Nested sequents for Conditional Logics: Preliminary Results

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Abstract. Nested sequent calculi are a useful generalization of ordinary sequent calculi, where sequents are allowed to occur within sequents. Nested sequent calculi have been profitably employed in the area of (multi)-modal logic to obtain analytic and modular proof systems for these logics. In this work, we extend the realm of nested sequents by providing nested sequent calculi for the basic conditional logic CK and some of its significant extensions. We provide also a calculus for Kraus Lehman Magidor cumulative logic C. The calculi are internal (a sequent can be directly translated into a formula), cut-free and analytic. Moreover, they can be used to design (sometimes optimal) decision procedures for the respective logics, and to obtain complexity upper bounds. Our calculi are an argument in favour of nested sequent calculi for modal logics and alike, showing their versatility and power.

1 Introduction

The recent history of the conditional logics starts with the work by Lewis\cite{lewis1973counterfactual,lewis1975counterfactual}, who proposed them in order to formalize a kind of hypothetical reasoning (if $A$ were the case then $B$), that cannot be captured by classical logic with material implication. One original motivation was to formalize counterfactual sentences, i.e. conditionals of the form “if $A$ were the case then $B$ would be the case”, where $A$ is false. Conditional logics have found an interest in several fields of artificial intelligence and knowledge representation. They have been used to reason about prototypical properties\cite{kuno1995analogical} and to model belief change\cite{levesque1987default,levesque1989default}. Moreover, conditional logics can provide an axiomatic foundation of nonmonotonic reasoning\cite{greenough2010weakening,van2007nonmonotonic}, here a conditional $A \Rightarrow B$ is read as “in normal circumstances if $A$ then $B$”. Recently, constructive conditional logics have been applied to reason about access control policies\cite{fagin2003access,chrisandrade2006conditional}: the statement $A$ says $B$, intuitively meaning that a user/program $A$ asserts $B$ to hold in the system, can be naturally expressed by a conditional $A \Rightarrow B$. Finally, a kind of (multi)-conditional logics\cite{baader2007description,baader2006conditional} have been used to formalize epistemic change in a multi-agent setting and in some kind of epistemic “games”, each conditional operator expresses the “conditional beliefs” of an agent.

Semantically, all conditional logics enjoy a possible world semantics, with the intuition that a conditional $A \Rightarrow B$ is true in a world $x$, if $B$ is true in the set of worlds
where $A$ is true and that are most similar/closest/“as normal as” $x$. Since there are different ways of formalizing “the set of worlds similar/closest/...” to a given world, there are expectedly rather different semantics for conditional logics, from the most general selection function semantics to the stronger sphere semantics.

From the point of view of proof-theory and automated deduction, conditional logics do not have however a state of the art comparable with, say, the one of modal logics, where there are well-established alternative calculi, whose proof-theoretical and computational properties are well-understood. This is partially due to the mentioned lack of a unifying semantics; as a matter of fact the most general semantics, the selection function one, is of little help for proof-theory, and the preferential/sphere semantics only captures a subset of (actually rather strong) systems. Similarly to modal logics and other extensions/alternative to classical logics two types of calculi have been studied: external calculi which make use of labels and relations on them to import the semantics into the syntax, and internal calculi which stay within the language, so that a “configuration” (sequent, tableau node...) can be directly interpreted as a formula of the language. Just to mention some work, to the first stream belongs [2] proposing a calculus for (unnested) cumulative logic $C$ (see below). More recently, [26] presents modular labeled calculi (of optimal complexity) for CK and some of its extensions, basing on the selection function semantics, and [19] presents modular labeled calculi for preferential logic PCL and its extensions. The latter calculi take advantage of a sort of hybrid modal translation. To the second stream belong the calculi by Gent [15] and by de Swart [29] for Lewis’ logic $\mathcal{VK}$ and neighbours. These calculi manipulate sets of formulas and provide a decision procedure, although they comprise an infinite set of rules. Very recently, some internal calculi for CK and some extensions (with any combination of MP, ID, CEM) have been proposed by Pattinson and Schröder [27]. The calculi are obtained by a general method for closing a set of rules (corresponding to Hilbert axioms) with respect to the cut rule. These calculi have optimal complexity; notice that some of the rules do not have a fixed number of premises. These calculi have been extended to preferential conditional logics [28], i.e. including cumulativity (CM) and or-axiom (CA), although the resulting systems are fairly complicated. Finally in [23] the authors provide internal calculi for Lewis’ conditional logic $\mathcal{V}$ and some extensions, their calculi are formulated for a language comprising the entrenchment connective, the strong and the weak conditional operator, both conditional operators can be defined in terms of the entrenchment connective. These calculi are actually an extension of Gent’s and de Swart’s ones. Basing on these calculi, the authors obtain a PSPACE decision procedure for the considered logics.

In this paper we begin to investigate nested sequents calculi for conditional logics. Nested sequents are a natural generalization of ordinary sequents where sequents are allowed to occur within sequents. However a nested sequent always corresponds to a formula of the language, so that we can think of the rules as operating “inside a formula”, combining subformulas rather than just combining outer occurrences of formulas as in ordinary sequents. In this sense, they are a special kind of “deep inference” calculi as proposed by Guglielmi and colleagues [10, 9]. Nested calculi have been provided for modal logics by Brünnler [7, 6, 8] and extended further by Fitting [12] who
has also clarified their relations with prefixed tableaux. In [20] nested sequent calculi have been provided for Bi-Intuitionistic Logic.

In this paper we treat the basic normal conditional logic CK (its role is the same as K in modal logic) and its extensions with ID, MP and CEM. We also consider the cumulative logic C introduced in [22] which corresponds to flat fragment (i.e., without nested conditionals) of CK+CSO+ID. The calculi are rather natural, all rules have a fixed number of premises. The completeness is established by cut-elimination, whose peculiarity is that it must take into account the substitution of equivalent antecedents of conditionals (a condition corresponding to normality). The calculi can be used to obtain a decision procedure for the respective logics by imposing some restrictions preventing redundant applications of rules. In all cases, we get a PSPACE upper bound, a bound that for CK and its extensions with ID and MP is optimal (but not for CK+CEM that is known to be coNP). For flat CK+CSO+ID = cumulative logic C we also get a PSPACE bound, we are not aware of better upper bound for this logic (although we may suspect that it is not optimal). We can see the present work as a further argument in favor of nested sequents as a useful tool to provide natural, yet computationally adequate, calculi for modal extensions of classical logics.

This paper extends and revises preliminary results proposed in [1].

2 Conditional Logics

A propositional conditional language L contains:

– a set of propositional variables ATM;
– the symbols of false ⊥ and true ⊤;
– a set of connectives ∧, ∨, ¬, →, ⇒.

We define formulas of L as follows:

– ⊥, ⊤ and the propositional variables of ATM are atomic formulas;
– if A and B are formulas, then ¬A and A ⊗ B are complex formulas, where ⊗ ∈ {∧, ∨, →, ⇒}.

We adopt the selection function semantics. We consider a non-empty set of possible worlds W. Intuitively, the selection function f selects, for a world w and a formula A, the set of worlds of W which are closer to w given the information A. A conditional formula A ⇒ B holds in a world w if the formula B holds in all the worlds selected by f for w and A.

**Definition 1 (Selection function semantics).** A model is a triple

\[ \mathcal{M} = (W, f, [\_]) \]

where:

– W is a non empty set of worlds;
– f is the selection function \( f : W \times 2^W \rightarrow 2^W \).
A formula \( F \in \mathcal{L} \) is valid in a model \( M = \langle W, f, \mathbb{S} \rangle \), and we write \( M \models F \), if \([F] = W\). A formula \( F \in \mathcal{L} \) is valid, and we write \( \models F \), if it is valid in every model, that is to say \( M \models F \) for every \( M \).

We have defined \( f \) taking \([A]\) rather than \( A \) (i.e. \( f(w, [A]) \) rather than \( f(w, A) \)) as an argument; this is equivalent to define \( f \) on formulas, i.e. \( f(w, A) \) but imposing that if \([A] = [A']\) in the model, then \( f(w, A) = f(w, A') \). This condition is called normality.

The semantics above characterizes the basic conditional system, called CK [25]. An axiomatization of CK is given by (\( \vdash \) denotes provability in the axiom system):

- any axiomatization of the classical propositional calculus (prop)
- If \( \vdash A \) and \( \vdash A \rightarrow B \), then \( \vdash B \) (Modus Ponens)
- If \( \vdash A \leftrightarrow B \) then \( \vdash (A \rightarrow C) \leftrightarrow (B \rightarrow C) \) (RCEA)
- If \( \vdash (A_1 \land \cdots \land A_n) \rightarrow B \) then \( \vdash (C \Rightarrow A_1 \land \cdots \land C \Rightarrow A_n) \rightarrow (C \Rightarrow B) \) (RCK)

Other conditional systems are obtained by assuming further properties on the selection function. In this work, we consider the following standard extensions of the basic system CK:

<table>
<thead>
<tr>
<th>System</th>
<th>Axiom</th>
<th>Model condition</th>
</tr>
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<tbody>
<tr>
<td>ID</td>
<td>( A \Rightarrow A )</td>
<td>( f(w, [A]) \subseteq [A] )</td>
</tr>
<tr>
<td>CEM</td>
<td>((A \Rightarrow B) \lor (A \Rightarrow \neg B))</td>
<td>( f(w, [A]) \leq 1 )</td>
</tr>
<tr>
<td>MP</td>
<td>((A \Rightarrow B) \rightarrow (A \rightarrow B))</td>
<td>( w \in [A] ) implies ( w \in f(w, [A]) )</td>
</tr>
<tr>
<td>CSO</td>
<td>((A \Rightarrow B) \land (B \Rightarrow A) \rightarrow (A \Rightarrow C) \rightarrow (B \Rightarrow C))</td>
<td>( f(w, [A]) \subseteq [B] ) and ( f(w, [B]) \subseteq [A] )</td>
</tr>
</tbody>
</table>

The above axiomatization is sound and complete with respect to the semantics [25]:

**Theorem 1 (Soundness and completeness of axiomatization with respect to the semantics of Definition 1 [25])**. Given a formula \( F \in \mathcal{L} \), \( F \) is valid in a conditional logic if and only if it is provable in the respective axiomatization, i.e. \( \models F \) if and only if \( \vdash F \).

## 3 Nested Sequent Calculi \( \mathcal{N}S \) for Conditional Logics

In this section we present nested sequent calculi \( \mathcal{N}S \), where \( S \) is an abbreviation for CK\{+X\}, and X=\{CEM, ID, MP, ID+MP, CEM+ID\}. We are able to deal with the basic normal conditional logic CK and its extensions with axioms ID, MP and CEM. We are
also able to deal with some combinations of them, namely the systems allowing ID with either MP or CEM. The problem of extending N\(S\) to the conditional logics allowing both MP and CEM is open at present. As usual, the completeness of the calculi is an easy consequence of the admissibility of cut. We are also able to turn N\(S\) into a terminating calculus, which gives us a decision procedure for the respective conditional logics.

**Definition 2.** A nested sequent \(\Gamma\) is defined inductively as follows:

1. A finite multiset of formulas of \(\mathcal{L}\) is a nested sequent;
2. if \(A\) is a formula and \(\Gamma\) is a nested sequent, then \([A : \Gamma]\) is a nested sequent;
3. a finite multiset of nested sequents is a nested sequent.

A nested sequent can be displayed as

\[A_1, \ldots, A_m, [B_1 : \Gamma_1], \ldots, [B_n : \Gamma_n]\]

where \(n, m \geq 0\), \(A_1, \ldots, A_m, B_1, \ldots, B_n\) are formulas and \(\Gamma_1, \ldots, \Gamma_n\) are nested sequents.

First of all, let us define the depth of a nested sequent:

**Definition 3 (Depth of a nested sequent).** The depth \(d(\Gamma)\) of a nested sequent \(\Gamma\) is defined as follows:

1. if \(\Gamma = A_1, \ldots, A_m\), then \(d(\Gamma) = 0\);
2. if \(\Gamma = [A : \Delta]\), then \(d(\Gamma) = 1 + d(\Delta)\);
3. if \(\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_n\), then \(d(\Gamma) = \max(d(\Gamma_1), d(\Gamma_2), \ldots, d(\Gamma_n))\).

A nested sequent can be directly interpreted as a formula, just replace “,” by \(\lor\) and “:” by \(\Rightarrow\). More explicitly, the interpretation of a nested sequent

\[A_1, \ldots, A_m, [B_1 : \Gamma_1], \ldots, [B_n : \Gamma_n]\]

is inductively defined by the formula

\[F(\Gamma) = A_1 \lor \ldots \lor A_m \lor (B_1 \Rightarrow F(\Gamma_1)) \lor \ldots \lor (B_n \Rightarrow F(\Gamma_n)).\]

For example, the nested sequent

\[A, B, [A : C], [B : E, F], [A : D]\]

denotes the formula

\[A \lor B \lor (A \Rightarrow C \lor (B \Rightarrow (E \lor F))) \lor (A \Rightarrow D).\]

The specificity of nested sequent calculi is to allow inferences that apply within formulas. In order to introduce the rules of the calculus, we need the notion of context. Intuitively a context denotes a “hole”, a unique empty position, within a sequent that can be filled by a formula/sequent. We use the symbol \(\langle \rangle\) to denote the empty context. A context is defined inductively as follows:

**Definition 4.** We define a context:
We conclude as follows:

- \( \Gamma(\ ) = \Delta, (\ ) \) is a context with depth \( d(\Gamma(\ )) = 0; \)
- if \( \Sigma(\ ) \) is a context, \( \Gamma(\ ) = \Delta; [A : \Sigma(\ )] \) is a context with depth \( d(\Gamma(\ )) = 1 + d(\Sigma(\ )); \)

Finally we define the result of filling “the hole” of a context by a sequent:

**Definition 5.** Let \( \Gamma(\ ) \) be a context and \( \Delta \) be a sequent, then the sequent obtained by filling the context by \( \Delta \), denoted by \( \Gamma(\Delta) \) is defined as follows:

- if \( \Gamma(\ ) = \Lambda, (\ ) \), then \( \Gamma(\Delta) = \Lambda, \Delta; \)
- if \( \Gamma(\ ) = \Lambda, [A : \Sigma(\ )] \), then \( \Gamma(\Delta) = \Lambda, [A : \Sigma(\Delta)]; \)

The calculi \( NS \) are shown in Figure 1. As usual, we say that a nested sequent \( \Gamma \) is derivable in \( NS \) if it admits a derivation. A derivation is a tree whose nodes are nested sequents. A branch is a sequence of nodes \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \). Each node \( \Gamma_i \) is obtained from its immediate successor \( \Gamma_{i-1} \) by applying backward a rule of \( NS \), having \( \Gamma_{i-1} \) as the conclusion and \( \Gamma_i \) as one of its premises. A branch is closed if one of its nodes is an instance of axioms \( (AX) \), \( (AX_\top) \), \( (AX_\bot) \), otherwise it is open. We say that a tree is closed if all its branches are closed. A nested sequent \( \Gamma \) has a derivation in \( NS \) if there is a closed tree having \( \Gamma \) as a root. As an example, Figure 2 shows a derivation of (an instance of) the axiom ID.

Before analyzing in detail the calculi \( NS \), let us define the complexity of a formula \( F \) of \( L \), that will be used in the proof of several properties of the calculi themselves.

