Proof Methods for Conditional and Preferential Logics of Nonmonotonic Reasoning

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Abstract. Conditional and Preferential Logics have been applied in order to formalize nonmonotonic reasoning. In spite of their significance, very few proof methods have been proposed for these logics. In my PhD thesis [10], whose content is summarized in this paper, I have tried to partially overwhelm this gap, by introducing sequent and tableau calculi for Conditional and Preferential Logics.

1 Introduction

Many systems exhibiting a nonmonotonic behavior have been studied in the literature; negation as failure, Circumscription, Default logic are only some examples. Each of these systems deserves an independent interest, however it is not clear that any one of them really captures the whole properties of nonmonotonic reasoning. An alternative solution to the formalization of nonmonotonic reasoning consists in the application of conditional and preferential logics.

Conditional logics have a long history. They have been studied first by Lewis and Stalnaker [7] in order to formalize a kind of hypothetical reasoning (“if A were the case then B”), that cannot be captured by classical logic with material implication. Moreover, they have been used to formalize counterfactual sentences, i.e. conditionals of the form “if A were the case then B would be the case”, where A is false.

In the last years, there has been a considerable amount of work on applications of conditional logics to various areas of artificial intelligence and knowledge representation. For instance, they have been used to formalize knowledge update and revision. Moreover, conditional logics have been used to model hypothetical queries in deductive databases and logic programming. In a related context, conditional logics have been used to model causal inference and reasoning about action execution in planning.

The application of conditional logics to nonmonotonic reasoning was firstly investigated by Delgrande [2] who proposed a conditional logic for prototypical reasoning: the understanding of a conditional A ⇒ B in his logic is “the A’s have typically the property B”. For instance, one could have: ∀x(Penguin(x) → Bird(x)), ∀x(Penguin(x) → ¬Fly(x)), ∀x(Bird(x) ⇒ Fly(x)). The last sentence states that birds typically fly. Observe that replacing ⇒ with the classical implication →, the above knowledge base is consistent only if there are no penguins.
The study of the relations between conditional logics and nonmonotonic reasoning has gone much further since the seminal work by Kraus, Lehmann, and Magidor [5, 6] (KLM framework), who proposed a formalization of the properties of a nonmonotonic consequence relation. According to KLM framework, a defeasible knowledge base is represented by a (finite) set of nonmonotonic conditionals of the form \( A \not\rightarrow B \), whose reading is normally (or typically) the \( A \)'s are \( B \)'s. The operator “\( \not\rightarrow \)” is nonmonotonic, in the sense that \( A \not\rightarrow B \) does not imply \( A \land C \not\rightarrow B \). For instance, a knowledge base \( K \) may contain football_lover \( \not\rightarrow \) bet, football_player \( \not\rightarrow \) football_lover, football_player \( \not\rightarrow \) ~bet, whose meaning is that people loving football typically bet on the result of a match, football players typically love football but they typically do not bet (especially on matches they are going to play...). If \( \not\rightarrow \) were interpreted as classical implication, one would get football_player \( \not\rightarrow \) ⊥, i.e. typically there are not football players, thereby obtaining a trivial knowledge base. In KLM framework, the set of adopted inference rules defines some fundamental types of inference systems, namely, from the strongest to the weakest: Rational (R), Preferential (P), Loop-Cumulative (CL), and Cumulative (C) logic. In all these systems one can infer new assertions without incurring the trivializing conclusions of classical logic: in the above example, in none of the systems can one infer football_player \( \not\rightarrow \) bet. In cumulative logics (both C and CL) one can infer football_lover \( \land \) football_player \( \not\rightarrow \) ~bet, giving preference to more specific information; in Preferential logic P one can also infer that football_lover \( \not\rightarrow \) ~football_player; in the rational case R, if one further knows that ~football_lover \( \not\rightarrow \) rich, that is to say it is not the case that football lovers are typically rich persons, one can also infer that football_lover \( \land \) ~rich \( \not\rightarrow \) bet. The logics of the KLM framework are also known as Preferential logics.

