mathsf{V}} N$ implies $M \Rightarrow_{\mathsf{V}} N_1 \Rightarrow_{\mathsf{V}} \dots \Rightarrow_{\mathsf{V}} N_n \Rightarrow_{\mathsf{V}} N$. So, by applying repeatedly Lemmas 19 and 21 we reach the proof. □

Now we can prove the main theorem, stating that the $\lambda\mathsf{V}$-calculus enjoy the standardization property, if the set $\mathsf{V}$ of input values is standard.

**Theorem 23.** *Let* $\mathsf{V}$ *be standard.*

$M \rightarrow^{*}_{\mathsf{V}} N$ *implies there is a standard reduction sequence from* $M$ *to* $N$.

**Proof.** By induction on $N$. From the Corollary 22, $M \rightarrow^{*}_{\mathsf{V}} N$ implies $M \rightarrow^{*p}_{\mathsf{V}} Q \rightarrow^{*i}_{\mathsf{V}} N$, for some $Q$. Obviously the reduction sequence $\sigma : M \rightarrow^{*p}_{\mathsf{V}} Q$ is standard by definition of $\rightarrow^{p}_{\mathsf{V}}$. Note that, by

definition of $\rightarrow_{\mathsf{V}}^{*i}$, $Q \rightarrow_{\mathsf{V}}^{*i} N$ implies that $Q$ and $N$ have the same structure, i.e., $Q \equiv \lambda x_1 \ldots x_n.\zeta Q_1 \ldots Q_n$ and $N \equiv \lambda x_1 \ldots x_n.\zeta' N_1 \ldots N_n$, where $Q_i \rightarrow_{\mathsf{V}}^{*} N_i$ $(i \leqslant n)$ and either $\zeta$ and $\zeta'$ are the same variable, or $\zeta \equiv (\lambda x.R)S$, $\zeta' \equiv (\lambda x.R')S'$, $R \rightarrow_{\mathsf{V}}^{*} R'$ and $S \rightarrow_{\mathsf{V}}^{*} S'$.
By induction there are standard reduction sequences $\sigma_i : Q_i \rightarrow_{\mathsf{V}}^{\circ} N_i$ $(1 \leqslant i \leqslant n)$, $\tau_R : R \rightarrow_{\mathsf{V}}^{\circ} R'$ and $\tau_S : S \rightarrow_{\mathsf{V}}^{\circ} S'$. Let $S \equiv S_0 \rightarrow_{\mathsf{V}} \cdots\cdots \rightarrow_{\mathsf{V}} S_k \equiv S'$ $(k \in \mathbb{N})$.
If $\forall i \leqslant k$ $S_i \notin V$ then the desired standard reduction sequence is $\sigma$ followed by $\tau_S, \tau_R, \sigma_1, \ldots, \sigma_n$.
Otherwise, $\exists S_h \in \mathsf{V}$ $(h \leqslant k)$. In this case, let $\tau_S^0 : S_0 \rightarrow_{\mathsf{V}} \cdots\cdots \rightarrow_{\mathsf{V}} S_h$ and $\tau_S^1 : S_k \rightarrow_{\mathsf{V}} \cdots\cdots \rightarrow_{\mathsf{V}} S_k$; the desired standard reduction sequence is $\sigma$ followed by $\tau_S^0, \tau_R, \tau_S^1, \sigma_1, \ldots, \sigma_n$.
The case $\zeta$ and $\zeta'$ are the same variable is simpler. □

**Theorem 24.** *The condition that* $\mathsf{V}$ *is standard is necessary and sufficient for the* $\lambda\mathsf{V}$*-calculus to enjoy the standardization property.*

**Proof.** The sufficiency of the condition is consequence of the Standardization Theorem. For proving its necessity, assume $\mathsf{V}$ is not standard: we can find a term $M \notin \mathsf{V}$ such that $M \rightarrow_{\mathsf{V}} N \in \mathsf{V}$, without reducing the principal redex. Hence $IM \rightarrow_{\mathsf{V}} IN \rightarrow_{\mathsf{V}} N$, by reducing first a redex of degree different from 0 and then a redex of degree 0. Clearly there is no way of commuting the reduction order. □

An important consequence of the standardization property is the fact that the reduction sequence reducing, at every step, the principal redex is normalizing, as shown in the next property.

**Corollary 25.** *Let* $\mathsf{V}$ *be standard.*
*If* $M \rightarrow_{\mathsf{V}}^{*} N$ *and* $N$ *is a normal form then* $M \rightarrow_{\mathsf{V}}^{*p} N$.