**Definition 6 (Complexity of a formula).** Given a formula \( F \in L \), we define the complexity \( cp(F) \) as follows:

- \( cp(P) = 1 \), if \( P \in ATM \)
- \( cp(\top) = 1 \)
- \( cp(\bot) = 1 \)
- \( cp(\neg A) = 1 + cp(A) \)
- \( cp(A \otimes B) = 1 + cp(A) + cp(B), where \otimes \in \{\land, \lor, \rightarrow, \Rightarrow\}. \)

The following lemma shows that axioms can be generalized to any formula \( F \):

**Lemma 1.** Given any formula \( F \), the sequent \( \Gamma(F, \neg F) \) is derivable in \( NS \).

**Proof.** By induction on the complexity of \( F \). For the base case, we distinguish three subcases: \( F \) is either (i) \( P \in ATM \) or (ii) \( \top \) or (iii) \( \bot \). Obviously, in all cases \( \Gamma(F, \neg F) \) is derivable: in case (i), \( \Gamma(F, \neg F) \) is an instance of \( (AX) \), in case (ii) it is an instance of \( (AX_\top) \), in case (iii) it is an instance of \( (AX_\bot) \).

For the inductive step, we apply the inductive hypothesis on the subformula(s) of \( F \) and then we conclude by applying the rule(s) manipulating the principal connective in \( F \). As an example, consider \( F = A \Rightarrow B \). By inductive hypothesis, \( \Gamma(A, \neg A) \) and \( \Gamma(B, \neg B) \) are derivable, then so are (1) \( \Gamma(\neg(A \Rightarrow B), [A : B, \neg B]) \) and (2) \( A, \neg A \). We conclude as follows:

\[
\begin{align*}
(1) \quad & \Gamma(\neg(A \Rightarrow B), [A : B, \neg B]) \\
& \quad \text{(2) } A, \neg A \\
& \quad \text{(2) } A, \neg A \\
& \quad \Rightarrow \text{ } (\Rightarrow) \\
\quad \text{ (2) } A, \neg A \\
\text{ (2) } A, \neg A \\
& \quad \text{ (2) } A, \neg A \\
& \quad \Rightarrow \text{ } (\Rightarrow) \\
\text{ (2) } A, \neg A \\
\text{ (2) } A, \neg A \\
\end{align*}
\]

\( \square \)
Fig. 1. The nested sequent calculi $\mathcal{NS}$.

Fig. 2. A derivation of the axiom ID.
In [27] the authors propose optimal sequent calculi for CK and its extensions by any combination of ID, MP and CEM. It is not difficult to see that the rules $CK_g$, $CKID_g$, $CKCEM_g$, $CKCEMID_g$ of their calculi are derivable in our calculi.

3.1 Basic structural properties of $\mathcal{NS}$

First of all, we show that weakening and contraction are height-preserving admissible in the calculi $\mathcal{NS}$. Furthermore, we show that all the rules of the calculi, with the exceptions of $(\Rightarrow\neg)$ and (CEM), are height-preserving invertible. However, for $(\Rightarrow\neg)$ and (CEM), we can prove a weaker version of invertibility. As usual, we define the height of a derivation as the height of the tree corresponding to the derivation itself.

**Lemma 2 (Admissibility of weakening).** Weakening is height-preserving admissible in $\mathcal{NS}$: if $\Gamma(\Delta)$ (resp. $\Gamma([A : \Delta])$) is derivable in $\mathcal{NS}$ with a derivation of height $h$, then also $\Gamma(\Delta, F)$ (resp. $\Gamma([A : \Delta, F])$) is derivable in $\mathcal{NS}$ with a proof of height $h' \leq h$, where $F$ is either a formula or a nested sequent $[B : \Sigma]$.

**Proof.** We proceed by induction on the height $h$ of the derivation of $\Gamma(\Delta)$. The base case corresponds to a derivation whose height is 0, namely $\Gamma(\Delta)$ is either an instance of $(AX)$ or an instance of $(AX\bot)$ or an instance of $(AX\perp)$, i.e. we are considering a sequent of the form either $\Gamma(\Delta', P, \neg P)$ or $\Gamma(\Delta', \top)$ or $\Gamma(\Delta', \bot)$. We can immediately conclude that also $\Gamma(\Delta', P, \neg P, F)$ or $\Gamma(\Delta', \top, F)$ or $\Gamma(\Delta', \bot, F)$ are instances of axioms, and we are done. The same in case we are considering axioms of the form $\Gamma([A : \Delta, P, \neg P])$, $\Gamma([A : \Delta, \top])$, and $\Gamma([A : \Delta, \bot])$. For the inductive step, we have to consider the last rule applied to derive (looking forward) $\Gamma(\Delta)$. We only show the most interesting case of $(\Rightarrow\neg)$, in which the proof is concluded as follows:

$$\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \quad A, \neg A' \quad A', \neg A \quad (\Rightarrow\neg)$$

We can apply the inductive hypothesis to the leftmost premise, obtaining a derivation (of no greater height) of $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B, F])$, from which we conclude $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, F])$ by applying (forward) the rule $(\Rightarrow\neg)$ with $A, \neg A'$ and $A', \neg A$. We can also apply the inductive hypothesis to obtain a derivation of $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B, F])$: again by applying $(\Rightarrow\neg)$ we have a derivation of $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, F])$.

**Lemma 3 (Invertibility).** All the rules of $\mathcal{NS}$, with the exceptions of $(\Rightarrow\neg)$ and (CEM), are height-preserving invertible: if $\Gamma$ has a derivation of height $h$ and it is an instance of a conclusion of a rule (R), then also $\Gamma_i$, $i = 1, 2$, are derivable in $\mathcal{NS}$ with derivations of heights $h_i \leq h$, where $\Gamma_i$ are instances of the premises of (R).

**Proof.** Let us first consider the rule (ID). In this case, we can immediately conclude because the premise $\Gamma([A : \Delta, \neg A])$ is obtained by weakening, which is height-preserving admissible (Lemma 2), from $\Gamma([A : \Delta])$. We proceed in the same way for the rule (MP): indeed, also in this case, both the premises $\Gamma(\neg(A \Rightarrow B), A)$ and $\Gamma(\neg(A \Rightarrow B), \neg B)$ are obtained by weakening from the conclusion $\Gamma(\neg(A \Rightarrow B))$. For the other

...
rules, we proceed by induction on the height of the derivation of $\Gamma$. We only show the most interesting case of $(\Rightarrow^+)$, the other cases are similar and left to the reader. For the base case, consider $(\ast)$ $\Gamma(\Rightarrow B, \Delta)$ where either (i) $P \in \Delta$ and $\neg P \in \Delta$, i.e. $(\ast)$ is an instance of $(AX)$, or (ii) $\top \in \Delta$, i.e. $(\ast)$ is an instance of $(AX\top)$, or (iii) $\neg \bot \in \Delta$, i.e. $(\ast)$ is an instance of $(AX\bot)$; we immediately conclude that also $\Gamma([A : B], \Delta)$ is an instance of either $(AX)$ in case (i) or $(AX\top)$ in case (ii) or $(AX\bot)$ in case (iii).

For the inductive step, we consider each rule ending (looking forward) the derivation of $\Gamma(A \Rightarrow B)$. If the derivation is ended by an application of $(\Rightarrow^+)$ to $\Gamma([A : B])$, we are done. Otherwise, we apply the inductive hypothesis to the premise(s) and then we conclude by applying the same rule. As an example, consider a derivation ended by an application of $(\Rightarrow^-)$ as follows:

\[
\frac{(1) \quad \Gamma(A \Rightarrow B, \neg(C \Rightarrow D), [C' : \Delta, \neg D])}{\Gamma(A \Rightarrow B, \neg(C \Rightarrow D), [C' : \Delta])}
\]

We apply the inductive hypothesis to (1), obtaining a derivation of no greater height of $(\gamma)$ $\Gamma([A : B], \neg(C \Rightarrow D), [C' : \Delta, \neg D])$. By an application of $(\Rightarrow^-)$ to (1'), (2), and (3) we obtain a derivation of $\Gamma([A : B], \neg(C \Rightarrow D), [C' : \Delta])$, and we are done. The other cases are similar and therefore omitted.

We can prove a “weak” version of invertibility also for the rules $(\Rightarrow^-)$ and $(CEM)$. Roughly speaking, if $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$, which is an instance of the conclusion of $(\Rightarrow^-)$, is derivable, then also the sequent $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$, namely the left-most premise in the rule $(\Rightarrow^-)$, is derivable too. Similarly for $(CEM)$, where if $\Gamma([A : \Delta], [B : \Sigma])$ is derivable, then so is the left-most premise in the rule $(CEM)$, namely $\Gamma([A : \Delta, \Sigma], [B : \Sigma])$.

**Lemma 4.** If $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ has a derivation of height $h$ in $\mathcal{N}$S, then also $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$ has a derivation of height $h' \leq h$. If $\Gamma([A : \Delta], [B : \Sigma])$ has a derivation of height $h$ in $\mathcal{N}$S, then also $\Gamma([A : \Delta, \Sigma], [B : \Sigma])$ has a derivation of height $h' \leq h$.

**Proof.** The property immediately follows since $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$ (resp. $\Gamma([A : \Delta, \Sigma], [B : \Sigma])$) is obtained by weakening from $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ (resp. $\Gamma([A : \Delta], [B : \Sigma])$), and weakening is height-preserving admissible (Lemma 2).

It is worth noticing that the height preserving invertibility also preserves the number of applications of the rules in a proof, that is to say: if $\Gamma_1$ is derivable since it is the premise of a backward application of an invertible rule to $\Gamma_2$, then it has a derivation containing the same rule applications of the derivation of $\Gamma_2$. For instance, if $\Gamma([A : \Delta])$ is derivable with a derivation $\Pi$, then $\Gamma([A : \Delta, \neg A])$ is derivable since $(ID)$ is invertible; moreover, there exists a derivation of $\Gamma([A : \Delta, \neg A])$ containing the same rules of $\Pi$, obtained by adding $\neg A$ in each $[A : \Delta]$ in the sequents of $\Pi$ from which $\Gamma([A : \Delta])$ descends. This fact will be systematically used throughout the paper, in the sense that we will assume that every proof transformation due to the invertibility preserves the number of rules applications in the initial proof.

Since the rules are invertible, it follows that contraction is admissible, that is to say:
Lemma 5 (Admissibility of contraction). Contraction is height-preserving admissible in NS: if \( \Gamma(F, F) \) has a derivation of height \( h \), then also \( \Gamma(F) \) has a derivation of height \( h' \leq h \), where \( F \) is either a formula or a nested sequent \([A : \Sigma]\). Moreover, the derivation of the contracted sequent does not add any rule application to the initial derivation\(^3\).

Proof. By induction on the height of the derivation of \( \Gamma(F, F) \). The base cases are as follows:

- \( \Gamma(F, F) \) is an instance of \((AX)\), i.e. either (i) \( \Gamma(F, F) = \Gamma'(F, F, P, \neg P) \), or (ii) \( \Gamma(F, F) = \Gamma''(F, F, \neg F) \). In case (i), obviously also \( \Gamma''(F, P, \neg P) \) is an instance of \((AX)\) and we are done. In case (ii), we conclude similarly by observing that also \( \Gamma''(F, \neg F) \) is an instance of \((AX)\);

- \( \Gamma(F, F) \) is an instance of \((AX\_\neg)\), that is to say either \( F = \top \) or \( \Gamma(F, F) = \Gamma''(F, F, \top) \). In both cases, obviously also \( \Gamma(F) \) is an instance of \((AX\_\neg)\) and we are done;

- \( \Gamma(F, F) \) is an instance of \((AX\_\top)\), that is to say either \( F = \bot \) or \( \Gamma(F, F) = \Gamma''(F, F, \bot) \). Again, in both cases also \( \Gamma(F) \) is an instance of \((AX\_\top)\).

For the inductive step, we distinguish two cases:

- \( F \) is not principal in the application of the rule ending (looking forward) the derivation of \( \Gamma(F, F) \). In all these cases, the two instances of \( F \) also belong to the premise(s) of the rule, to which we can apply the inductive hypothesis and then conclude by an application of the same rule. As an example, consider a derivation ended by an application of \((CEM)\) as follows:

\[
\begin{array}{c}
(1) \Gamma(F, F, [A : \Delta, \Sigma], [B : \Sigma]) A, \neg B & B, \neg A \\
\Gamma(F, F, [A : \Delta, [B : \Sigma])
\end{array}
\]

\((CEM)\)

We can apply the inductive hypothesis on (1) to obtain a derivation (of no greater height) of \( \Gamma'(F, [A : \Delta], [B : \Sigma]) \), then we conclude by an application of \((CEM)\) as follows:

\[
\begin{array}{c}
(1') \Gamma(F, [A : \Delta, \Sigma], [B : \Sigma]) A, \neg B & B, \neg A \\
\Gamma(F', [A : \Delta], [B : \Sigma])
\end{array}
\]

\((CEM)\)

- \( F \) is principal in the application of the rule ending (forward) the derivation of \( \Gamma(F, F) \). We further distinguish between two groups of rules:
  - \((\Rightarrow \neg)\), \((ID)\), \((MP)\), \((CEM)\): in these rules, the principal formula is copied into one of the premises, therefore we apply the inductive hypothesis to such premise and conclude by applying the same rule. Let us first consider the case of \((\Rightarrow \neg)\) and a derivation ending as follows:

\[
\begin{array}{c}
(1) \Gamma'(-(A \Rightarrow B), -(A \Rightarrow B), [A' : \Delta, \neg B]) \\
(2) A, \neg A' & (3) A', \neg A \\
\Gamma'(-(A \Rightarrow B), -(A \Rightarrow B), [A' : \Delta])
\end{array}
\]

\((\Rightarrow \neg)\)

\(^3\) In this case we say that contractions are rule-preserving admissible.
We can apply the inductive hypothesis to (1), obtaining a derivation of no greater height than \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \). Together with (2) and (3), we conclude with an application of \( (\Rightarrow \neg) \).

If the derivation ends as follows:

\[
\begin{array}{c}
(1) \quad \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B], [A' : \Delta]) \\
(2) \quad A, \neg A' \\
(3) \quad A', \neg A \\
\hline
\Gamma(\neg(A \Rightarrow B), [A' : \Delta], [A' : \Delta])
\end{array}
\]

we observe that also (1') \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B], [A' : \Delta, \neg B]) \) has a derivation of no greater height than (1) since weakening is admissible (Lemma 2), to which we can apply the inductive hypothesis to obtain a derivation of (1'') \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \). We conclude from (1''), (2) and (3) with an application of \( (\Rightarrow \neg) \).

Concerning \( (ID) \), we have a derivation ending as follows:

\[
\begin{array}{c}
(1) \quad \Gamma([A : \Delta, \neg A], [A : \Delta]) \\
\hline
\Gamma([A : \Delta], [A : \Delta])
\end{array}
\]

By Lemma 2, from (1) we have a derivation of no greater height of (1') \( \Gamma([A : \Delta, \neg A], [A : \Delta, \neg A]) \), to which we apply the inductive hypothesis to obtain a derivation of no greater height of \( \Gamma([A : \Delta, \neg A]) \), from which we conclude by an application of \( (ID) \).

Let us now consider \( (MP) \). The derivation is ended as follows:

\[
\begin{array}{c}
(1) \quad \Gamma(\neg(A \Rightarrow B), \neg(A \Rightarrow B), A) \\
(2) \quad \Gamma(\neg(A \Rightarrow B), \neg(A \Rightarrow B), B) \\
\hline
\Gamma(\neg(A \Rightarrow B), \neg(A \Rightarrow B))
\end{array}
\]

We can apply the inductive hypothesis to (1), obtaining a derivation of (1') \( \Gamma(\neg(A \Rightarrow B), A) \), and to (2), obtaining a derivation of (2') \( \Gamma(\neg(A \Rightarrow B), B) \), then we can conclude by an application of \( (MP) \) to (1') and (2').