The connection between conditional and preferential logics has been largely investigated. It turns out that all forms of inference studied in KLM framework are particular cases of well-known conditional axioms [1]. In this respect the KLM language is just a fragment of conditional logics.

In spite of their significance, very few proof systems have been proposed for conditional and preferential logics. In my PhD thesis [10], whose content is summarized in this paper, I have tried to (partially) overwhelm this gap, by providing analytic calculi for some standard conditional logics and for all the four KLM systems.

2 Conditionals Logics

Conditional logics are extensions of classical logic by the conditional operator \( \Rightarrow \). We restrict our concern to propositional conditional logics.

A propositional conditional language \( \mathcal{L} \) is defined from a set of propositional variables \( \text{ATM} \), the symbols of false \( \bot \) and true \( \top \), and a set of connectives \( \neg, \land, \lor, \rightarrow, \) and \( \Rightarrow \). Formulas of \( \mathcal{L} \) are defined as usual: \( \bot, \top \) and the propositional variables of \( \text{ATM} \) are atomic formulas; if \( A \) and \( B \) are formulas, then \( \neg A, A \land B, A \lor B, A \rightarrow B \) and \( A \Rightarrow B \) are complex formulas.
Similarly to modal logics, the semantics of conditional logics can be defined in terms of possible world structures. In this respect, conditional logics can be seen as a generalization of modal logics where the conditional operator is a sort of modality indexed by a formula of the same language. We adopt the selection function semantics: intuitively, the selection function selected for a world \( w \) and a formula \( A \), the set of worlds of \( W \) which are closer to \( w \) given \( A \). A conditional formula \( A \Rightarrow B \) holds in a world \( w \) if the formula \( B \) holds in all the worlds selected by \( f \) for \( w \) and \( A \).

Formally, a selection function model is a triple \( (W,f,[ ]) \) where \( W \) is a non-empty set of worlds, \( f : W \times 2^W \rightarrow 2^W \) is the selection function, and \( [ ] \) is the evaluation function, which assigns to an atom \( P \in ATM \) the set of worlds where \( P \) is true, and is extended to the other formulas as follows: \([\bot] = \emptyset; \[\top] = W; \[\neg A \] = W - [A]; \[A \wedge B \] = [A] \cap [B]; \[A \vee B \] = [A] \cup [B]; \[A \rightarrow B \] = (W - [A]) \cup [B]; \[A \Rightarrow B \] = \{w \in W \mid f(w, [A]) \subseteq [B]\}.

Notice that we have defined \( f \) taking \([A]\) rather than \( A \); this is equivalent to define \( f \) on formulas, i.e. \( f(w,A) \) but imposing that if \([A]=[A']\) in the model, then \( f(w,A)=f(w,A') \). This condition is called normality.

The semantics above characterizes the basic normal conditional system, called CK. As in modal logic, extensions of the basic system CK are obtained by assuming further properties on the selection function. We consider the following standard extensions of the basic system CK:

<table>
<thead>
<tr>
<th>System</th>
<th>Axiom</th>
<th>Model condition</th>
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<tbody>
<tr>
<td>ID</td>
<td>( A \Rightarrow A )</td>
<td>( f(w, [A]) \subseteq [A] )</td>
</tr>
<tr>
<td>MP</td>
<td>( (A \Rightarrow B) \rightarrow (A \rightarrow B) )</td>
<td>( w \in [A] \Rightarrow w \in f(w, [A]) )</td>
</tr>
<tr>
<td>CS</td>
<td>( (A \wedge B) \rightarrow (A \Rightarrow B) )</td>
<td>( w \in [A] \Rightarrow f(w, [A]) \subseteq {w} )</td>
</tr>
<tr>
<td>CEM</td>
<td>( (A \Rightarrow B) \vee (A \Rightarrow \neg B) )</td>
<td>(</td>
</tr>
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### 2.1 Sequent Calculi SeqS

In Figure 1 sequent calculi SeqS for conditional logics are presented. S stands for \{CK,ID,MP,CS,CEM\} and all their combinations (except for those containing both CEM and MP). The calculi make use of labelled formulas, where the labels are drawn from a denumerable set \( L \); there are two kinds of formulas: world formulas, denoted by \( x : A \), where \( x \in L \) and \( A \in L \); transition formulas, denoted by \( x \xrightarrow{A} y \), where \( x, y \in L \) and \( A \in L \). A world formula \( x : A \) is used to represent that \( A \) holds in the possible world represented by the label \( x \); a transition formula \( x \xrightarrow{A} y \) represents that \( y \in f(x, [A]) \).