**Proof.** By Corollary 12 and by Theorem 23. □

**Example 26.** Let $\mathsf{V} = \Lambda$. The term $KI(DD)$ has $\Lambda$-normal form $I$. In fact the standard $\Lambda$-reduction is $KI(DD) \rightarrow_{\Lambda} (\lambda y.I)(DD) \rightarrow_{\Lambda} I$, while the $\Lambda$-reduction sequence choosing at every step the rightmost $\Lambda$-redex never stops. Notice that, if we choose $\mathsf{V} = \Lambda_v$, $KI(DD)$ has not $\Lambda_v$-normal form.

## 5. Technical Remarks

It is natural to ask if the closure conditions on input values, given in Definition 1, are necessary in order to assure the confluence and standardization property of the calculus. In order to discuss this topic, in this section we will implicitly extend to any subset of $\Lambda$ all the notions defined in the previous sections for sets of input values.

As far as the confluence property is concerned, it can be observed that a weaker version of both the closure conditions is needed.

**Definition 27.** Let $\Delta \subseteq \Lambda$ and let $\mathsf{Var} \subseteq \Delta$.

- $\Delta$ is *weakly closed under substitution* if and only if
  $P, Q \in \Delta$ implies $P[Q/x] \to_\Delta^* R$, for some $R \in \Delta$;
- $\Delta$ is *weakly closed under reduction* if and only if
  $M \in \Delta$ and $M \to_\Delta^* N \notin \Delta$ implies there is $R \in \Delta$ such that $N \to_\Delta^* R$.

It is immediate to check that every set of input values satisfies the previous conditions.

**Theorem 28.** *Let $\Delta \subseteq \Lambda$ and let $\mathsf{Var} \subseteq \Delta$. In order to the $\Delta$-reduction be confluent, it is necessary for $\Delta$ to be weakly closed under substitution and reduction.*

**Proof.** Let $P \in \Delta$, but, for every $Q \notin P$ such that $P \to_\Delta^* Q$, $Q \notin \Delta$. Then $(\lambda x.M)P$ reduces both to $M[P/x]$ and to $(\lambda x.M)Q$, which do not have a common reduct, since the last term will be never a redex.
On the other hand, let $N, P \in \Delta$ but for all $Q$ such that $N[P/x] \to_\Delta^* Q$, $Q \notin \Delta$. Thus $(\lambda x.(\lambda y.M)N)P$ reduces both to $(\lambda y.M[P/x])N[P/x]$ and to $(M[N/y])[P/x]$, which do not have a common reduct. $\square$

As far as the standardization property is concerned, it is easy to see that the substitution closure of input values, given in Definition 1, is necessary.

**Theorem 29.** *Let $\Delta \subseteq \Lambda$ and let $\mathsf{Var} \subseteq \Delta$. In order for the $\Delta$-reduction enjoy the standardization property it is necessary for $\Delta$ to be closed under substitution.*

**Proof.** Let $M, N \in \Delta$ and $M[N/x] \notin \Delta$. The following non-standard reduction sequence $(\lambda x.IM)N \to_\Delta (\lambda x.M)N \to_\Delta M[N/x]$ has not a standard counterpart, in fact $I(M[N/x]) \not\to_\Delta M[N/x]$. $\square$

The investigation on the reduction closure is more complex and it needs some additional definitions and remarks. In fact we will prove that the reduction closure is necessary, but in some *degenerated* cases of input values, that are excluded by the next definition.

**Definition 30.** Let $\Delta \subseteq \Lambda$ and let $\mathsf{Var} \subseteq \Delta$.
$\Delta$ is *suitable* if and only if $\Delta$ not closed under $\Delta$-reduction implies that there are $P_0 \in \Delta$, $P_1 \notin \Delta$ such that $P_0 \to_\Delta P_1$ and one of the following two cases arises:
- the number of redexes in $P_1$ is less than the number of redexes in $P_0$;
- there is $P_2 \in \Lambda$ such that $P_1 \to_\Delta P_2$, and:
  - every $\Delta$-reduction sequences from $P_0$ to $P_2$ has length at least 2 and, if all terms in it belong to $\Delta$, then it is not standard;
  - there is $r \in \mathbb{N}$ greater than the maximum number of occurrences of $\Delta$-redexes in all the terms occurring in all reduction sequences from $P_0$ to $P_2$.

