Intersection Types: a Proof-Theoretical Approach

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Abstract. The goal of this work is to present a proof-theoretical justification for **IT**. In particular, we discuss the relationship between the intersection connective and the intuitionistic conjunction. For this purpose, we define a new logical system called Intersection Synchronous Logic (**ISL**), that proves properties of sets of deductions of the implication-conjunction fragment of NJ. The main idea behind **ISL** is the decomposition of the intuitionistic conjunction into two connectives, one with synchronous and the other with asynchronous behavior. Then we show how proofs of **ISL** can be decorated with terms in a way that it matches the standard IT assignment system when only the synchronous conjunction is taken into account, and the simple types assignment with pairs and projections when the asynchronous conjunction is considered. Finally, we prove that **ISL** enjoys both the strong normalization and subformula properties.

1 Introduction

The intersection type assignment system (**IT**) [4] is a deductive system that assigns formulae (built from the intuitionistic implication $\rightarrow$ and the intersection $\cap$) as types to the untyped λ-calculus. **IT** has been used as an investigation tool for a large variety of problems, like, for example, characterizations of the strongly normalizing λ-terms [10].

The main goal of this work is to to give a proof-theoretical justification to **IT**. [1] To this aim a basis step is to clarify, within a pure logical system, the difference between the connectives intersection ($\cap$) and intuitionistic conjunction

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[1] This goal sounds very much alike, for example, to the one of giving a proof-theoretical characterization of linear functions. To that purpose one could use λ-terms with exactly a single occurrence of every free and bound name. However, proof-theoretically equivalently, the same set, under the "derivations-as-programs" analogy, is characterized by a deductive system of second order propositional logic without weakening and contraction.

($\wedge$), by imposing constraints on the use of the logical and structural rules of intuitionistic logic.

Recall that derivations of **IT** form a strict subset of derivations of the implicative and conjunctive fragment of Intuitionistic logic (**NJ**), in the sense that the λ-terms to which **IT** gives types to are used as meta-theoretical modalities. More specifically, for every $\Pi : \Gamma \vdash_{\mathrm{IT}} M : \sigma$ of **IT**, the term M records where $\rightarrow$-introductions and eliminations are used inside Π. Then the intersection can be introduced only between formulas typing the *same* term. Hence the rule for the introduction of the intersection ($\cap I$) can be seen, roughly speaking, as a "mistaken decoration" of the rule for the introduction of the conjunction ($\wedge I$) of **NJ**, where pais are forgotten:

$$\frac{\Gamma \vdash N : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash (M, N) : \sigma \wedge \tau}\ (\wedge I) \qquad\qquad \frac{\Gamma \vdash_{\mathbf{IT}} M : \sigma \quad \Gamma \vdash_{\mathbf{IT}} M : \tau}{\Gamma \vdash_{\mathbf{IT}} M : \sigma \cap \tau}\ (\cap I)$$

In order to evidence, at the level of λ-terms, the difference between the usual conjunction $\wedge$ of **NJ** and the intersection $\cap$ of **IT**, we start by defining a non standard decoration for **NJ** (called **NJr**) that has explicit structural rules and where the original conjunction $\wedge$ is split into two conjunctions $\cap$ and &, in the spirit of Linear Logic [6]. Note that this decomposition cannot be expressed directly inside **NJ** without collapsing $\cap$ and &.

From **NJr**, we build **ISL**, a logical system that internalizes this decomposition, maintaining explicit the structural rules. The rules of **ISL** inductively build *equivalence classes*. The basis of the inductive definition is any set of relevant axioms. The inductive steps, represented by the rules of **ISL**, preserve the structural invariant as follows: (A) $\rightarrow$-introduction and elimination must be applied *synchronously* on all the components of a set of already equivalent deductions; (B) weakening must be applied *synchronously* on all the assumptions of the judgments of already equivalent deductions of **NJ**; (C) the order of assumptions matters as far as two derivations must be declared structurally equivalent, or not. So the explicit use of the exchange rule is required. Without the careful management of structural equivalences, **ISL** would collapse to **NJ**— see Subsection 4.1 for a better understanding of the role played by structural rules in this setting.

Given the notion of equivalence classes above, the intersection operator of **IT** becomes a conjunction of **ISL** that can be introduced only between the components of a given class. That conjunction is dubbed as *synchronous*, to recall the kind of building process we use for the derivations it is applied to.

We should conclude by saying that the present work gives a complete proof-theoretical justification for **IT** since **ISL**: (i) highlights the role of structural rules to delineate **IT** inside intuitionistic logic; (ii) reinterprets the intersection operator $\cap$ of **IT** in terms of an operator that can be used among sets of structurally equivalent deductions of intuitionistic logic; (iii) reformulates the tree structures that **IL** [13] required in order to characterize **IT**. The reformulation is in terms of (simultaneous) logical and structural operations on the equivalence classes. Finally, **ISL** is technically good, since it enjoys strong normalization and sub-formula properties.

The rest of the paper is organized as follows: Section 2 recalls the implicative and conjunctive fragment of Intuitionistic Logic (**NJ**) and introduces the system **NJr**, a refinement of the standard decoration for **NJ**. The Intersection Types Assignment System (**IT**), with explicit weakening and exchange rules, is then introduced as a subsystem of **NJr**. Section 3 introduces **ISL**, which embodies all our intuitions into a formal system. Section 4 formalizes how **ISL**, **NJ** and **IT** correspond. Also in this section, we give a technical justification for the conditions on the explicit structural rules, required to reformulate **IT** in terms of **ISL**. Section 5 proves that **ISL** is a good deductive system and describes the behavior of the two **ISL** conjunctions with respect to the implication. Finally, Section 6 describes the relationship between this and some related work.