For \( (CEM) \) we proceed similarly to what done for \( (\Rightarrow \neg) \). If the derivation ends as follows:

\[
\begin{array}{c}
(1) \quad \Gamma([A : \Delta, \Sigma], [B : \Sigma], [B : \Sigma]) \\
(2) \quad A, \neg B \\
(3) \quad B, \neg A \\
\hline
\Gamma([A : \Delta], [B : \Sigma], [B : \Sigma])
\end{array}
\]

we apply the inductive hypothesis to (1), obtaining a derivation of \( \Gamma([A : \Delta, \Sigma], [B : \Sigma]) \), then we conclude by applying \( (CEM) \) together with (2) and (3). If the derivation ends as follows:

\[
\begin{array}{c}
(1) \quad \Gamma([A : \Delta, \Sigma], [A : \Delta], [B : \Sigma]) \\
(2) \quad A, \neg B \\
(3) \quad B, \neg A \\
\hline
\Gamma([A : \Delta], [A : \Delta], [B : \Sigma])
\end{array}
\]

we first observe that also (1') \( \Gamma([A : \Delta, \Sigma], [A : \Delta, \Sigma], [B : \Sigma]) \) is derivable (with a derivation of no greater height than (1)) by Lemma 2. We apply the inductive hypothesis on (1') to obtain a derivation of (1'') \( \Gamma([A : \Delta, \Sigma], [B : \Sigma]) \). We conclude by an application of \( (CEM) \) to (1''), (2) and (3).
To improve readability, we slightly abuse the notation identifying a sequent \(\Gamma\): in these rules, the principal formula is not copied into the premise(s), therefore we cannot apply the inductive hypothesis. However, since all these rules are height-preserving invertible (Lemma 3), we apply the inductive hypothesis to the premise of a (further) application of the same rule to the premise, as in the example of \((\Rightarrow^+)\) when the derivation ends as follows (the other cases are similar and left to the reader):

\[
\begin{align*}
(1)\quad & \Gamma([A : B], A \Rightarrow B) \\
& \Gamma(A \Rightarrow B, A \Rightarrow B) \quad (\Rightarrow^+)
\end{align*}
\]

By Lemma 3, we have a derivation of no greater height than (1) of \(\Gamma([A : B], [A : B])\), to which we can apply the inductive hypothesis to obtain a derivation of \(\Gamma([A : B])\), from which we conclude by an application of \((\Rightarrow^+)\).

### 3.2 Soundness of the Calculi \(\mathcal{N}, S\)

To improve readability, we slightly abuse the notation identifying a sequent \(\Gamma\) with its interpreting formula \(\mathcal{F}(\Gamma)\), thus we shall write \(A \Rightarrow \Delta, \Gamma \land \Delta\), etc. instead of \(A \Rightarrow \mathcal{F}(\Gamma), \mathcal{F}(\Gamma) \land \mathcal{F}(\Delta)\). First of all we prove that nested inference is sound. The proof is inspired to the one of Lemma 2.8 in [7].

**Lemma 6.** Let \(\Gamma(\ )\) be any context. If the formula \(A_1 \land \ldots \land A_n \Rightarrow B\), with \(n \geq 0\), is \((\text{CK} \{+X\})\)-valid, then also \(\Gamma(A_1) \land \ldots \land \Gamma(A_n) \Rightarrow \Gamma(B)\) is \((\text{CK} \{+X\})\) valid.

**Proof.** By induction on the depth of a context \(\Gamma(\ )\). Let \(d(\Gamma(\ )) = 0\), then \(\Gamma = A(\ )\).

Since \(A_1 \land \ldots \land A_n \Rightarrow B\) is valid, by propositional reasoning, we have that also \((A \lor A_1) \land \ldots \land (A \lor A_n) \Rightarrow (A \lor B)\) is valid, that is \(\Gamma(A_1) \land \ldots \land \Gamma(A_n) \Rightarrow \Gamma(B)\) is valid. Let \(d(\Gamma(\ )) > 0\), then \(\Gamma(\ ) = \Delta, [C : \Sigma(\ )]\). By inductive hypothesis, we have that \(\Sigma(A_1) \land \ldots \land \Sigma(A_n) \Rightarrow \Sigma(B)\) is valid. By (RCK), we obtain that also \((C \Rightarrow \Sigma(A_1)) \land \ldots \land (C \Rightarrow \Sigma(A_n)) \Rightarrow (C \Rightarrow \Sigma(B))\) is valid. Then, we get that \(\Gamma(A_1) \land \ldots \land \Gamma(A_n) \Rightarrow \Gamma(B)\) is valid.

**Theorem 2 (Soundness of \(\mathcal{N}, S\)).** If \(\Gamma\) is derivable in \(\mathcal{N}, S\), then \(\Gamma\) is valid.

**Proof.** By induction on the height of the derivation of \(\Gamma\). If \(\Gamma\) is an axiom, that is \(\Gamma = \Gamma(P, \neg P)\), then trivially \(P \lor \neg P\) is valid; by Lemma 6 (case \(n = 0\)), we get \(\Gamma(P, \neg P)\) is valid. Similarly for \(\Gamma(\top)\) and \(\Gamma(\bot)\). Otherwise \(\Gamma\) is obtained by a rule (R):

- (R) is a propositional rule, say

\[
\begin{array}{c}
\Gamma_1 \\
\hline
\Delta
\end{array} \quad \Gamma_2 (R)
\]

We first prove that \(\Gamma_1 \land \Gamma_2 \Rightarrow \Delta\) is valid. All rules are easy, since for the empty context they are nothing else than trivial propositional tautologies. We can then use Lemma 6 to propagate them to any context. For instance, let the rule (R) be \((\lor^\land)\). Then \((\neg A \land \neg B) \Rightarrow \neg (A \lor B)\) and, by the previous Lemma 6, we get that \(\Gamma(\neg A) \land \Gamma(\neg B) \Rightarrow \Gamma(\neg (A \lor B))\). Thus if \(\Gamma\) is derived by (R) from \(\Gamma_1, \Gamma_2\), we use the inductive hypothesis that \(\Gamma_1\) and \(\Gamma_2\) are valid and the above fact to conclude.
Completeness is an easy consequence of the admissibility of the following rule \( \text{cut} \):

\[
\begin{array}{c}
\Gamma(F) \\
\Gamma(\neg F)
\end{array} \qquad \text{(cut)}
\]

where \( F \) is a formula. The standard proof of admissibility of cut proceeds by a double induction over the complexity of \( F \) and the sum of the heights of the derivations of the two premises of \( \text{(cut)} \), in the sense that we replace one cut by one or several cuts on formulas of smaller complexity, or on sequents derived by shorter derivations.

However, in \( N^S \) the standard proof does not work in the following case, in which the cut formula \( F \) is a conditional formula \( A \Rightarrow B \):
This will prove that we show the following facts: (obtained from Subproof of (iv) (that is that Sub and such that the sum of the heights of derivation of the premises is \(\sum\)).

**Theorem 3.** Cut in two propositions:

\[
\begin{align*}
(1) & \quad \Gamma([A : B], [A' : \Delta]) \\
(2) & \quad \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \\
(3) & \quad \Gamma(\neg(A \Rightarrow B), [A' : \Delta]) \\
\hline
& \quad \Gamma([A' : \Delta])
\end{align*}
\]

Indeed, even if we apply the inductive hypothesis on the heights of the derivations of the premises to cut (2) and (3), obtaining (modulo weakening, which is admissible by Lemma 2) a derivation of \(\Gamma'([A' : \Delta, \neg B], [A' : \Delta])\), we cannot apply the inductive hypothesis on the complexity of the cut formula to (2') and (1') \(\Gamma'([A : \Delta, B], [A' : \Delta])\) (obtained from (1) again by weakening). Such an application would be needed in order to obtain a derivation of \(\Gamma'([A' : \Delta], [A' : \Delta])\) and then to conclude \(\Gamma'([A' : \Delta])\) since contraction is admissible (Lemma 5).

We first consider the calculi without MP.

**Admissibility of cut for systems without MP** In order to prove the admissibility of cut for \(N\mathcal{S}\), we proceed as follows. First, we show that if \(A, \neg A'\) and \(A', \neg A\) are derivable, then if \(\Gamma([A : \Delta])\) is derivable, then \(\Gamma([A' : \Delta])\), obtained by replacing \([A : \Delta]\) with \([A' : \Delta]\), is also derivable. We prove that cut is admissible by “splitting” the notion of cut in two propositions:

**Theorem 3.** In \(N\mathcal{S}\), the following propositions hold:

- (A) If \(\Gamma(F)\) and \(\Gamma(\neg F)\) are derivable, so is \(\Gamma(\emptyset)\), i.e. (cut) is admissible in \(N\mathcal{S}\);
- (B) if (I) \(\Gamma([A : \Delta])\), (II) \(A, \neg A'\) and (III) \(A', \neg A\) are derivable, then \(\Gamma([A' : \Delta])\) is derivable.

**Proof.** The proof of both is by mutual induction. To make the structure of the induction clear call: \(\text{Cut}(c, h)\) the property (A) for any \(\Gamma\) and any formula \(F\) of complexity \(c\) and such that the sum of the heights of derivation of the premises is \(h\). Similarly call \(\text{Sub}(c)\) the assertion that (B) holds for any \(\Gamma\) and any formula \(A\) of complexity \(c\). Then we show the following facts:

\[
\begin{align*}
(i) & \quad \forall h \text{Cut}(0, h) \\
(ii) & \quad \forall c \text{Cut}(c, 0) \\
(iii) & \quad \forall c < c \text{Sub}(c') \rightarrow (\forall c' < c \, \forall h' \text{Cut}(c', h') \land \forall h < h \, \text{Cut}(c, h') \rightarrow \text{Cut}(c, h)) \\
(iv) & \quad \forall h \, \text{Cut}(c, h) \rightarrow \text{Sub}(c)
\end{align*}
\]

This will prove that \(\forall c \, \forall h \text{Cut}(c, h)\) and \(\forall c \, \text{Sub}(c)\), that is (A) and (B) hold. The proof of (iv) (that is that \(\text{Sub}(c)\) holds) in itself is by induction on the height \(h\) of the derivation of the premise (I) of (B).

- Base for (A): (at least) one of the premises of \(\text{cut}\) is an instance of the axioms. Suppose that the left premise is an instance of \((AX)\). In case it has the form \(\Gamma(P, \neg P, F)\), then also \(\Gamma(P, \neg P)\) is an instance of \((AX)\) and we are done. Otherwise, we have \(\Gamma(F, \neg F)\). In this case, the right premise of \(\text{cut}\) has the form
\( \Gamma(\neg F, \neg F) \), whereas the conclusion is \( \Gamma(\neg F) \): since contraction is admissible (Lemma 5), we conclude that \( \Gamma(\neg F) \) is derivable and we are done. The other cases are easy and left to the reader;

- Base for (B): \( \Gamma(\{A : \Delta\}) \) is an instance of the axioms, that is to say either \( \Delta = A(\neg P) \) or \( \Delta = A(\neg \top) \): we immediately conclude that also \( \Gamma(\{A' : \Delta\}) \) is an instance of the axioms, and we are done;

- Inductive step for (A): we distinguish the following two cases:
  - (case 1) the last step of one of the two premises is obtained by a rule (R) in which \( F \) is not the principal formula. This case is standard, we can permute (R) over the cut, i.e. we cut the premise(s) of (R) and then we apply (R) to the result of cut. As an example, we present the case of \( \Rightarrow^{-} \):

\[
\begin{array}{ccc}
(1) \Gamma(\neg (A \Rightarrow B), \{A' : \Delta, \neg B\}, F) & (2) A, \neg A' & (3) A', \neg A \\
& \Gamma(\neg (A \Rightarrow B), \{A' : \Delta\}, F) & (\Rightarrow^{-}) \\
& \Gamma(\neg (A \Rightarrow B), \{A' : \Delta\}, F) & (\text{cut})
\end{array}
\]

Since weakening is height-preserving admissible (Lemma 2), from (4) we obtain a derivation of no greater height of \( \Gamma(\neg (A \Rightarrow B), \{A' : \Delta, \neg B\}, \neg F) \). We then apply the inductive hypothesis on the height to cut \( (1) \) and \( (4') \), obtaining a derivation of \( \Gamma(\neg (A \Rightarrow B), \{A' : \Delta, \neg B\}, \neg F) \); we conclude by an application of \( \Rightarrow^{-} \) to \( (5), (2), \) and \( (3) \);

- (case 2) \( F \) is the principal formula in the last step of both derivations of the premises of the cut inference. There are seven subcases: \( F \) is introduced (a) by \( \land^{-} \) - \( \land^{+} \), (b) by \( \lor^{-} \) - \( \lor^{+} \), (c) by \( \rightarrow^{-} \) - \( \rightarrow^{+} \), (d) by \( \Rightarrow^{-} \) - \( \Rightarrow^{+} \), (e) by \( \Rightarrow^{-} \) - \( \text{ID} \), (f) by \( \Rightarrow^{-} \) - \( \text{CEM} \), (g) by \( \text{CEM} \) - \( \text{ID} \). The list is exhaustive. Let us consider each case.

For case (a), the derivation is ended as follows:

\[
\begin{array}{ccc}
(1) \Gamma(\neg A, \neg B) & (2') \Gamma(A) & (3) \Gamma(B) \\
& \Gamma(A \land B) & (\text{cut})
\end{array}
\]

First of all, since weakening is admissible (Lemma 2), from (2) we obtain a derivation of \( (2') \) \( \Gamma(A, \neg B) \). We can then apply the inductive hypothesis for Proposition (A) on the complexity of the cut formula to \( (1) \) and \( (2') \) to obtain a derivation of \( \Gamma(\neg B) \). We conclude by applying again the inductive hypothesis for Proposition (A) on the complexity of the cut formula to cut \( (4) \) and \( (3) \).

For case (b), the derivation is as follows:

\[
\begin{array}{ccc}
(1) \Gamma(\neg A) & (2) \Gamma(\neg B) & (3) \Gamma(A, B) \\
& \Gamma(\neg (A \lor B)) & (\lor^{-}) \\
& \Gamma(\neg (A \lor B)) & (\text{cut})
\end{array}
\]
By Lemma 2, from (1) we obtain a derivation of $(1') \Gamma(\neg A, B)$. We proceed similarly to what done for case a). We first apply the inductive hypothesis for (A) on the complexity of the cut formula to cut $(1')$ and (3), obtaining a derivation of $(4) \Gamma(B)$. We conclude by applying again the inductive hypothesis for (A) on the cut formula to cut $(4)$ and $(2)$.

Case c) is similar to the previous ones:

$$
\frac{(1) \Gamma(A) \quad (2) \Gamma(\neg B)(\neg \rightarrow) \quad (3) \Gamma(\neg A, B)(\neg \rightarrow)}{\Gamma(\neg (A \rightarrow B)) \quad \Gamma(A \rightarrow B) \quad \Gamma(\emptyset) (\text{cut})}
$$

From (1), we obtain a derivation of $(1') \Gamma(A, B)$ since weakening is admissible (Lemma 2). We apply the inductive hypothesis for (A) on the complexity of the cut formula to cut $(1')$ and $(3)$, obtaining a derivation of $(4) \Gamma(B)$. We conclude by applying again the inductive hypothesis for (A) on the cut formula to cut $(4)$ and $(2)$.