In the previous definition $P_0 \not\equiv P_1$, since $P_0 \in \Delta$ while $P_1 \notin \Delta$. Furthermore, note that if the number of redexes in $P_1$ is greater than or equal to the number of redexes in $P_0$ then there is a $P_2 \in \Lambda$ such that $P_1 \to_\Delta P_2$ with a standard reduction sequence.

**Example 31.**

(1) Let $\Delta_0 = \mathsf{Var} \cup \{\lambda x.P \mid P \notin \Lambda\text{-NF}\ \}$. $\Delta_0$ is closed under substitution, it is not closed under $\Delta_0$-reduction but it is suitable. Note that $\Delta_0$ it is not weakly closed under reduction.

(2) Let $I \equiv \lambda x.x$, $D \equiv \lambda x.xx$ and $\Delta_1 = \mathsf{Var} \cup \{D, (ID)D\}$.
$\Delta_1$ is closed under substitution, but it is not closed under $\Delta_1$-reduction, in fact $(ID)D \to_{\Delta_1} DD$. Note that $(ID)D, DD$ both contain one redex and by reducing this unique redex in $DD$ we obtain $DD$ too, so there is a reduction sequence from $(ID)D$ to $DD$ having length less than 2.
Hence $\Delta_1$ is not suitable. Note that $\Delta_1$ it is not weakly closed under reduction.

(3) Let $\Delta_2 = \mathsf{Var} \cup \{M, MM, \lambda z.MM\}$ where $M \equiv \lambda x.(\lambda u.ux)(\lambda y.xx)$. Thus $\Delta_2$ is closed under substitution, while it is not closed under $\Delta_2$-reduction, since it is easy to check that both $MM \to_{\Delta_2} (\lambda u.uM)(\lambda y.MM) \notin \Delta_2$ and $\lambda z.MM \to_{\Delta_2} \lambda z.(\lambda u.uM)(\lambda y.MM) \notin \Delta_2$.
In $MM$ there is a unique redex, while in $(\lambda u.uM)(\lambda y.MM)$ there are two redexes, in particular
$(\lambda u.uM)(\lambda y.MM) \to_{\Delta_2} (\lambda y.MM)M$
$(\lambda u.uM)(\lambda y.MM) \to_{\Delta_2} (\lambda u.uM)(\lambda y.(\lambda u_0.u_0 M)(\lambda y_0.MM))$.
But $(\lambda y.MM)M \to_{\Delta_2} MM$, $(\lambda u.uM)(\lambda y.(\lambda u_0.u_0 M)(\lambda y_0.MM)) \to^*_{\Delta_2} MM$, moreover it is easy to see that for all $n \in \mathbb{N}$, there is $P_n \in \Lambda$ such that $P_n$ contains at least $n$ redexes and $MM \to^*_{\Delta_2} P_n \to^*_{\Delta_2} MM$. By reasoning in the same way on $\lambda x.MM$ it follows that $\Delta_2$ is not suitable.

**Theorem 32.** *Let $\Delta \subseteq \Lambda$ and let $\mathsf{Var} \subseteq \Delta$. If $\Delta$ is suitable then, in order for the $\Delta$-reduction enjoy the standardization property it is necessary for $\Delta$ to be closed under substitution.*

**Proof.** Let $\Delta$ be not closed under substitution; since $\Delta$ is suitable, there are two cases.

- There are $P_0 \in \Delta$ , $P_1 \notin \Delta$, $P_0 \to_\Delta P_1$ and the number of redexes in $P_1$ is less than the number of redexes in $P_0$. Let $P_0 \to_\Delta P_1$ by reducing a redex of degree $k \in \mathbb{N}$, $M \equiv IP_0(Ix)$ and $N \equiv IP_1 x$. Assume $m \in \mathbb{N}$ be such that $k + m$ is the maximum between all the degrees of redexes in $P_0$. There are two possible $\Delta$-reduction sequences from $M$ to $N$, and no one of these is standard, as showed in the next figure, where to every reduction arrow the degree of the reduced redex is associated.

$$\begin{tikzcd}
 & IP_0(Ix) \arrow[dl, "k+1"', "\Delta"] \arrow[dr, "(m+k+1)+1", "\Delta"'] & \\
IP_1(Ix) \arrow[dr, "k"', "\Delta"] & & IP_0 x \arrow[dl, "k+1", "\Delta"'] \\
 & IP_1 x &
\end{tikzcd}$$