A sequent is a pair \( \Gamma, \Delta \), usually denoted with \( \Gamma \vdash \Delta \), where \( \Gamma \) and \( \Delta \) are multisets of labelled formulas. The intuitive meaning of \( \Gamma \vdash \Delta \) is: every model that satisfies all labelled formulas of \( \Gamma \) in the respective worlds (specified by the labels) satisfies at least one of the labelled formulas of \( \Delta \) (in those worlds).

In [10] it is shown that SeqS calculi are sound and complete with respect to the selection function semantics, i.e. a sequent \( \Gamma \vdash \Delta \) is valid if and only if \( \Gamma \vdash \Delta \) is derivable in SeqS. Moreover, SeqS calculi can be used to describe a
We consider a propositional language which is shown that provability in the studied logics is decidable in polynomial space.

### 3 Preferential Logics

We consider a propositional language $\mathcal{L}$ defined from a set of propositional variables $ATM$, the boolean connectives and the conditional operator $\vdash$. We use $A, B, C, ...$ to denote propositional formulas, whereas $F, G, ...$ are used to denote all formulas (including conditionals). The formulas of $\mathcal{L}$ are defined as follows: if $A$ is a propositional formula, $A \in \mathcal{L}$; if $A$ and $B$ are propositional formulas, $A \vdash B \in \mathcal{L}$; if $F$ is a boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$.

In general, the semantics of KLM logics is defined by considering possible world (or possible states) structures with a preference relation $w < w'$ among worlds (or states), whose meaning is that $w$ is preferred to $w'$. $A \vdash B$ holds in a model $\mathcal{M}$ if $B$ holds in all minimal worlds (states) where $A$ holds. This definition makes sense provided minimal worlds for $A$ exist whenever there are $A$-worlds ($A$-states): this is ensured by the smoothness condition defined below.

We recall the semantics of KLM logics [5, 6] from the strongest $\mathbf{R}$ to the weakest $\mathbf{C}$. A rational model is a triple $\mathcal{M} = (W, <, V)$, where $W$ is a non-empty set of items called worlds, $<$ is an irreflexive, transitive and modular^1 relation on $W$, and $V$ is an evaluation function $V : W \rightarrow 2^{ATM}$, which assigns to every world $w$ the set of atoms holding in that world. The truth conditions for a formula $F$ are as follows: - if $F$ is a boolean combination of formulas, $\mathcal{M}, w \models F$ is defined as for propositional logic; - let $A$ be a propositional formula; we define $\mathcal{M} \models \Delta(A)$.

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<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
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<tbody>
<tr>
<td>$(\text{A})$</td>
<td>$\Gamma, x : \bot \vdash \Delta$</td>
<td>$(\text{AX})$</td>
</tr>
<tr>
<td>$(\rightarrow L)$</td>
<td>$\Gamma, x : \Delta, x : A$</td>
<td>$\Gamma, x : B \vdash \Delta$</td>
</tr>
<tr>
<td>$(\rightarrow L)$</td>
<td>$\Gamma, x : A \Rightarrow B \vdash \Delta, x : y$</td>
<td>$\Gamma, x : A \Rightarrow B \vdash \Delta$</td>
</tr>
<tr>
<td>$(\text{EQ})$</td>
<td>$(u : A \vdash u : B)$</td>
<td>$u : B \vdash u : A$</td>
</tr>
<tr>
<td>$(\text{CEM})$</td>
<td>$\Gamma, x \vdash \Delta, x \vdash z$</td>
<td>$(\text{CEM})$</td>
</tr>
<tr>
<td>$(\text{CS})$</td>
<td>$\Gamma, x \vdash \Delta, x : A$</td>
<td>$(\text{CS})$</td>
</tr>
</tbody>
</table>

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^{1} A relation $<$ is modular if, for each $u, v, w$, if $u < v$, then either $w < v$ or $u < w$. 