For case d), the derivation is as follows:

$$
\frac{(1) \Gamma(\neg (A \Rightarrow B), [A' : \Delta, \neg B]) \quad A, \neg A' \quad A', \neg A}{\Gamma(\neg (A \Rightarrow B), [A' : \Delta]) \quad (\Rightarrow) \quad (2) \Gamma([A : B], [A' : \Delta]) \quad (\Rightarrow +) \quad (3) \Gamma(A \Rightarrow B, [A' : \Delta]) \quad \Gamma([A' : \Delta]) (\text{cut})}
$$

First of all, since we have proofs for $A, \neg A'$ and for $A', \neg A$ and $\text{cp}(A) < \text{cp}(A \Rightarrow B)$, we can apply the inductive hypothesis for (B) to (2), obtaining a derivation of $(2'') \Gamma([A' : B], [A' : \Delta])$. By Lemma 2, from (3) we obtain a derivation of at most the same height of $(3') \Gamma(A \Rightarrow B, [A' : \Delta, \neg B])$. We can then conclude as follows: we first apply the inductive hypothesis on the height for (A) to cut (1) and (3'), obtaining a derivation of $(4) \Gamma([A' : \Delta, \neg B])$. By Lemma 2, we have also a derivation of $(4') \Gamma([A' : \Delta, B], [A' : \Delta])$. Again by Lemma 2, from (2') we obtain a derivation of $(2'') \Gamma([A' : \Delta, B], [A' : \Delta])$. We then apply the inductive hypothesis on the complexity of the cut formula to cut $(2''')$ and $(4')$, obtaining a derivation of $\Gamma([A' : \Delta], [A' : \Delta])$, from which we conclude since contraction is admissible (Lemma 5).

For case e), the derivation is as follows:

$$
\frac{(1) \Gamma(\neg (A \Rightarrow B), [A' : \Delta, F, \neg B]) \quad A, \neg A' \quad A', \neg A}{\Gamma(\neg (A \Rightarrow B), [A' : \Delta, F]) \quad (\Rightarrow) \quad (2) \Gamma(\neg (A \Rightarrow B), [A' : \Delta, \neg F, \neg A']) \quad \Gamma(\neg (A \Rightarrow B), [A' : \Delta]) \quad (\text{cut}) \quad (\text{ID})}
$$

Since weakening is height-preserving admissible (Lemma 2), from (1) we obtain a derivation, of at most the same height, of $(1') \Gamma(\neg (A \Rightarrow B), [A' : \Delta, F, \neg A'])$; we conclude as follows, by applying the inductive hypothesis for (A) on the height:

$$
\frac{(1') \Gamma(\neg (A \Rightarrow B), [A' : \Delta, F, \neg A']) \quad (2) \Gamma(\neg (A \Rightarrow B), [A' : \Delta, \neg F, \neg A'])}{\Gamma(\neg (A \Rightarrow B), [A' : \Delta, \neg A']) \quad (\text{cut}) \quad (\text{ID})}
$$
For case (f), the derivation is as follows:

1. \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, F, \neg B], [C : \Sigma]) \)
2. \( A, \neg A' \)
3. \( A', \neg A \)
4. \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, F, \neg B], [C : \Sigma]) \) (\( \Rightarrow \))
5. \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg F], [C : \Sigma]) \) (\( \text{CEM} \))
6. \( (4') \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg F, B], [C : \Sigma]) \)
7. \( (2') A, \neg A' \quad (3') A', \neg A \)

By Lemma 2, from (4) we obtain a derivation of at most the same height for

\[ (4') \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg F, \neg B], [C : \Sigma]) \]

then we can conclude as follows by applying the inductive hypothesis for (A) on the height:

1. \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B], [C : \Sigma]) \) (\( \text{cut} \))
2. \( (2') A, \neg A' \quad (3') A', \neg A \)

For case (g), the derivation is as follows:

1. \( \Gamma([A : \Delta, \neg A], [B : \Sigma]) \) (\( ID \))
2. \( \Gamma([A : \Delta, \neg F, \Sigma]) \) (\( \text{CEM} \))
3. \( \Gamma([A : \Delta, F], [B : \Sigma]) \)

By Lemma 2, from (2) we obtain a derivation of at most the same height of \( (2') \Gamma([A : \Delta, \neg F], [B : \Sigma]) \). We can then apply the inductive hypothesis on the height to cut (1) and (2') to obtain \( \Gamma([A : \Delta, \neg A], [B : \Sigma]) \), from which we conclude by an application of \( ID \).

Inductive step for (B) (that is statement (iv) of the induction): we have to consider all possible rules ending (looking forward) the derivation of \( \Gamma([A : \Delta]) \). We only show the most interesting case, when \( (\Rightarrow \neg) \) is applied by using \( [A : \Delta] \) as principal formula. The derivation ends as follows:

1. \( \Gamma(\neg(C \Rightarrow D), [A : \Delta, \neg D]) \)
2. \( C, \neg A \)
3. \( A, \neg C \)

We can apply the inductive hypothesis to (1) to obtain a derivation of \( (1') \Gamma(\neg(C \Rightarrow D), [A' : \Delta, \neg D]) \). Since weakening is admissible (Lemma 2), from (II) we obtain a derivation of \( (II') \Gamma,C, \neg A', \) from which we obtain a derivation of \( (III') \neg A', \neg A, \neg C. \) Again by weakening, from (2) and (3) we obtain derivations of \( (2') C, \neg A', \neg A' \) and \( (3') A', \neg A, \neg C, \) respectively. We apply the inductive hypothesis of (A) that is that cut holds for the formula \( A' \) of (a given complexity c) and conclude as follows:

1. \( \Gamma(\neg(C \Rightarrow D), [A' : \Delta, \neg D]) \)
2. \( C, \neg A' \)
3. \( (III') A', \neg A, \neg C \quad (3') A', \neg A, \neg C \)

\( \Rightarrow \neg \)

Let us now consider the case of the calculi for systems allowing the axiom MP.
Admissibility of cut for systems CK+MP and CK+ID+MP The proof is slightly different from the one of Theorem 3 above, that is why we prove it in a separate theorem. To make cut elimination work, we need to show the admissibility of the following rule:

$$\frac{\Gamma(A) \Gamma([A : \Delta])}{\Gamma(\Delta)}.$$  

This rule corresponds to a generalization of the MP axiom and it is needed to handle the elimination of a cut in which one of the premises is a negative conditional $\neg(A \Rightarrow B)$ introduced by an instance of the MP rule.

**Theorem 4.** In systems CK+MP and CK+ID+MP the following propositions hold:

(a) If $\Gamma(A)$ and $\Gamma(\neg A)$ are derivable, then so is $\Gamma(\emptyset)$.

(b) If (I) $\Gamma([A : \Delta])$, (II) $\neg A$, $A'$, and (III) $\neg A'$, $A$ are derivable, then so is $\Gamma([A' : \Delta])$.

(c) If $\Gamma([A : \Delta])$ and $\Gamma(A)$ is derivable, then so is $\Gamma(\Delta)$.

**Proof.** The proof of the three statements is by mutual induction. To make the structure of the induction clear call:

- $Cut(c, h)$ the property (a) for any $\Gamma$ and any formula $A$ of complexity $c$ and such that the sum of the heights of derivation of the premises is $h$;
- $Sub(c)$ the assertion that (b) holds for any $\Gamma$ and any formula $A$ of complexity $c$ and, finally,
- $MP(c)$ the assertion that (c) holds for any $\Gamma, \Delta$ and any formula $A$ of complexity $c$.

Then we show that following facts:

(i) $\forall h Cut(0, h)$

(ii) $\forall c Cut(c, 0)$

(iii) $[\forall c' < c Sub(c')] \land \forall c' < c MP(c') \land \forall c' < c \forall h' < h Cut(c', h') \rightarrow Cut(c, h)$

(iv) $\forall h Cut(c, h) \rightarrow Sub(c)$

(v) $\forall h Cut(c, h) \rightarrow MP(c)$.

This will prove that $\forall c \forall h Cut(c, h), \forall c Sub(c)$ and $\forall c MP(c)$ that is (a), (b), (c) hold. The proof of (i), (ii) and (iii) is the same as the one of Theorem 3 except in the case the principal formula $\neg (C \Rightarrow D)$ is introduced by the (MP) rule, namely we have the following:

$$\frac{(1) \Gamma(\neg(C \Rightarrow D), C) (2) \Gamma(\neg(C \Rightarrow D), \neg D) (3) \Gamma([C : D]) (\Rightarrow R) (4) \Gamma(C \Rightarrow D) (cut)}{\Gamma(\emptyset)}.$$

In this case we proceed as follows: first of all, since weakening is height-preserving admissibile (Lemma 2), from (4) we obtain a derivation of no greater height of $(4') \Gamma(C \Rightarrow D, C)$ and of $(4'') \Gamma((C \Rightarrow D, \neg D)$; then, we cut $(4')$ with $(1)$ and $(4'')$ with $(2)$ (inductive hypothesis for (a) on the height), obtaining:

$$\frac{(5) \Gamma(C) (6) \Gamma(\neg D)}{\Gamma(\emptyset)}.$$
Since $cp(C) = c' < cp(C \Rightarrow D) = c$ we can use the inductive hypothesis on (c), namely $MP(c')$, so that from (5) and (3) we get

$\Gamma(D)$

(7) $\Gamma(D)$

Since $cp(D) < cp(C \Rightarrow D) = c$ we can use the inductive hypothesis on (a), that is to say we cut (6) and (7) obtaining $\Gamma(\emptyset)$.

Concerning (b), (statement (iv)) the proof is the same as Theorem 3, except that there is an additional case where $\Gamma[A : \Delta]$ is derived by the $(MP)$ rule; we have two subcases: first $\Gamma'[A : \Delta] = \Gamma'[(C \Rightarrow D), [A : \Delta]]$ obtained by:

$\Gamma'[(C \Rightarrow D), C, [A : \Delta]] \quad \Gamma'[(C \Rightarrow D), \neg D, [A : \Delta]]$

$(MP)$

We can obviously apply the inductive hypothesis to the two premises obtaining $\Gamma'[(C \Rightarrow D), C, [A' : \Delta]]$ and $\Gamma'[(C \Rightarrow D), \neg D, [A' : \Delta]]$ and then we conclude by an application of $(MP)$. The second subcase is when $\Gamma'[A : \Delta] = \Gamma'[A : \neg(C \Rightarrow D), \Delta']$ and is obtained by $(MP)$ as follows:

$\Gamma'[A : \neg(C \Rightarrow D), C, \Delta'] \quad \Gamma'[A : \neg(C \Rightarrow D), \neg D, \Delta']$

$(MP)$

and we proceed in the same way.

Concerning (c), that is statement (v), it is proved by induction on the height $h$ of the derivation of $\Gamma'[A : \Delta]$. For the base case $h = 0$, we immediatly conclude that, if $\Gamma'[A : \Delta]$ is an axiom, so is $\Gamma(\Delta)$. For the inductive step, we distinguish the following subcases:

- $h > 0$: all cases, except $(\Rightarrow \neg)$ and $(ID)$, are straightforward, for instance if $\Gamma'[A : \Delta]$ is obtained by:

$\Gamma_1[A : \Delta] \quad \Gamma_2[A : \Delta]$

$(R)$

where $(R)$ is any rule different from $(\Rightarrow \neg)$ and $(ID)$, then we just apply the inductive hypothesis to the premises obtaining $\Gamma_1(\Delta)$ and $\Gamma_2(\Delta)$ and then we apply $(R)$, we proceed similarly if $\Gamma'[A : \Delta]$ is derived from say $\Gamma'[A : \Delta_1]$ and $\Gamma'[A : \Delta_2]$.

- $h > 0$ $(\Rightarrow \neg)$, if $[A : \Delta]$ is not principal in the application of $(\Rightarrow \neg)$, we proceed as above; the non-trivial case is when $\Gamma'[A : \Delta] = \Gamma'[(C \Rightarrow D), [A : \Delta]]$ and it is obtained as follows:

$\Gamma'[\neg(C \Rightarrow D), [A : \Delta], \neg D] \quad \Gamma'[(C \Rightarrow D), [A : \Delta]]$

$(\Rightarrow \neg)$

In this case, from (2) by weakening we obtain:

$\Gamma'[(C \Rightarrow D), \neg A, C]$

$\Gamma'[(C \Rightarrow D), [A : \Delta]]$
We have that $\Gamma(A)$ is derivable (second condition in proposition (c)), that is to say $(\ast)\ \Gamma' (\lnot (C \Rightarrow D), A)$ is derivable. By Lemma 2 (weakening), also $(4)\ \Gamma' (\lnot (C \Rightarrow D), A, C)$ is derivable. We apply the inductive hypothesis on the complexity of the cut formula to cut $(4)$ and $(3)$ to obtain:

$$(4')\ \Gamma' (\lnot (C \Rightarrow D), C),$$

and by weakening:

$$(4'')\ \Gamma' (\lnot (C \Rightarrow D), \Delta, C),$$

Now we apply the inductive hypothesis of (v) to (1) using the premise $(\ast)\ \Gamma' (\lnot (C \Rightarrow D), A)$ and we get

$$(5)\ \Gamma (\lnot (C \Rightarrow D), \Delta, \lnot D)$$

Now we apply the $(MP)$ rule to $(4'')$ and $(5)$ and we finally get:

$\Gamma (\lnot (C \Rightarrow D), \Delta)$

- $h > 0\ (ID)$, again the case is trivial if the inference step does not involve $[A : \Delta]$, the significant case is the following:

$$(1)\ \Gamma ([A : \Delta, \lnot A])$$

We apply the inductive hypothesis of (v) to (1) with the premise $\Gamma (A)$ and we get:

$$(2)\ \Gamma (\Delta, \lnot A)$$

From $\Gamma (A)$, by weakening we have a derivation of $(3)\ \Gamma (\Delta, A)$, and we apply the inductive hypothesis on the complexity of the cut formula to cut $(2)$ and $(3)$ and we finally get $\Gamma (\Delta)$. $\square$

**Completeness of $\mathcal{NS}$**

We can now prove the completeness of the calculi $\mathcal{NS}$:

**Theorem 5 (Completeness of $\mathcal{NS}$).** If $\Gamma$ is valid, then it is derivable in $\mathcal{NS}$.

**Proof.** We prove that the axioms are derivable and that the set of derivable formulas is closed under (Modus Ponens), (RCEA), and (RCK). A derivation of an instance of ID has been shown in Figure 2. A derivation of an instance of MP is as follows:

$$
\frac{\lnot (A \Rightarrow B), \lnot A, B, A}{\lnot (A \Rightarrow B), \lnot A, B, \lnot B} (AX)
\frac{\lnot (A \Rightarrow B), \lnot A, B}{\lnot (A \Rightarrow B), A \Rightarrow B} (\neg^+) \quad \frac{\lnot (A \Rightarrow B), A \Rightarrow B}{(A \Rightarrow B) \Rightarrow (A \Rightarrow B)} (\neg^+)
$$

$\frac{\lnot (A \Rightarrow B), \lnot A, B}{\lnot (A \Rightarrow B), A \Rightarrow B} (\neg^+) \quad \frac{\lnot (A \Rightarrow B), A \Rightarrow B}{(A \Rightarrow B) \Rightarrow (A \Rightarrow B)} (\neg^+)$
Here is a derivation of an instance of CEM:

\[
\frac{[A \vdash B, \neg B]}{(AX)}\frac{A, \neg A}{(AX)}\frac{\neg A, A}{(AX)}\frac{[A \vdash B, [A \vdash \neg B]}{(CEM)}\frac{[A \vdash B, A \Rightarrow \neg B]}{(\Rightarrow +)}\frac{A \Rightarrow B, A \Rightarrow \neg B}{(\Rightarrow +)}\frac{(A \Rightarrow B) \lor (A \Rightarrow \neg B)}{(\lor +)}
\]