- There are $P_0, P_1, P_2$ be such that $P_0 \in \Delta$, $P_1 \notin \Delta$, $P_0 \to_\Delta P_1 \to_\Delta P_2$; moreover if $\mathcal{R}$ is the set of all the $\Delta$-reduction sequences from $P_0$ to $P_2$ and $P_0 \equiv Q_0 \to_\Delta Q_1 \to_\Delta \cdots \to_\Delta Q_{n-1} \to_\Delta Q_n \equiv P_2$ is a sequence in $\mathcal{R}$ then
  - $n \geqslant 2$ and if $\forall i < n\ \ Q_i \in \Delta$ then $Q_0 \to_\Delta Q_1 \to_\Delta \cdots \to_\Delta Q_{n-1} \to_\Delta Q_n$ is not standard;
  - there is $r \in \mathbb{N}$ greater than the maximum number of occurrences of $\Delta$-redexes in all the terms occurring in all reduction sequences in $\mathcal{R}$.

  Let $T \equiv \lambda x. \underbrace{(Ix) \ldots\ldots (Ix)}_{r}$.

  If $\forall i < n\ \ Q_i \in \Delta$ then $TQ_0 \to_\Delta TQ_1 \to_\Delta \cdots \to_\Delta TQ_n$ is not standard too. Let $j < n$ be the minimum index such that $Q_j \notin \Delta$, let $m_0$ be the degree of the redex reduced in the reduction step

$Q_{j-1} \to_{\Delta} Q_j$ and let $m_1$ be the degree of the redex reduced in the reduction step $Q_j \to_{\Delta} Q_{j+1}$. Hence $TQ_{j-1} \to_{\Delta} TQ_j$ by reducing a redex of degree $r + m_0$, while $TQ_j \to_{\Delta} TQ_{j+1}$ by reducing a redex of degree $m_1$. So $m_1 + 1 \leqslant r \leqslant r + m_0$ implies that $TQ_0 \to_{\Delta} TQ_1 \to_{\Delta} \cdots \to_{\Delta} TQ_{n-1} \to_{\Delta} TQ_n$ is not standard too. □

In conclusion, since we are interested in calculi enjoying both the confluence and the standardization property, the two closure conditions we impose on the set of input values are not too restrictive.

## 6. Operational Semantics

In order to define an operational semantics for the $\mathsf{V}$-calculus, besides the notion of input values, also a notion of *output values* is necessary. In fact the evaluation process is a procedure trying to transform a term into an output value, by applying a sequence of $\mathsf{V}$-reductions.

The notion of set of output results is parametric with respect to the set of input values, as shown in the next definition.

**Definition 33.** Let $\mathsf{V}$ be a set of input values.

A set of *output values with respect to* $\mathsf{V}$ is any set $\mathsf{V}_{\text{out}} \subseteq \Lambda$ such that:

(i) $\mathsf{V}_{\text{out}}$ contains all the $\mathsf{V}$-normal forms;

(ii) if $M =_{\mathsf{V}} N$ and $N \in \mathsf{V}_o$ then, there is $P \in \mathsf{V}_{\text{out}}$ such that $M \to_{\mathsf{V}}^{*p} P$ (*principality condition*).

The first condition of the previous definition takes into account that the set of normal forms is in some sense the most "natural" set of output values, corresponding to the complete evaluation of terms. Remember that Corollary 25 assures us that, for reaching the normal form of a term, if it exists, is sufficient to perform at every step the principal redex. So the second condition simply says that we are interested in the evaluations that are an initial step of the complete one. As we will show in the sequel, each interesting evaluation is of this kind.

The next property shows some examples of set of output values.

**Property 34.**

(1) *$\Lambda$, $\Lambda$-$\mathsf{NF}$, and the set of head normal forms are sets of output values with respect to $\Lambda$.*

(2) *$\Lambda$ and $\Lambda_v$-$\mathsf{NF}$ are sets of output values with respect to $\Lambda_v$.*

(3) *$\Lambda_v$ is not a set of output values with respect to $\Lambda$.*

(4) *$\Lambda_v$ is not a set of output values with respect to $\Lambda_v$.*

**Proof.** Easy. □

So we can define a reduction machine, parametric with respect to both the sets of input and output values, performing the principal reduction for the $\lambda\mathsf{V}$-calculus. The interest of such a machine is that almost all the known reduction machines for $\lambda$-calculi can be obtained from it, by a suitable instantiation of both $\mathsf{V}$ and $\mathsf{V}_{\text{out}}$.