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Fig. 1. Sequent calculi SeqS. Rules (ID), (MP), (CEM) and (CS) are only used in corresponding extensions of the basic system SeqCK. To save space, we omit the standard rules for the other boolean connectives.
Sequent calculi SeqS for some standard conditional logics

In this paper I have briefly summarized the contents of my PhD thesis [10].

3.1 Tableau calculi $\mathcal{TST}$

In Figure 2 we present the tableaux calculi $\mathcal{TST}$ for KLM logics, where $S$ stands for \{R, P, CL, C\}. The basic idea is simply to interpret the preference relation as an accessibility relation. The calculi for R and P implement a sort of runtime translation into (extensions of) Gödel-Löb modal logic of provability G.

This is motivated by the fact that we assume the smoothness condition, which ensures that minimal $A$-worlds exist whenever there are $A$-worlds, by preventing infinitely descending chains of worlds. This condition therefore corresponds to the finite-chain condition on the accessibility relation (as in modal logic G). This approach is extended to the cases of CL and C by using a second modality $L$ which takes care of states. The rules of the calculi manipulate sets of formulas $\Gamma$. We write $\Gamma, F$ as a shorthand for $\Gamma \cup \{F\}$. Moreover, given $\Gamma$ we define the following sets: $\Gamma^\Box = \{\Box \neg A \mid \Box \neg A \in \Gamma\}$; $\Gamma^\Box^L = \{\neg A \mid \Box \neg A \in \Gamma\}$; $\Gamma^{\bowtie-k} = \{A \bowtie B \mid A \bowtie B \in \Gamma\}$; $\Gamma^L = \{A \mid LA \in \Gamma\}$.

As mentioned, the calculus for rational logic R makes use of labelled formulas, where the labels are drawn from a denumerable set $\mathcal{A}$; there are two kinds of formulas: 1. world formulas, denoted by $x : F$, where $x \in \mathcal{A}$ and $F \in \mathcal{L}$; 2. relation formulas, denoted by $x < y$, where $x, y \in \mathcal{A}$, representing the preference relation. We define $\Gamma^M_{\bowtie-k} = \{y : \neg A, y : \Box \neg A \mid x : \Box \neg A \in \Gamma\}$.

The calculi $\mathcal{TST}$ are sound and complete wrt the semantics, i.e. given a set of formulas $\Gamma$ of $\mathcal{L}$, it is unsatisfiable if and only if there is a closed tableau in $\mathcal{TST}$ having $\Gamma$ as a root [10]. Furthermore, it can be shown that the calculi $\mathcal{TST}$ ensure termination only by preventing that the ($\bowtie^\ast$) rule is applied more than once in the same world. In [10] it is also shown that the problem of deciding validity for preferential logics is $\text{coNP}$-complete.

4 Conclusions

In this paper I have briefly summarized the contents of my PhD thesis [10]. Sequent calculi SeqS for some standard conditional logics and tableau calculi...
Fig. 2. Table systems $TST^T$. To save space, we omit the standard rules for boolean connectives. For $TCL^T$ and $TC^T$ the axiom $(AX)$ is as in $TP^T$.

$TST^T$ for preferential logics have been presented. These calculi are sound, complete and terminating. They have been also implemented in SICStus Prolog [8, 4]. Recently, a free-variable version of the tableau calculus $TP^T$ for $P$ has been presented [3]. Moreover, the calculi Seq$S$ have been used as the base of a goal-directed proof search mechanisms for conditional logics [9]. This extension might be the base of a language for reasoning hypothetically about change and actions in a logic programming framework.

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References