For (Modus Ponens), we have to show that, if \((1) A \rightarrow B\) and \((2) A\) are derivable, then also \(B\) is derivable. Since weakening is admissible (Lemma 2), we have also derivations for \((1') A \rightarrow B, B, \neg A\) and \((2') A, B\). Furthermore, observe that \((3) A, B, \neg A\) and \((4) \neg B, B, \neg A\) are both instances of \((AX)\). Since cut is admissible (Proposition A in Theorem 3), the following derivation shows that \(B\) is derivable:

\[
\frac{(3) A, B, \neg A}{(AX)}\frac{(4) \neg B, B, \neg A}{(⇒ −)}\frac{(1') A \rightarrow B, B, \neg A}{(⇒ −)}\frac{(2') A, B}{(⇒ −)}\frac{B, \neg A}{(cut)}\frac{B}{(cut)}
\]

For (RCEA), we have to show that if \(A \leftrightarrow B\) is derivable, then also \((A \Rightarrow C) \leftrightarrow (B \Rightarrow C)\) is derivable. As usual, \(A \leftrightarrow B\) is an abbreviation for \((A \rightarrow B) \land (B \rightarrow A)\). Since \(A \leftrightarrow B\) is derivable, and since \((\land +)\) and \((⇒ +)\) are invertible (Lemma 3), we have a derivation for \(A \rightarrow B\), then for \((1) \neg A, B\), and for \(B \rightarrow A\), then for \((2) A, \neg B\). We derive \((A \Rightarrow C) \rightarrow (B \Rightarrow C)\) (the other half is symmetric) as follows:

\[
\frac{(AX)}{\neg (A \Rightarrow C), [B : C, \neg C]}\frac{(1) A, B}{(⇒ −)}\frac{(2) A, \neg B}{(⇒ −)}\frac{\neg (A \Rightarrow C), [B : C]}{(⇒ +)}\frac{(⇒ +)}{\neg (A \Rightarrow C), B \Rightarrow C} \frac{(⇒ −)}{(A \Rightarrow C) \rightarrow (B \Rightarrow C)}
\]

For (RCK), suppose that we have a derivation in \(\mathcal{N}S\) of \((A_1 \land \ldots \land A_n) \rightarrow B\). Since \((⇒ +)\) is invertible (Lemma 3), we have also a derivation of \(B, \neg (A_1 \land \ldots \land A_n)\). Since \((\land −)\) is also invertible, then we have a derivation of \(B, \neg A_1, \ldots, \neg A_n\) and, by weakening (Lemma 2), of \((1) \neg (C \Rightarrow A_1), \ldots, \neg (C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2, \ldots, \neg A_n]\).
from which we conclude as follows:

(1) \( \neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2, \ldots, \neg A_n] \)

\[
\begin{align*}
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2] & \quad \Rightarrow^+ \quad \neg C, C \\
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1] & \quad \Rightarrow^- \quad C, \neg C \\
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B] & \quad \Rightarrow^- \quad -\neg C, C \\
\neg(C \Rightarrow A_1 \land \ldots \land C \Rightarrow A_n), C \Rightarrow B & \quad \Rightarrow^+ \quad \neg C, C \\
(C \Rightarrow A_1 \land \ldots \land C \Rightarrow A_n) \Rightarrow (C \Rightarrow B) &
\end{align*}
\]

3.4 **Termination and complexity of \( \mathcal{N}S \)**

The rules \( \Rightarrow^- \), \( MP \), \( CEM \), and \( ID \) may be applied infinitely often. In order to obtain a terminating calculus, we have to put some restrictions on the application of these rules. We first consider the case of the calculi \( \mathcal{N}S \) for systems without CEM, for which standard restrictions on the application of rules allow us to turn the calculi into terminating ones. For the systems allowing CEM, we need a more sophisticated machinery.

**Termination and complexity of \( \mathcal{N}CK, \mathcal{N}CK+ID, \mathcal{N} CK + MP, \) and \( \mathcal{N}CK+ID+MP \)**

We put the following restrictions:

- apply \( \Rightarrow^- \) to \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta]) \) only if \( \Rightarrow^- \) has not been applied to the formula \( \neg(A \Rightarrow B) \) with the context \( [A' : \Delta] \) in the current branch;
- apply \( ID \) to \( \Gamma([A : \Delta]) \) only if \( ID \) has not been applied to the context \( [A : \Delta] \) in the current branch;
- apply \( MP \) to \( \Gamma(\neg(A \Rightarrow B)) \) only if \( MP \) has not been applied to the formula \( \neg(A \Rightarrow B) \) in the current branch.

These restrictions impose that \( \Rightarrow^- \) can be applied only once to each formula \( \neg(A \Rightarrow B) \) with a context \( [A' : \Delta] \) in each branch, that \( ID \) can be applied only once to each context \( [A : \Delta] \) in each branch, and that \( MP \) can be applied only once to each formula \( \neg(A \Rightarrow B) \) in each branch.

In order to obtain a terminating version of \( \mathcal{N}S \), we need to implement the above restrictions. These restrictions can be effectively implemented by adding a suitable bookkeeping mechanism. To give an idea, sequents are equipped with three lists, let us call them \textsc{Cond}, \textsc{Id} and \textsc{Mp}, containing formulas and/or contexts already used in the current branch for an application of the three mentioned rules. For instance, the list \textsc{Cond} contains pairs \( (\neg(A \Rightarrow B), [A' : \Delta]) \) to keep track that, in the current branch, the rule \( \Rightarrow^- \) has been already applied to \( \neg(A \Rightarrow B) \) by using \( [A' : \Delta] \), thus introducing the sequent \( \Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B]) \). The rule \( \Rightarrow^- \) can be then applied
to a sequent $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ only if the pair $\langle \neg(A \Rightarrow B), [A' : \Delta] \rangle$ does not belong to COND. When the rule $(\Rightarrow^-)$ is applied to some $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$, then the pair $\langle \neg(A \Rightarrow B), [A' : \Delta] \rangle$ is added to the list COND. The lists ID and MP, containing contexts $[A : \Delta]$ and negated conditional $\neg(A \Rightarrow B)$, respectively are used in a similar way to implement the restriction for (ID) and (MP).

**Theorem 6.** The calculi $N S$ with the termination restrictions are sound and complete for their respective logics.

**Proof.** We show that it is useless to apply the rules $(\Rightarrow^-)$, (ID) and (MP) without the restrictions.

- $(\Rightarrow^-)$: suppose it is applied twice on $\Gamma(\neg(A \Rightarrow B), [A' : \Delta])$ in a branch. Since $(\Rightarrow^-)$ is “weakly” invertible (Lemma 4), we can assume, without loss of generality, that the two applications of $(\Rightarrow^-)$ are consecutive, starting from $\Gamma'(\neg(A \Rightarrow B), [A' : \Delta, \neg B, \neg B])$. By Lemma 5 (contraction), we have a derivation of $\Gamma(\neg(A \Rightarrow B), [A' : \Delta, \neg B])$, and we can conclude with a (single) application of $(\Rightarrow^-)$. Remember that contraction is rule-preserving admissible, therefore the obtained derivation does not add any application of $(\Rightarrow^-)$;

- (ID): similarly to the case of $(\Rightarrow^-)$ above, suppose that the rule (ID) is applied twice on $\Gamma([A : \Delta])$ in a branch. Since (ID) is invertible (Lemma 3), we can assume, without loss of generality, that the two applications of (ID) are consecutive, starting from $\Gamma([A : \Delta, \neg A, \neg A])$. As we have done for $(\Rightarrow^-)$, we conclude that the second application is useless, since we obtain a derivation of $\Gamma([A : \Delta, \neg A])$ since contraction is admissible (remember again that contraction is rule-preserving admissible), and we get $\Gamma([A : \Delta])$ with a single application of (ID);

- (MP): similarly to the cases above, suppose that the rule (MP) is applied twice on $\Gamma(\neg(A \Rightarrow B))$ in a branch. Since (MP) is invertible (Lemma 3), we can assume, without loss of generality, that the two applications of (MP) are consecutive, starting from $\Gamma(\neg(A \Rightarrow B), A, A)$ and from $\Gamma(\neg(A \Rightarrow B), \neg B, \neg B)$. As we have done for $(\Rightarrow^-)$ and (ID), we conclude that the second application is useless: we obtain derivations of $\Gamma(\neg(A \Rightarrow B), A)$ and $\Gamma(\neg(A \Rightarrow B), \neg B)$ since contraction is rule-preserving admissible, then we get $\Gamma(\neg(A \Rightarrow B))$ with a single application of (MP).

The above restrictions ensure a terminating proof search for the systems under consideration, in particular:

**Theorem 7.** The calculi $NCK, NCK+ID, NCK+MP$, and $NCK+ID+MP$ with the termination restrictions give a PSPACE decision procedure for their respective logics.

To prove Theorem 7, we first show that the size of any sequent occurring in a derivation is polynomially bounded by the size of the initial sequent. We then give a polynomial bound on the number of rule applications in any branch of the derivation. By these results, we obtain that the size of every branch in a derivation is polynomially bounded by the size of the sequent at the root. As a consequence, the calculi $N S$ with the termination restriction provides a PSPACE decision procedure.
From now on, we denote by $\text{sub}(\Gamma)$ the set of subformulas of $\Gamma$. First of all, we need to define the \textit{conditional depth} of a formula $F \in \text{sub}(\Gamma)$. Intuitively, we say that $F$ has conditional depth $n$ (or simply has depth $n$) if $F$ or its negation appears in $\Gamma$ with a conditional nesting of $n$. Before introducing the depth of a formula we further need to define the notion of \textit{propositional subformula}:

\textbf{Definition 7 (Propositional subformula).} Given $F, G \in \mathcal{L}$, we say that $F$ is a propositional subformula of $G$ if either

(i) $F \equiv G$

or

(ii) $G \equiv \neg F$

or

(iii) $G \equiv A \otimes B$ and either $F \equiv A$ or $F \equiv B$, with $\otimes \in \{\land, \lor, \rightarrow\}$.

Let $\Gamma$ be a sequent and let $F$ be a formula in $\text{sub}(\Gamma)$. We say that $F$ is a propositional subformula of $\Gamma$ if there is $G \in \Gamma$ such that $F$ is a propositional subformula of $G$.

\textbf{Definition 8 (Conditional depth of a formula (depth of a formula)).} Let $\Gamma$ be a sequent. Let $F$ be a formula in $\text{sub}(\Gamma)$. We define the conditional depth (or simply depth) of $F$ as follows:

- if $F$ is a propositional subformula of $\Gamma$ (Definition 7), then $F$ has conditional depth 0;
- given $A \Rightarrow B$ (resp. $\neg(A \Rightarrow B)$) a propositional subformula of $\Gamma$ with conditional depth $n$, if $F$ is a propositional subformula of either $A$ or $B$, then $F$ has conditional depth $n + 1$.

Note that a formula can have multiple depths: for instance if $\Gamma = B \lor ((A \lor C) \Rightarrow (B \lor A \lor (C \Rightarrow B)))$, then $B$ has depths 0, 1 and 2, whereas $A$ has depth 1 and $C$ has depths 1 and 2. Let $\text{depth}(\Gamma, n)$ be the set of formulas of $\text{sub}(\Gamma)$ of depth $n$. We have $\sum_{i=0}^n |\text{depth}(\Gamma, i)| = O(|\Gamma|)$.

Now to give bounds to the size of a sequent, and to the size of a derivation, we have to give a bound to the number of contexts (possibly nested) that can appear in a sequent $A$ occurring in $\Pi$.

By induction on the level of nesting $n$ of a context $[A : \Delta]$, we can show that $\Delta$ can only contain formulas whose conditional depth in $\Gamma$ is $n + 1$. Indeed, observe that a formula $B \in \Delta$ can only come from i) an application of a boolean rule which does not change the depth of a formula ii) an application of $(\Rightarrow^+)\,$, hence it comes from a formula $A \Rightarrow B$ with a nesting level of $n - 1$, iii) an application of $(\Rightarrow^-)$ or $(MP)$, hence it comes from a formula $\neg(A' \Rightarrow B)$ with a nesting level of $n - 1$, iv) $(MP)$, here $A = \neg B$ and since the context has been necessarily introduced by an application of $(\Rightarrow^+)$ to a formula $(A \Rightarrow C)$ of depth $n - 1$, we easily conclude that $\neg B$ has depth $n$. Therefore, since contexts are only created by the rule $(\Rightarrow^+)$, the number of immediate sub-contexts of a context $[A : \Delta]$ at a depth $n$ is bounded by the number of positive conditional formulas of depth $n + 1$ occurring in $\text{sub}(\Gamma)$. By an easy induction, we can show that the maximum number of contexts that can appear in a sequent $A \in \Pi$ is bounded by $\sum_{i=0}^{\infty} |\text{depth}(\Gamma, i)| \times |\text{depth}(\Gamma, i + 1)|$, which is in $O(|\Gamma|^2)$.
Lemma 7. Let $\Pi$ be a derivation of a sequent $\Gamma$ in $NS$ with the termination restrictions, and $\Lambda$ be a sequent occurring in $\Pi$. Then $|\Lambda| \in O(|\Gamma|^3)$.

Proof. We first give a bound on the size of each context. Let $[A : \Delta]$ be a context with a nesting level of $n$. By our previous discussion, $\Delta$ can only contain formulas of depth $n$. The termination restrictions preventing redundant application of the rules, $|\Delta|$ is linearly bounded by $|\text{depth}(\Gamma, n)|$, which is itself bounded by $|\Gamma|$.

We also have to take into account the extra space that may be needed to implement the termination restrictions. Observe that these restrictions can be implemented by some bookkeeping mechanism, that is for each context we store the list of (negative conditional) formulas to which the rule ($\Rightarrow -$) has been applied with this context. Since this rule can only be applied once to each formula with respect to a given context, the extra space overhead is in $O(|\Gamma|)$. For the rules ($ID$) and ($MP$), we can simply use a flag to record whether the rules have been applied to a given context/negated conditional. This gives a constant space overhead for each context. Therefore, the space needed to store a single context (taking the termination restrictions into account) is in $O(|\Gamma|)$. The number of (possibly nested) contexts that can appear in $\Lambda$ being in $O(|\Gamma|^2)$, we conclude that $|\Lambda| \in O(|\Gamma|^3)$. □

We now bound the length of a branch in a derivation.

Lemma 8. Let $\Pi$ be a derivation of a sequent $\Gamma$ in $NS$ with the termination restrictions. Then the number of rule applications in any given branch of $\Pi$ is in $O(|\Gamma|^4)$.

Proof. Let $[A : \Delta]$ be a context occurring in $\Lambda$ with a nesting level of $n$. We estimate the number of rules that can be applied within this context, i.e. the number of rules that can be applied to formulas of $\Delta$. We observe that: 1. As usual, the boolean rules cannot be applied redundantly and the number of application of these rules is linearly bounded by the number of boolean subformulas occurring in $\Delta$, which is itself linearly bounded by $|\Gamma|$. The rules ($\Rightarrow ^+$) can be applied at most once for each positive conditional formula of depth $n$ appearing in subformulas of $\Delta$. Hence its number of applications is linearly bounded by $|\Gamma|$. Since this is the only rule which can introduce new contexts, this also gives a (linear) bound to the number of contexts that can appear in $\Delta$. 3. Due to the termination restrictions, the rule ($\Rightarrow -$) can be applied once for each negative conditional formula of depth $n$ appearing in subformulas of $\Delta$, and for each context occurring in $\Delta$. Both numbers being linearly bounded by $|\Gamma|$, the number of applications of the rule ($\Rightarrow -$) is linearly bounded by $|\Gamma|^2$. 4. The rule ($ID$) can be applied at most once for each context in $\Delta$ (due to termination restrictions), and thus its number of applications is linearly bounded by $|\Gamma|$. 5. The rule ($MP$) can be applied at most once for each negative conditional formula of depth $n$ appearing in subformulas of $\Delta$ (due to termination restrictions), and thus its number of applications is linearly bounded by $|\Gamma|$. Therefore the number of rule applications in a given context is linearly bounded by $|\Gamma|^2$. The number of (possibly nested) contexts that can appear in $\Lambda$ being bounded by $|\Gamma|^2$, we easily obtain the linear bound of $|\Gamma|^4$ to the number of application of rules in any branch of the derivation. □
By the bounds on the size of a sequent and the number of rule applications in a given branch of a derivation, we obtain that the size of each branch is in $O(|\Gamma|^7)$ and Theorem 7 follows.