$$\frac{M \to^e_{\mathsf{V}} N}{\lambda x.M \to^e_{\mathsf{V}} \lambda x.N}\ 1$$

$$\frac{i = \min\{j \leq m | M_i \notin \mathsf{V\text{-}NF}\} \quad M_i \to^e_{\mathsf{V}} N_i}{x M_1 ... M_m \to^e_{\mathsf{V}} x M_1 ... N_i ... M_m}\ 2$$

$$\frac{Q \in \mathsf{V}}{M \equiv (\lambda x.P) Q M_1 ... M_m \to^e_{\mathsf{V}} P[Q/x] M_1 ... M_m}\ 3$$

$$\frac{Q \notin \mathsf{V} \quad Q \notin \mathsf{V\text{-}NF} \quad Q \to^e_{\mathsf{V}} Q'}{M \equiv (\lambda x.P) Q M_1 ... M_m \to^e_{\mathsf{V}} (\lambda x.P) Q' M_1 ... M_m}\ 4$$

$$\frac{Q \notin \mathsf{V} \quad Q \in \mathsf{V\text{-}NF} \quad P \notin \mathsf{V\text{-}NF} \quad P \to^e_{\mathsf{V}} P'}{M \equiv (\lambda x.P) Q M_1 ... M_m \to^e_{\mathsf{V}} (\lambda x.P') Q M_1 ... M_m}\ 5$$

$$\frac{Q \notin \mathsf{V} \quad P, Q \in \mathsf{V\text{-}NF} \quad i = \min\{j \leq m | M_i \notin \mathsf{V\text{-}NF}\} \quad M_i \to^e_{\mathsf{V}} N_i}{M \equiv (\lambda x.P) Q M_1 ... M_m \to^e_{\mathsf{V}} (\lambda x.P) Q M_1 ... N_i ... M_m}\ 6$$

Fig. 3. The Principal $\mathsf{V}$-machine.

The fact that the set of output values satisfy the principality condition allows us to define a universal evaluation relation, parametric both in the set of input and output values, from which many interesting evaluation relations can be derived by suitable instantiations. Such an evaluation relation is based on a formal system, defining the principal evaluation of a term of the $\lambda\mathsf{V}$-calculus.

The one-step reduction machine which reduces at every step the principal redex is defined in Fig. 3.

**Definition 35.** Let $M \in \Lambda$ and $N \in \mathsf{V}_{\text{out}}$. The statement $M \Downarrow_{\mathsf{V},\mathsf{V}_o} N$ (*read it as "M converges to the output value N"*) is formalized as follows:

$$\frac{M \in \mathsf{V}_{\text{out}}}{M \Downarrow_{\mathsf{V},\mathsf{V}_o} M}\ 0 \qquad\qquad \frac{M \to^e_{\mathsf{V}} P \qquad P \Downarrow_{\mathsf{V},\mathsf{V}_o} N}{M \Downarrow_{\mathsf{V},\mathsf{V}_o} N}\ 0'.$$

The following theorem proves that the notion of convergence is coherent with respect to the reduction. As usual, we will write $M \Downarrow_{\mathsf{V},\mathsf{V}_o}$ if and only if there exists $P \in \mathsf{V}_{\text{out}}$ such that $M \Downarrow_{\mathsf{V},\mathsf{V}_o} P$.

**Theorem 36.** *Let $\mathsf{V}_{\text{out}}$ be a set of output values with respect to $\mathsf{V}$ and let $N \in \mathsf{V}_{\text{out}}$. $M \Downarrow_{\mathsf{V},\mathsf{V}_o} N$ if and only if $M \to^*_{\mathsf{V}} N$.*

**Proof.** Easy. □

The machine performing the left-most reduction, defined in [3], can be obtained from the principal $\mathsf{V}$-machine by posing $\mathsf{V} \equiv \Lambda$ and $\mathsf{V}_{\text{out}}$ as the set of $\beta$-normal forms. By choosing $\mathsf{V} = \Lambda$ and

$\mathsf{V}_{\text{out}}$ as the set of head-normal-forms the principal machine performs the head reduction strategy, while if $\mathsf{V} = \Lambda$ and $\mathsf{V}_{\text{out}}$ is the set of weak head normal forms it performs the lazy strategy (a machine performing the lazy strategy on closed terms has been defined in [1]). Moreover by choosing $\mathsf{V} = \Lambda_v$ and $\mathsf{V}_{\text{out}}$ as the union of $\Lambda_v$ and the set of normal forms with respect to the $\Lambda_v$-reduction, the principal $\mathsf{V}$-machine becomes (the extension to open terms of the ) SECD machine of Landin [13,18]. Let us recall the classical definition of the operational semantics, as given by Plotkin in [18].