**Termination and complexity of $\mathcal{NCK+CEM}$ and $\mathcal{NCK+CEM+ID}$**

In order to prove that we can turn the calculus into a terminating one, we need the following definitions.

**Definition 9.** Given a nested sequent $\Gamma = A_1, \ldots, A_n, [B_1 : \Gamma_1], \ldots, [B_m : \Gamma_m]$ and a formula $F$, we define

$$F \in^* \Gamma$$

if either $F = A_i$ for some $i \in \{1, 2, \ldots, n\}$ or if $F \in^* \Gamma_j$ for some $j \in \{1, 2, \ldots, m\}$.

**Definition 10 (CEM-reduced sequent).** Given a nested sequent $\Gamma = A_1, \ldots, A_n, [B_1 : \Gamma_1], \ldots, [B_m : \Gamma_m]$, we say that $\Gamma$ is CEM-reduced if the following conditions hold:

- for each formula $F$ such that $F \in^* \Gamma$, either $F = P$ or $F = \neg P$, where $P \in \text{ATM}$, or $F$ is a negated conditional $\neg (C \Rightarrow D)$;
- for each negated conditional $\neg (C \Rightarrow D) \in^* \Gamma$ and for each (sub)context $[C' : \Delta]$ occurring in $\Gamma$, the rule $(\Rightarrow \neg)$ has been applied in a backward proof search to $\neg (C \Rightarrow D)$ and $[C' : \Delta]$.

The intuitive idea is that, given a CEM-reduced sequent, we have already applied $(\Rightarrow \neg)$ as much as possible to each negated conditional $\neg (C \Rightarrow D)$. Furthermore, since it contains only atomic formulas and negated conditionals, all subformulas of $\neg D$ have been introduced in the derivation.

Let us now introduce a nested inclusion among nested sequents:

**Definition 11.** Given two sequents $\Gamma = A_1, \ldots, A_n, [B_1 : \Gamma_1], \ldots, [B_m : \Gamma_m] \subseteq \Delta$, we define

$$\Gamma \subseteq^* \Delta$$

if the following conditions hold:

- $A_i \in \Delta$, for each $i = 1, 2, \ldots, n$
- there is $[B_i : \Delta_i] \in \Delta$ such that $\Gamma_i \subseteq^* \Delta_i$, for each $i = 1, 2, \ldots, m$.

As an example, the following sequents:

$$\Gamma = C, C, [D : E, E, F], [D : E, F], [G : [H : K]];$$

$$\Delta = C, [D : E, E, F, H], [G : [H : K, M]]$$

are such that $\Gamma \subseteq^* \Delta$.

In order to obtain a terminating procedure, we have to put the following restrictions on the application of some rules:

- apply $(\Rightarrow \neg)$ to $\Gamma(\neg (A \Rightarrow B), [A' : \Delta])$ only if $(\Rightarrow \neg)$ has not been applied to the formula $\neg (A \Rightarrow B)$ with the context $[A' : \Delta]$ in the current branch;
- apply $(ID)$ to $\Gamma([A : \Delta])$ only if $(ID)$ has not been applied to $[A : \Delta]$ in the current branch;
– apply \((CEM)\) to \(\Gamma([A : \Delta], [B : \Sigma])\) only if
1. \(\Gamma([A : \Delta], [B : \Sigma])\) is CEM-reduced
2. not \(\Sigma \subseteq A\).

These restrictions impose that \((\Rightarrow)\) is applied only once to a given formula \(\neg (A \Rightarrow B)\) with a context \([A' : \Delta]\) in each branch and that \((ID)\) is applied only once to a given context \([A : \Delta]\). Furthermore, also \((CEM)\) is applied a finite number of times in a given branch. This is stated in a rigorous manner in the following theorems.

**Theorem 8.** The calculi \(NCK+CEM\) and \(NCK+CEM+ID\) with the termination restrictions is sound and complete for their respective logics.

**Proof.** We show that it is useless to apply the rules \((\Rightarrow), (ID)\) and \((CEM)\) without the restrictions. We proceed by induction on the height of the derivation of a valid sequent \(\Gamma\). First of all, consider the rule \((\Rightarrow)\). Suppose to have an application of such a rule not respecting the corresponding termination restriction, as follows:

\[
\begin{array}{c}
(1) \Gamma(\neg(A \Rightarrow B), [A' : \Delta', \neg B]) \\
(1') \Gamma(\neg(A \Rightarrow B), [A : \Delta, \neg B]) \\
\end{array}
\]

Since contraction is height-preserving admissible (Lemma 5), we have also a derivation of at most the same height of \((1)\) of the sequent \((1')\) \(\Gamma(\neg(A \Rightarrow B), [A' : \Delta', \neg B])\), to which we can apply the inductive hypothesis to conclude that there is also a derivation of \((1')\) where the three rules are applied following the termination restrictions.

The case of \((ID)\) is similar. Consider a derivation ending as follows:

\[
\begin{array}{c}
(1) \Gamma([A : \Delta, \neg A]) \\
(1') \Gamma([A : \Delta', \neg A]) \\
\end{array}
\]

By Lemma 5 of height-preserving admissibility of contraction, from \((1)\) we obtain a derivation of \((1')\) with a lower height, therefore we can apply the inductive hypothesis to obtain a derivation with no applications of the three rules without the termination restrictions.

Let us now consider the non-trivial case of \((CEM)\). We distinguish two cases, namely an application of \((CEM)\) violating the termination condition number 1 and an application of \((CEM)\) violating the termination condition number 2.

1. let us consider a derivation containing an application of \((CEM)\) to a non CEM-reduced sequent, as follows:

\[
\begin{array}{c}
\Pi \Gamma([A : \Delta, \Sigma], [B : \Sigma]) \\
\Pi \Gamma([A : \Delta, [B : \Sigma]) \\
\end{array}
\]

Since the rules are invertible (Lemma 3) or, at least, “weakly” invertible\(^4\), we can permute the application of the rules in \(\Pi\) over \((CEM)\). As an example, consider a derivation \(\Pi\) ending with an application of \((\Rightarrow)\) as follows:

\[^4\text{The rules } (\Rightarrow) \text{ and } (CEM) \text{ are not invertible, however, their leftmost premises can be obtained by weakening from the respective conclusions.}\]
We obtain the following derivation:

\[
\Gamma^\prime(\neg(C \Rightarrow D), [A : \Delta, \Sigma, \neg D], [B : \Sigma]) \quad A, \neg B \quad B, \neg A
\]

\[
\Gamma(\neg(C \Rightarrow D), [A : \Delta, \Sigma], [B : \Sigma])
\]

\[
\Gamma(\neg(C \Rightarrow D), [A : \Delta, \neg D], [B : \Sigma]) 
\]

\[
\Gamma(\neg(C \Rightarrow D), [A : \Delta], [B : \Sigma])
\]

We can therefore permute all the rules, obtaining a derivation in which \((CEM)\) is applied to CEM-reduced sequents only.

2. let us consider a derivation containing an application of \((CEM)\) to \(\Gamma([A : \Delta], [B : \Sigma])\) such that \(\Sigma \subseteq^{*} \Delta\). We have a derivation as follows:

\[
(1) \quad \Gamma([A : \Delta, \Sigma], [B : \Sigma]) \quad A, \neg B \quad B, \neg A 
\]

\[
(1') \quad \Gamma([A : \Delta], [B : \Sigma])
\]

Since \(\Sigma \subseteq^{*} \Delta\), from (1) and the fact that contraction is height-preserving admissible (Lemma 5), we immediately get that there is a derivation, whose height is no greater than (1)’s, also of (1’), therefore the application of \((CEM)\) is useless. □

One can have the suspect that the above restrictions on the application of \((CEM)\) are not sufficient to ensure termination due to the fact that such a rule can be applied to \(\Gamma([A : \Delta], [B : \Sigma])\) in two different ways, namely:

\[
\Gamma([A : \Delta, \Sigma], [B : \Sigma]) \quad A, \neg B \quad B, \neg A 
\]

\[
\Gamma([A : \Delta], [B : \Sigma])
\]

\[
\Gamma([A : \Delta, \Sigma], [B : \Delta, \Sigma]) \quad A, \neg B \quad B, \neg A 
\]

\[
\Gamma([A : \Delta], [B : \Sigma])
\]

We observe that this is not the case. Indeed, let us consider the former derivation: the restriction 2 does not avoid a further application of \((CEM)\) (the “symmetric one”), as follows:

\[
(\ast) \quad \Gamma([A : \Delta, \Sigma], [B : \Delta, \Sigma, \Sigma]) \quad A, \neg B \quad B, \neg A 
\]

\[
\Gamma([A : \Delta, \Sigma], [B : \Sigma])
\]

\[
\Gamma([A : \Delta], [B : \Sigma])
\]
However, the rule \((CEM)\) is no longer applicable to \((\ast)\), because both \(\Delta, \Sigma \subseteq \ast \Delta, \Sigma, \Sigma\) and \(\Delta, \Sigma, \Sigma \subseteq \ast \Delta, \Sigma\) hold.

Now we have all the ingredients to prove that the calculi \(\mathcal{NCK} + CEM\) and \(\mathcal{NCK} + CEM + ID\) ensure a terminating proof search. Moreover, we are able to give a \(PSPACE\) complexity upper bound.

**Theorem 9.** The calculi \(\mathcal{NCK} + CEM\) and \(\mathcal{NCK} + CEM + ID\) with the termination restrictions give a \(PSPACE\) decision procedure for their respective logics.

Proceeding similarly to what done for systems not allowing \(CEM\), we can prove that:

**Lemma 9.** Let \(\Pi\) be a derivation of a sequent \(\Gamma\) in \(\mathcal{NCK} + CEM\) and \(\mathcal{NCK} + CEM + ID\) with the termination restrictions, and \(\Lambda\) be a sequent occurring in \(\Pi\). Then \(|\Lambda| \in O(|\Gamma|^3)|\).

**Proof.** The proof is very similar to the one of Lemma 7 above. We first give a bound on the size of each context. Let \([A : \Delta]\) be a context with a nesting level of \(n\). By our previous discussion, \(\Delta\) can only contain formulas of depth \(n\). The termination restrictions preventing redundant application of the rules, \(|\Delta|\) is linearly bounded by \(|\text{depth}(\Gamma, n)|\), which is itself bounded by \(|\Gamma|\).

We also have to take into account the extra space that may be needed to implement the termination restrictions. Observe that these restrictions can be implemented by some bookkeeping mechanism, that is for each context we store the list of (negative conditional) formulas to which the rule \((\Rightarrow^-)\) has been applied with this context. Since this rule can only be applied once to each formula with respect to a given context, the extra space overhead is in \(O(|\Gamma|)\). Similarly for the rule \((CEM)\): since this rule can only be applied once to each pair of contexts \([A : \Gamma'] - [B : \Sigma]\), the extra space overhead is in \(O(|\Gamma'|)\). For the rule \((ID)\), we can simply use a flag to record whether the rule has been applied to a given context. This gives a constant space overhead for each context. Therefore, the space needed to store a single context (taking the termination restrictions into account) is in \(O(|\Gamma'|)\).

The number of (possibly nested) contexts that can appear in \(\Lambda\) being bounded by \(|\Gamma'|\), we conclude that \(|\Lambda| \in O(|\Gamma|^3)|\).

**Lemma 10.** Let \(\Pi\) be a derivation of a sequent \(\Gamma\) in \(\mathcal{NCK} + CEM\) and \(\mathcal{NCK} + CEM + ID\) with the termination restrictions. Then the number of rule applications in any given branch of \(\Pi\) is in \(O(|\Gamma|^4)\).

**Proof.** The proof is the same as the one for Lemma 8. We just further observe that the rule \((CEM)\) merges contexts occurring in \(\Delta\), and thus its number of application is linearly bounded by the number of contexts that can appear in \(\Delta\), that is it is linearly bounded by \(|\Gamma|\). Therefore the number of rule applications in a given context is linearly bounded by \(|\Gamma|^2\). The number of (possibly nested) contexts that can appear in \(\Lambda\) being bounded by \(|\Gamma|^2\), we easily obtain the linear bound of \(|\Gamma|^4\) to the number of application of rules in any branch of the derivation.

By the bounds on the size of a sequent and the number of rule applications in a given branch of a derivation, we obtain that the size of each branch is in \(O(|\Gamma|^7)\) and Theorem 9 follows. It is worth noticing that our calculi match the \(PSPACE\) lower-bound of
the logics CK and CK+ID, and are thus optimal with respect to these logics. On the contrary the calculi for CK+CEM(+ID) are not optimal, since validity in these logics is known to be decidable in CONP. In future work we shall try to devise an optimal decision procedure by adopting a suitable strategy.

4 A calculus for the KLM Cumulative Logic C

In this section we show another application of nested sequents to give an analytic calculus for the flat fragment, i.e. without nested conditionals ⇒, of CK+CSO+ID. This logic is well-known and it corresponds to logic C, the logic of cumulativity, the weakest system in the family of KLM logics [22]. Formulas are restricted to boolean combinations of propositional formulas and conditionals A ⇒ B where A and B are propositional. A sequent has then the form:

\[ A_1, \ldots, A_m, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m] \]

where \( B_i \) and \( \Delta_i \) are propositional. The logic has also an alternative semantics in terms of weak preferential models, as recalled in the following Definition 12:

**Definition 12 (Semantics of C, Definitions 5, 6, 7 in [22]).** A cumulative model is a tuple

\[ \mathcal{M} = (S, W, l, <, V) \]

where \( S, W, l, \) and \( V \) are defined as follows:

- \( S \) is a set of elements are called states;
- \( W \) is a set of possible worlds;
- \( l : S \rightarrow 2^W \) is a function that labels every state with a nonempty set of worlds;
- \( < \) is an irreflexive relation on \( S \);
- \( V \) is a valuation function

\[ V : W \rightarrow 2^{\text{ATM}} \]

which assigns to every world \( w \) the atoms holding in that world.

For \( s \in S \) and \( A \) propositional, we let \( \mathcal{M}, s \models A \) if \( \forall w \in l(s), \mathcal{M}, w \models A \), where \( \mathcal{M}, w \models A \) is defined as in propositional logic. Let \( \text{Min}_<(A) \) be the set of minimal states \( s \) such that \( \mathcal{M}, s \models A \). We define \( \mathcal{M}, s \models A \models B \) if \( \forall s' \in \text{Min}_<(A), \mathcal{M}, s' \models B \). The relation \( \models \) is extended to boolean combinations of conditionals in the standard way. We assume that \( < \) satisfies the following smoothness condition: given a propositional formula \( A \), a model \( \mathcal{M} \) and a state \( s \in S \), if \( \mathcal{M}, s \models A \), then either \( s \in \text{Min}_<(A) \) or there exists \( s' \in S \) such that \( s' < s \) and \( s' \in \text{Min}_<(A) \).

Given a model \( \mathcal{M} = (S, W, l, <, V) \), we say that a formula \( F \) is valid in \( \mathcal{M} \) if \( \mathcal{M}, s \models F \) for every \( s \in S \), and we write \( \mathcal{M} \models F \). We say that a formula \( F \) is valid if it is valid in every model, i.e. \( \mathcal{M} \models F \) for every model \( \mathcal{M} \), and we write \( \models F \).

Actually, Kraus, Lehmann and Magidor have given a different, but equivalent, axiomatization for the cumulative logic C, as follows (\( \vdash_{PC} \) denotes provability in the propositional logic):

- any axiomatization of the classical propositional calculus (prop)
Given a formula $F$, $F$ is valid in $C$ if and only if it is provable in the axiomatization, i.e. $\models F$ if and only if $\vdash F$.

In [11] it has been shown that KLM logic $C$ corresponds to the flat fragment of conditional logic $CK+CSO+ID$, thus showing an alternative axiomatization of $C$ given by the axioms and rules (prop), (Modus Ponens), (RCEA), (RCK), ID, and CSO introduced in Section 2 above.

The rules of $NC_{KLM}$ are those ones of $NC_{CK+ID}$ (restricted to the flat fragment) where the rule ($\Rightarrow^\sim$) is replaced by the rule (CSO):

\[ \Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D] \quad \Gamma, \neg(C \Rightarrow D), [A : C] \quad \Gamma, \neg(C \Rightarrow D), [C : A] \]

The calculi $NC_{KLM}$ are shown in Figure 3.

A derivation of an instance of CSO is shown in Figure 4. More interestingly, in Figure 5 we give an example of derivation of the cumulative axiom $CM$

\[ ((A \Rightarrow B) \land (A \Rightarrow C)) \Rightarrow (A \land B \Rightarrow C) \]
Fig. 4. A derivation of the axiom CSO.

Fig. 5. A derivation of the cumulative axiom $((A \Rightarrow B) \land (A \Rightarrow C)) \rightarrow (A \land B) \Rightarrow C$. We omit the first propositional steps, and we let $\Sigma = \neg(A \Rightarrow B), \neg(A \Rightarrow C)$. 
Exactly as we have made for the calculi $\mathcal{NS}$, we can prove that weakening is height preserving admissible also in the calculus $\mathcal{NC}_{KLM}$ for $C$:

**Lemma 11 (Admissibility of weakening in $\mathcal{NC}_{KLM}$).** Weakening is height-preserving admissible in $\mathcal{NC}_{KLM}$: if $\Gamma(\Delta)$ (resp. $\Gamma([A : \Delta])$) is derivable in $\mathcal{NS}$ with a derivation of height $h$, then also $\Gamma(\Delta, F)$ (resp. $\Gamma([A : \Delta, F])$) is derivable in $\mathcal{NS}$ with a proof of height $h' \leq h$, where $F$ is either a formula or a nested sequent $[B : \Sigma]$.

Proof. Exactly as we have made for $\mathcal{NS}$, we proceed by induction on the height $h$ of the derivation of $\Gamma(\Delta)$. The only case we have to consider further is when the derivation is ended by an application of (CSO), as follows:

$$
\frac{\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D] \quad \Gamma, \neg(C \Rightarrow D), [A : C] \quad \Gamma, \neg(C \Rightarrow D), [C : A]}{\Gamma, \neg(C \Rightarrow D), [A : \Delta]} \quad (\text{CSO})
$$

We can apply the inductive hypothesis to the three premises, to obtain derivations of $(i)$ $\Gamma, F, \neg(C \Rightarrow D), [A : \Delta, \neg D]$ (respectively, of $(i')$ $\Gamma, \neg(C \Rightarrow D), [A : \Delta, F, \neg D]$), $(ii)$ $\Gamma, F, \neg(C \Rightarrow D), [A : C]$, and $(iii)$ $\Gamma, F, \neg(C \Rightarrow D), [C : A]$, from which we obtain a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta]$ (respectively of $\Gamma, \neg(C \Rightarrow D), [A : \Delta]$) by applying (CSO) to $(i)$ (respectively $(i')$), $(ii)$ and $(iii)$.

Furthermore, it is easy to observe that the rule (CSO) is “weakly” invertible, since its leftmost premise is obtained by weakening from the conclusion.

Our goal is to prove that contraction is admissible also in the calculus $\mathcal{NC}_{KLM}$. As we have done in Section 3.1 for the calculi without CSO, the standard proof is by induction on the height of the derivation of a sequent $\Gamma(F, F)$ (see Lemma 5). However, such a standard proof does not work in the following case, in which we are considering a proof of $\Gamma, \neg(C \Rightarrow D), [A : \Delta], [A : \Delta]$ and the rule ending the derivation is (CSO):

$$
\frac{\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D], [A : \Delta] \quad \Gamma, \neg(C \Rightarrow D), [A : C] \quad \Gamma, \neg(C \Rightarrow D), [C : A]}{\Gamma, \neg(C \Rightarrow D), [A : \Delta], [A : \Delta]} \quad (\text{CSO})
$$

In order to obtain a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta]$, it is tempting to apply the inductive hypothesis to the leftmost premise$^5$, obtaining a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D]$. However, the inductive hypothesis is not applicable to the other two premises, then we are not able to get a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta]$ by applying (CSO).

The proof of contraction in $\mathcal{NC}_{KLM}$ relies on a kind of disjunction property that holds only on a particular form of sequents, that do not contain other formulas than literals or negated conditionals. We call these sequents CSO-reduced sequents, and we define them in Definition 13:

---

$^5$ We first obtain a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D], [A : \Delta, \neg D]$ since weakening is height-preserving admissible, then we apply the inductive hypothesis.
Definition 13 (CSO-reduced sequent). A sequent $\Gamma$ is CSO-reduced if it has the form $\Gamma = \Lambda, \Pi, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$, where $\Lambda$ is a multiset of literals and $\Pi$ is a multiset of negative conditionals.

The following proposition is a kind of disjunctive property for CSO-reduced sequents.

Proposition 1. Let $\Gamma = \Lambda, \Pi, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$ be CSO-reduced, if $\Gamma$ is derivable then for some $i$, the sequent $\Lambda, \Pi, [B_i : \Delta_i]$ is (height-preserving) derivable.

Proof. By induction on the height of the derivation of $\Gamma$: if $\Gamma$ is an axiom then the claim is easy and left to the reader. If $\Gamma$ is derived by any rule (R) applied to some $[B_j : \Delta_j]$, we proceed as follows. We illustrate the most difficult case of (CSO), the other are similar, but simpler. Thus suppose that $\Gamma$ is derived by (CSO) applied to $\neg(C \Rightarrow D)$ and $[B_j : \Delta_j]$, say $j = 1$ to keep indexing easy, then $\Gamma = \Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$ and the following are derivable with smaller height:

(a) $\Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : \Delta_1, \neg D], [B_2 : \Delta_2], \ldots, [B_m : \Delta_m]$
(b) $\Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : C], [B_2 : \Delta_2], \ldots, [B_m : \Delta_m]$
(c) $\Lambda, \Pi', \neg(C \Rightarrow D), [C : B_1], [B_2 : \Delta_2], \ldots, [B_m : \Delta_m]$

By inductive hypothesis, if for any of (a), (b), (c) $\Lambda, \Pi', \neg(C \Rightarrow D), [B_i : \Delta_i]$, with $i \neq 1$ is derivable, we are done. Otherwise, by induction hypothesis the following are derivable (with no greater height):

(a') $\Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : \Delta_1, \neg D]$
(b') $\Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : C]$
(c') $\Lambda, \Pi', \neg(C \Rightarrow D), [C : B_1]$

By applying the (CSO) rule we derive: $\Lambda, \Pi', \neg(C \Rightarrow D), [B_1 : \Delta_1]$. □

Proposition 2. Let $\Gamma = \Sigma, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$ be any sequent, if $\Gamma$ is derivable then it can be (height-preserving) derived from some CSO-reduced sequents $\Gamma_i = \Sigma_i, [B_1 : \Delta_1], \ldots, [B_m : \Delta_m]$. Moreover all rules applied to derive $\Gamma$ from $\Gamma_i$ are either propositional rules or the rule ($\Rightarrow^+$).

Proposition 2 can be proved by permuting (downwards) all the applications of propositional and ($\Rightarrow^+$) rules.

Let us now prove the following Proposition 3, which in turn is based on Proposition 1 above. Such a proposition will allow us to conclude the proof of admissibility of contraction in $K_{C_1K_{C_2LM}}$ in the above mentioned case the contracted formula is a context $[A : \Delta]$. The same argument does not extend immediately to the full language with nested conditionals.

Proposition 3. Let $\Gamma = \Sigma, [A : \Delta], [A : \Delta]$ be derivable, then $\Gamma = \Sigma, [A : \Delta]$ is (height-preserving) derivable.

Proof. By Proposition 2, $\Gamma$ is height-preserving derivable from a set of CSO-reduced sequents $\Sigma_i, [A : \Delta], [A : \Delta]$. By Proposition 1, each $\Sigma_i, [A : \Delta]$ is derivable; we then obtain $\Sigma, [A : \Delta]$ by applying the same sequence of rules. □
Lemma 12 (Admissibility of contraction in $\mathcal{N}C_{\text{KLM}}$). Contraction is admissible in $\mathcal{N}C_{\text{KLM}}$: if $\Gamma(F, F)$ is derivable, then $\Gamma(F)$ is (height-preserving) derivable, where $F$ is either a formula or a nested sequent $[A : \Delta]$.

Proof. If $F$ is a formula the proof is exactly the same as the one for Lemma 5 in $\mathcal{N}S$, except for the following case of $(CSO)$: if $F = \neg(C \Rightarrow D)$, and the derivation is ended as follows:

$$\Gamma, \neg(C \Rightarrow D), \neg(C \Rightarrow D), [A : \Delta, \neg D], \neg(C \Rightarrow D), [A : C], \Gamma, \neg(C \Rightarrow D), \neg(C \Rightarrow D), [C : A] (CSO)$$

we apply the inductive hypothesis on the three premises, obtaining derivations of (i) $\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D]$, of (ii) $\Gamma, \neg(C \Rightarrow D), [A : C]$ and of (iii) $\Gamma, \neg(C \Rightarrow D), [C : A]$. We obtain a derivation of $\Gamma, \neg(C \Rightarrow D), [A : \Delta]$ by applying $(CSO)$ to (i), (ii) and (iii).

If $F = [A : \Delta]$ we conclude by applying Proposition 3.

As usual, we obtain completeness by cut-elimination. As in case of $\mathcal{N}S$, the proof is by mutual induction together with a substitution property:

Theorem 11 (Admissibility of cut in $\mathcal{N}C_{\text{KLM}}$). In $\mathcal{N}C_{\text{KLM}}$, the following propositions hold:

- (A) If $\Gamma(F)$ and $\Gamma(\neg F)$ are derivable, then so is $\Gamma$;
- (B) if (I) $\Gamma([A : \Delta])$, (II) $\Gamma([A : A'])$, and (III) $\Gamma([A' : A])$ are derivable, then so is $\Gamma([A' : \Delta])$.

Proof. The proof of both is by mutual induction. To make the structure of the induction clear call: $Cut(c, h)$ the property (A) for any $\Gamma$ and any formula $F$ of complexity $c$ and such that the sum of the heights of derivation of the premises is $h$. Similarly call $Sub(c)$ the assertion that (B) holds for any $\Gamma$ and any formula $F$ of complexity $c$. Then we show that following facts:

(i) $\forall h Cut(0, h)$
(ii) $\forall c Cut(c, 0)$
(iii) $\forall c' < c Sub(c') \rightarrow (\forall h' < c \forall h Cut(c', h') \land \forall h < h Cut(c, h))$.
(iv) $\forall h Cut(c, h) \rightarrow Sub(c)$

This will prove that $\forall c \forall h Cut(c, h)$ and $\forall c Sub(c)$, that is (A) and (B) hold. The proof of (i) and (ii) and (iii) is the same as the one of Theorem 3, except in the following cases in which the principal formula is introduced by $(CSO)$. We have two cases: (d’) in which the cut formula is introduced by $(CSO) - (\Rightarrow^+)$ and $(e')$ in which the cut formula is introduced by $(CSO) - (ID)$.

- (d’) the derivation is as follows:

1. $\Gamma, [A : \Delta, \neg D], \neg(C \Rightarrow D)$
2. $\Gamma, [A : C], \neg(C \Rightarrow D)$
3. $\Gamma, [C : A], \neg(C \Rightarrow D)$
4. $\Gamma, [A : \Delta], [C : D] (CSO)$
5. $\Gamma, [A : \Delta], C \Rightarrow D (\Rightarrow^+)$

$$\Gamma, [A : \Delta, \neg D], \neg(C \Rightarrow D) (cut)$$

\text{Proof.} The proof of both is by mutual induction.
First of all, since weakening is admissible (Lemma 11), from (5) we obtain a derivation, of no greater height, also for (5') \( \Gamma, [A : \Delta, \neg D], C \Rightarrow D \). We then apply the inductive hypothesis on (A) to cut (1) and (5'), to obtain a derivation of (6) \( \Gamma, [A : \Delta, \neg D] \), from which we obtain a derivation of (6') \( \Gamma, [A : \Delta], [A : \Delta, \neg D] \) by weakening. Notice that we can do it because we are applying the inductive hypothesis on the sum of the heights of the derivations. Since we have derivations for (2) and (3), we can also apply the inductive hypothesis on (B), that is to say \( \forall c' < c \) \( \text{Sub}(c) \), to obtain a derivation of (4') \( \Gamma, [A : \Delta], [A : D] \) from (4), from which we obtain a derivation of (4'') \( \Gamma, [A : \Delta], [A : \Delta, D] \) by weakening. We conclude by applying the inductive hypothesis for (A) on the complexity of the cut formula to cut (6') and (4''): we obtain a derivation of \( \Gamma, [A : \Delta], [A : \Delta] \), from which we conclude that \( \Gamma, [A : \Delta] \) is derivable since contraction is admissible (Lemma 12);

- (e') the derivation is as follows:

\[
\begin{align*}
&\Gamma, [A : \Delta, F, \neg D], \neg(C \Rightarrow D) \\
&\Gamma, [C : \neg A] \quad \text{(CSO)} \\
&\Gamma, [C : A] \quad \text{(ID)} \\
&\Gamma, \neg(C \Rightarrow D), [A : \Delta, F] \\
&\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg F] \\
&\Gamma, \neg(C \Rightarrow D), [A : \Delta]
\end{align*}
\]

First of all, since weakening is admissible (Lemma 11), from (1) we obtain a derivation, of no greater height, also of (1') \( \Gamma, \neg(C \Rightarrow D), [A : \Delta, F, \neg A] \). We can apply the inductive hypothesis on (A) to cut (1') and (2) (on the sum of the heights of the derivations), then we obtain a derivation of \( \Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg A] \), to which we apply (ID) to obtain a derivation of \( \Gamma, \neg(C \Rightarrow D), [A : \Delta] \), and we are done.

To prove (iv), we first introduce the notion of rank: the rank \( r \) of an instance of (B) is the 3-multiset \( r = (h_1, h_2, h_3) \) of the respective heights of the premises of (B): \( h_1 \) is the height of \( \Gamma([A : \Delta]) \), \( h_2 \) of \( \Gamma([A : A']) \), and \( h_3 \) of \( \Gamma([A' : A]) \). Given two ranks \( r = (h_1, h_2, h_3) \) and \( r' = (h'_1, h'_2, h'_3) \) we define the following ordering (namely a case of multi-set ordering): \( r < r' \) iff \( r \) is obtained by replacing at least one \( h'_j \) with 1 to 3 strictly smaller values \( h_j \). This relation is transitive and well-founded, so that we will use it to prove statement (iv) by induction on the rank \( r \), where we suppose that the complexity of formula \( A \) is \( c \). All cases are easy except when \( \Gamma([A : \Delta]) \) is derived by (CSO) and by (ID).