**Definition 37.** Let $M, N \in \Lambda$, let $\mathsf{V}$ be a standard set of input values and $\mathsf{V}_{\text{out}}$ a set of output values. $M \approx_{\mathsf{V},\mathsf{V}_{\text{out}}} N$ if and only if, for all contexts $C[\_]$ such that $C[M], C[N] \in \Lambda^0$, $C[M] \Downarrow_{\mathsf{V},\mathsf{V}_o} \;\Leftrightarrow\; C[N] \Downarrow_{\mathsf{V},\mathsf{V}_o}$.

It is easy to check that the $\mathsf{V}$-equivalence is contained in $\approx_{\mathsf{V},\mathsf{V}_{\text{out}}}$: this means that the operational semantics is correct with respect to calculus.

**Lemma 38.** *Let $M, N \in \Lambda$. $M =_{\mathsf{V}} N$ implies $M \approx_{\mathsf{V},\mathsf{V}_{\text{out}}} N$.*

**Proof.** $M =_{\mathsf{V}} N$ implies $C[M] =_{\mathsf{V}} C[N]$, for all context $C[\_]$. So the proof follows by the principality condition. $\square$

## 7. Related papers

The first notion of standardization has been given, for the $\lambda\Lambda$-calculus, by Curry and Feys [6]. With respect to their notion, if $M \to_{\Lambda}^{*} N$ then there is a standard reduction sequence from $M$ to $N$, but this reduction sequence is not necessarily unique. For instance, $\lambda x.x(II)(II) \to_{\Lambda} \lambda x.xI(II) \to_{\Lambda} \lambda x.II$ and $\lambda x.x(II)(II) \to_{\Lambda} \lambda x.x(II)I \to_{\Lambda} \lambda x.II$ are both standard reduction sequences. The most known formal definition of standard reduction sequence is given using the notion of residuals of a given redex: this notion induces a partial order between redexes, and a reduction sequence is standard if and only if, for every pair of redexes $(R, R')$, if $R$ follows $R'$ in the partial order, then it cannot be reduced before it. Inductive formalizations of this notion have been given in [7,14].

Klop [12] introduced a notion of strong stardardization, according to which, if $M \to_{\Lambda}^{*} N$, then there is a unique strongly standard reduction sequence from $M$ to $N$, and he designed an algorithm for transforming a reduction sequence into a strongly standard one. According to his notion, in the example before only the first reduction sequence is standard. The algorithm uses again the notion of residual. A further definition of strong standardization is due to Takahashi, which introduces a total order between the redexes in a reduction sequence, in a similar way as we do. This total order is defined on the structure of terms, skipping the difficult notion of residual.

Our definition, when restricted to the $\lambda\Lambda$-calculus, is quite similar to the strong standardization. In fact, according to our definition, the standard reduction sequence is unique, but in some degenerated case: e.g., for $\mathsf{V} = \Lambda$, there are infinite reduction sequences from $x(DD)$ to $x(DD)$, each one performing a different number of $\Lambda$-reductions.

Plotkin [18] extended the notion of standardization to the $\lambda\Lambda_v$-calculus. His notion of standardization is not strong, using Klop's terminology. In fact, both the reduction sequences: $(\lambda x.II)(II) \to_{\Lambda_v} (\lambda x.II)I \to_{\Lambda_v} (\lambda x.I)I$ and $(\lambda x.II)(II) \to_{\Lambda_v} (\lambda x.I)(II) \to_{\Lambda_v} (\lambda x.I)I$ are standard, according to its definition. Our definition, when restricted to $\lambda\Lambda_v$-calculus, is a strong version of Plotkin's standardization. Indeed, only the first of the two previous reduction sequences is standard, in our terminology.

However, it is important to notice that, if we extend Plotkin's definition of standardization by replacing the set $\Lambda_v$ of input values by $\Lambda_I$, we obtain the same result we proved, namely that the standardization does not hold. So the fact that not all sets of input values enjoy the standardization property is not consequence of our definition, based on a total order between redexes, but is an intrinsic property of a call-by-value evaluation.

The advantage of our notion of standardization is that it implies immediately Corollary 25, i.e., the fact that the principal reduction is $\mathsf{V}$-normalizing.

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