- \( \text{(CSO)} \). The derivation is as follows:

\[
\begin{align*}
&\Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D] \\
&\Gamma, \neg(C \Rightarrow D), [A : C] \\
&\Gamma, \neg(C \Rightarrow D), [C : A] \\
&\Gamma, \neg(C \Rightarrow D), [A : \Delta] \quad \text{(CSO)}
\end{align*}
\]

Let \( h_1 \) be the height of the conclusion, the two other hypothesis are: (4) \( \Gamma, \neg(C \Rightarrow D), [A : A'] \) with height \( h_2 \) and (5) \( \Gamma, \neg(C \Rightarrow D), [A' : A] \) with height \( h_3 \), so that the instance of (B) with \( \Gamma, \neg(C \Rightarrow D), [A : \Delta] \), (4) has rank \( r = (h_1, h_2, h_3) \). Let \( k_1, k_2, k_3 < h_1 \) be the heights of the three premises (1), (2), (3) in that order. We have that an instance of (B) with premise (1), (4), (5) has rank \( (k_1, h_2, h_3) < r \).
and the instance of (B) with (2), (4), (5) has rank \((k_2, h_2, h_3) < r\). By inductive hypothesis we obtain:

\[
\begin{align*}
(1') & \Gamma, \neg(C \Rightarrow D), [A' : \Delta, \neg D] \\
(2') & \Gamma, \neg(C \Rightarrow D), [A' : C].
\end{align*}
\]

Moreover an instance of (B) with (4), (2) and (3) has rank \((h_2, k_2, k_3) < (h_1, h_2, h_3) = r\), so that we can apply the induction hypothesis and get

\[
(4') \Gamma, \neg(C \Rightarrow D), [C : A'].
\]

Now we apply \((\text{CSO})\) to \((1'), (2'), (4')\) and we obtain

\[
\Gamma, \neg(C \Rightarrow D), [A' : \Delta].
\]

– \((\text{ID})\) In this case, we have

\[
(1) \Gamma([A : \Delta, \neg A]) \quad (\text{ID})
\]

The height of the premise (1), say \(h_1'\) is smaller than \(h_1\) the height of the conclusion, so that \((h_1', h_2, h_3) < (h_1, h_2, h_3)\), thus by inductive hypothesis we get:

\[
(6) \Gamma([A' : \Delta, \neg A]).
\]

From the hypothesis of (B) that \(\Gamma([A' : A])\) is derivable, we get by weakening:

\[
(7) \Gamma([A' : \Delta, A]).
\]

Since by hypothesis we have cut on a formula of complexity \(c\), by cutting (6) and (7) we obtain \(\Gamma([A' : \Delta])\).

**Theorem 12.** The calculus \(\mathcal{N}C_{KLM}\) is sound and complete for KLM logic \(C\) (corresponding to the flat fragment of \(\text{CK+CSO+ID}\)).

**Proof.** For soundness we have just to check the validity of the \((\text{CSO})\) rule. In this case, \(\Gamma, \neg(C \Rightarrow D), [A : \Delta]\) is derived from \((i) \Gamma, \neg(C \Rightarrow D), [A : \Delta, \neg D]\), \((ii) \Gamma, \neg(C \Rightarrow D), [A : C]\), and \((iii) \Gamma, \neg(C \Rightarrow D), [C : A]\). We show that

\[
(\ast) (\neg(C \Rightarrow D) \lor (A \Rightarrow (\Delta \lor \neg D))) \land (\neg(C \Rightarrow D) \lor (A \Rightarrow C)) \land (\neg(C \Rightarrow D) \lor (C \Rightarrow A)) \rightarrow (\neg(C \Rightarrow D) \lor (A \Rightarrow \Delta))
\]

is valid in \(\text{CK+CSO+ID}\), then we conclude by propositional reasoning and the inductive hypothesis. The mentioned formula is derivable as follows: by \((\text{RCK})\) we have

\[
(1) (A \Rightarrow D) \land (A \Rightarrow (\Delta \lor \neg D)) \rightarrow (A \Rightarrow \Delta)
\]

By \((\text{CSO})\) we have:

\[
(2) (A \Rightarrow C) \land (C \Rightarrow A) \land (C \Rightarrow D) \rightarrow (A \Rightarrow D)
\]

Thus by (1) and (2) we get:

\[
(3) (A \Rightarrow C) \land (C \Rightarrow A) \land (C \Rightarrow D) \land (A \Rightarrow (\Delta \lor \neg D)) \rightarrow (A \Rightarrow \Delta)
\]
Let us abbreviate by $F$ the premise of (3), by propositional reasoning we get that:

$$(\neg(C \Rightarrow D) \lor F) \rightarrow (\neg(C \Rightarrow D) \lor (A \Rightarrow \Delta))$$

But then we get (*) by distributing $\neg(C \Rightarrow D)$ over the conjuncts of $F$ and obvious simplification.

For completeness, one can derive all instances of CSO axioms as shown in Figure 4. Moreover the rules RCK and RCEA are derivable too, as follows:

For (RCEA), we have to show that if $A \rightarrow B$ is derivable, then also $(A \Rightarrow C) \rightarrow (B \Rightarrow C)$ is derivable. Since $A \rightarrow B$ is derivable, and since $(\land^+)$ and $(\rightarrow^+)$ are invertible, we have a derivation for $A \rightarrow B$, then for (1) $\neg A, B$, and for $B \rightarrow A$, then for (2) $A, \neg B$; therefore, since weakening is admissible (Lemma 11), we have derivations for (1') $[A : \neg A, B]$, and for (2') $[B : A, \neg B]$. We derive $(A \Rightarrow C) \rightarrow (B \Rightarrow C)$ (the other half is symmetric) as follows:

\[
\begin{array}{c}
\hline
(AX) \quad (ID) \quad (CSO) \\
\hline
\neg(A \Rightarrow C), [B : C, \neg C] \quad [A : B] \quad [B : A] \\
\hline
\neg(A \Rightarrow C), B \Rightarrow C \quad \Rightarrow^+) \\
\hline
A \Rightarrow C \rightarrow (B \Rightarrow C) \quad (-^+) \\
\end{array}
\]

For (RCK), suppose that we have a derivation of $(A_1 \land \ldots \land A_n) \rightarrow B$. Since $(-^+)$ is invertible, we have also a derivation of $B, \neg(A_1 \land \ldots \land A_n)$. Since $(\land^-)$ is also invertible, then we have a derivation of $B, \neg A_1, \ldots, \neg A_n$, and, by weakening (Lemma 11), of (1) $\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2, \ldots, \neg A_n]$, from which we conclude as follows:

\[
\begin{array}{c}
\hline
(AX) \quad (CSO) \quad (CSO) \\
\hline
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1, \neg A_2, \ldots, \neg A_n] \\
\hline
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B, \neg A_1] \\
\hline
\neg(C \Rightarrow A_1), \ldots, \neg(C \Rightarrow A_n), [C : B] \\
\hline
\neg(C \Rightarrow A_1 \land \ldots \land C \Rightarrow A_n), [C : B] \\
\hline
(C \Rightarrow A_1 \land \ldots \land C \Rightarrow A_n) \rightarrow (C \Rightarrow B) \\
\end{array}
\]

For closure under (Modus Ponens), as usual we use the previous Theorem 11 exactly as we made to show that such rule is derivable in the calculus $N^S$ in the proof of Theorem 5.

**Termination and complexity of $N^C_{KLM}$** Termination of the calculus $N^C_{KLM}$ can be proved similarly to Theorem 7: in this case, the rules $(CSO)$ and $(ID)$ can be applied
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infinitely often, however we can control the application of such rules in order to obtain a terminating calculus. For (ID), the restriction is exactly the same adopted for NS, namely it is applied only once to each context \([A : \Delta]\) in each branch. As mentioned, the rule (CSO) in NC\textsubscript{KLM} replaces the rule \((\Rightarrow)\) of NS: for such rule, we show that we can impose a restriction similar to the one imposed to \((\Rightarrow)\) in NS, that is to say (CSO) is applied only once to each formula \(\neg(A \Rightarrow B)\) with a context \([A' : \Delta]\) in each branch.

Formally, in order to obtain a terminating calculus, we put the following restrictions on the application of (CSO) and (ID):

- apply (CSO) to \(\Gamma, \neg(A \Rightarrow B), [A' : \Delta]\) only if (CSO) has not been applied to the formula \(\neg(A \Rightarrow B)\) with the context \([A' : \Delta]\) in the current branch;
- apply (ID) to \(\Gamma([A : \Delta])\) only if (ID) has not been applied to \([A : \Delta]\) in the current branch.

Theorem 13. The calculus NC\textsubscript{KLM} with the termination restrictions is sound and complete for KLM logic C (corresponding to the flat fragment of CK+CSO+ID).

Proof. We show that it is useless to apply the rules (CSO) and (ID) without the restrictions.

- (CSO): suppose it is applied twice on \(\Gamma, \neg(A \Rightarrow B), [A' : \Delta]\) in a branch. Since (CSO) is “weakly” invertible, we can assume, without loss of generality, that the two applications of (CSO) are consecutive, starting from \(\Gamma, \neg(A \Rightarrow B), [A' : \Delta, \neg B, \neg \neg B]\). By Lemma 12 (contraction), we have a derivation of \(\Gamma, \neg(A \Rightarrow B), [A' : \Delta, \neg B]\), and we can conclude with a (single) application of (CSO). Remember that contraction is rule-preserving admissible, therefore the obtained derivation does not add any application of (CSO);
- (ID): similarly to the case of (CSO) above, suppose that the rule (ID) is applied twice on \(\Gamma, [A : \Delta]\) in a branch. Since (ID) is invertible (Lemma 3), we can assume, without loss of generality, that the two applications of (ID) are consecutive, starting from \(\Gamma, [A : \Delta, \neg \neg A, \neg A]\). We conclude that the second application is useless, since we obtain a derivation of \(\Gamma, [A : \Delta, \neg A]\) since contraction is admissible (remember again that contraction is rule-preserving admissible), and we get \(\Gamma, [A : \Delta]\) with a single application of (ID).

The above restrictions ensure a terminating proof search for the systems under consideration, in particular:

Theorem 14. The calculus NC\textsubscript{KLM} with the termination restrictions give a PSPACE decision procedure for KLM logic C (corresponding to the flat fragment of CK+CSO+ID).

Proof. We proceed as we have done in the proof of Theorem 7 for proving termination of NS, namely we give bounds to the size of a sequent, and to the size of a derivation.

In order to do that, we first give a bound to the number of contexts that can appear in a sequent occurring in a derivation.

Let \(\Pi\) be a derivation of a sequent \(\Gamma\) in NC\textsubscript{KLM}, and \(\Lambda\) be a sequent occurring in \(\Pi\). Since NC\textsubscript{KLM} takes into account only the flat fragment, i.e. without nested conditionals, of the logic CK+CSO+ID, we can easily observe that, given a context \([A : \Delta]\)
occurring in a sequent, $\Delta$ only contains propositional formulas; in other words, as a difference with NS, contexts are no longer nested. Since contexts are only created by the rule $(\Rightarrow^+)$, the maximum number of contexts that can appear in $A$ in II is in $O(|\Gamma|)$. Furthermore, given a context $[A : \Delta]$, by the termination restrictions we have that $|\Delta|$ is linearly bounded by $|\Gamma|$.

We also have to take into account the extra space that may be needed to implement the termination restrictions. Exactly as we made in the proof of Lemma 7 for NS, we can observe that these restrictions can be implemented by some bookkeeping mechanism, that is for each context we store the list of (negative conditional) formulas to which the rule (CSO) has been applied with this context. Since this rule can only be applied once to each formula with respect to a given context, the extra space overhead is in $O(|\Gamma|)$. For the rule (ID), we can simply use a flag to record whether the rule has been applied to a given context conditional. This gives a constant space overhead for each context. Therefore, the space needed to store a single context (taking the termination restrictions into account) is in $O(|\Gamma|)$. The number of contexts that can appear in $A$ being in $O(|\Gamma|^2)$, we conclude that $|A| \in O(|\Gamma|^3)$.

We now bound the length of a branch in a derivation. First, observe that the rules $(\Rightarrow^+)$ and $(\Rightarrow^-)$ can only be applied to conditional formulas appearing in $A$ (contexts only contain propositional formulas). Hence, the number of applications of $(\Rightarrow^+)$ is linearly bounded by $|\Gamma|$. Since this is the only rule which can introduce new contexts, this also gives a (linear) bound to the number of contexts that can appear in $\Delta$. Due to the termination restrictions, the rule $(\Rightarrow^-)$ can be applied once for each negative conditional formula and for each context occurring in $\Lambda$. Both numbers being linearly bounded by $|\Gamma|$, the number of applications of the rule $(\Rightarrow^-)$ is linearly bounded by $|\Gamma|^2$.

Let $[A : \Delta]$ be a context occurring in $A$. We estimate the number of rules that can be applied within this context, i.e. the number of rules that can be applied to formulas of $\Delta$. We observe that: 1. As usual, the boolean rules cannot be applied redundantly and the number of application of these rules is linearly bounded by the number of boolean subformulas occurring in $\Delta$, which is itself linearly bounded by $|\Gamma|$. 2. The rules $(\Rightarrow^+)$ and $(\Rightarrow^-)$ can no longer be applied to conditional formulas appearing in subformulas of $\Delta$, since $\Delta$ contains only propositional formulas. 3. The rule (ID) can be applied at most once for each context in $\Delta$ (due to termination restrictions), and thus its number of applications is linearly bounded by $|\Gamma|$. Therefore, the number of rule applications in a given context is linearly bounded by $|\Gamma|^2$. The number of contexts that can appear in $A$ being bounded by $|\Gamma|$, we easily obtain the linear bound of $|\Gamma|^3$ to the number of application of rules in any branch of the derivation.

By the bounds on the size of a sequent and the number of rule applications in a given branch of a derivation, we conclude that the size of each branch is in $O(|\Gamma|^5)$. □

We do not know whether this bound is optimal. The study of the optimal complexity for CK+CSO+ID is still open. A NEXP tableau calculus for cumulative logic C has been proposed in [17]. In [28] the authors provide calculi for full (i.e. with nested conditionals) CK+ID+CM and CK+ID+CM+CA; these logics are related to CK+ID+CSO, but they do not coincide with it, even for the flat fragment, i.e. cumulative logic C as CSO
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Their calculi are internal, but rather complex as the make use of ingenious yet highly combinatorial rules. They obtain a PSPACE bound in all cases.

5 Conclusions and Future Works

In this work we have provided nested sequent calculi for the basic normal conditional logic CK and a few extensions of it with combinations of ID, MP, and CEM, namely all combinations except CK+MP+CEM(+ID). The calculi are analytic and their completeness is easily established via cut-elimination. The calculi can be used to obtain a decision procedure, in some cases of optimal complexity. We have also provided a nested sequent calculus for cumulative logic C of the KLM framework corresponding to the flat fragment of CK+CSO+ID for which no internal calculus seems to be known so far. Even if for some of the logics considered in this paper there exist other proof systems, we think that nested sequents provide internal calculi that are particularly natural and simple. Obviously, it is our goal to extend them to a wider spectrum of conditional logics. First of all we intend to complete the cube of the extensions of CK by ID, MP, and CEM by covering the missing combinations CK+MP+CEM(+ID). More importantly, we would like to provide internal calculi for preferential conditional logics, as they still lack “simple” and internal calculi. Since for these logics the semantics is given in terms of ordered models, it might turn out that the basic connective to consider is a kind of entrenchment operator (as it is done in [23]).

We also intend to study refinements of the calculi aiming at efficiency, based on a tighter control of formula duplication. Finally, we wish to take advantage of the calculi to study logical properties of the corresponding systems (disjunction property, interpolation) in a constructive way.

References