2 Splitting the conjunction

In this section we first recall the implicative and conjunctive fragment of Intuitionistic Logic (**NJ**) in natural deduction style. We then present **NJr**, a type assignment system based on the splitting of the conjunction into two connectives, each one grasping a particular aspect of its behavior. Finally, the Intersection Types Assignment System (**IT**) will be presented as a subsystem of **NJr**.

Definition 1 ($\{\wedge \to\}$-fragment of NJ).

i) The formulae *of the implicative and conjunctive fragment of **NJ** are generated by the grammar: $\sigma ::= a \mid \sigma \to \sigma \mid \sigma \wedge \sigma$, where a belongs to a denumerable set of constants. As usual, $\to$ is right-associative while $\wedge$ is left-associative. Formulae of **NJ** will be denoted by Greek small letters.*

*ii) We will denote by $\mathcal{F}_{\boldsymbol{NJ}}$ the set of formulae of **NJ**. A* context *is a* finite sequence $\sigma_1,\ldots,\sigma_m$ *of formulae. Contexts are denoted by Γ and Δ.*

*iii) The implicative and conjunctive fragment of **NJ** proves statements $\Gamma \vdash_{\mathrm{NJ}} \sigma$, where Γ is a context and σ a formula. It consists of the following rules:*

$$(A)\frac{}{\sigma \vdash_{\mathrm{NJ}} \sigma} \qquad (W)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma, \tau \vdash_{\mathrm{NJ}} \sigma}$$

$$(X)\frac{\Gamma_1, \sigma_1, \sigma_2, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma}{\Gamma_1, \sigma_2, \sigma_1, \Gamma_2 \vdash_{\mathrm{NJ}} \sigma} \qquad (\wedge I)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \quad \Gamma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}$$

$$(\wedge E^l)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma} \qquad (\wedge E^r)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}{\Gamma \vdash_{\mathrm{NJ}} \tau}$$

$$(\to I)\frac{\Gamma, \sigma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \to \tau} \qquad (\to E)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \to \tau \quad \Gamma \vdash_{\mathrm{NJ}} \sigma}{\Gamma \vdash_{\mathrm{NJ}} \tau}$$

$\Pi : \Gamma \vdash_{\mathrm{NJ}} \sigma$ means that the deduction Π concludes by proving $\Gamma \vdash_{\mathrm{NJ}} \sigma$. $\vdash_{\mathrm{NJ}} \sigma$ is a short notation for $\emptyset \vdash_{\mathrm{NJ}} \sigma$.

*By somewhat abusing the name, **NJ** will always name the implicative and conjunctive fragment of **NJ**.*

NJr is a type assignment for λ-terms with pairs. It splits the original conjunction $\wedge$ of **NJ**

$$(\wedge I)\frac{\Gamma \vdash_{\mathrm{NJ}} \sigma \quad \Gamma \vdash_{\mathrm{NJ}} \tau}{\Gamma \vdash_{\mathrm{NJ}} \sigma \wedge \tau}$$

into two conjunctions, depending on the form of the λ-terms M and N that could be typed by the premises $\Gamma \vdash_{\mathrm{NJ}} \sigma$ and $\Gamma \vdash_{\mathrm{NJ}} \tau$. In particular, when the conclusion of the two premises is the type of the same λ-term, it is possible to replace $\wedge$ by a *synchronous conjunction* ($\cap$) that keeps giving type to M, identical to N. Otherwise, the refinement of $\wedge$ consists of the *asynchronous conjunction* (&) that gives type to the pair (M, N).

Definition 2 (NJr).

i) The grammar defining the set of formulae *of **NJr** ($\mathcal{F}_{\boldsymbol{NJr}}$) is obtained from that of Definition 1 by replacing the rule $\sigma ::= \sigma \wedge \sigma$ by the following rules: $\sigma ::= \sigma \cap \sigma \mid \sigma \& \sigma$, where $\cap$ and & and are, respectively* synchronous *and* asynchronous *conjunctions.*

*ii) A **NJr**-context is a finite sequence of pairs $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ that assigns formulae to variables so that $i \neq j$ implies $x_i \not\equiv x_j$. By abusing the notation, **NJr**-contexts will be denoted by Γ. If $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, then $dom(\Gamma) = \{x_1, ..., x_n\}$.*

iii) Terms of the λ-calculus (Λ) are defined by the following grammar: $M ::= x \mid \lambda x.M \mid MM$, where x belongs to a countable set of variables. Terms of the λ-calculus with pairs (Λ_p) are obtained by adjoining to the previous grammar the following terms: $M ::= (M, M) \mid \pi_l(M) \mid \pi_r(M)$. As usual, terms will be considered modulo α-conversion and application is left associative.

*iii) **NJr** derives judgments $\Gamma \vdash_{\mathrm{NJr}} M : \sigma$ where $M \in \Lambda_p$, Γ is an **NJr**-context, and σ is a formula. The rules of **NJr** are:*

$$(A)\frac{}{x : \sigma \vdash_{\mathrm{NJr}} x : \sigma}$$

$$(W)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \quad x \notin dom(\Gamma)}{\Gamma, x : \tau \vdash_{\mathrm{NJr}} M : \sigma} \qquad (X)\frac{\Gamma_1, x : \sigma_1, y : \sigma_2, \Gamma_2 \vdash_{\mathrm{NJr}} M : \sigma}{\Gamma_1, y : \sigma_2, x : \sigma_1, \Gamma_2 \vdash_{\mathrm{NJr}} M : \sigma}$$

$$(\cap I)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \quad \Gamma \vdash_{\mathrm{NJr}} M : \tau}{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \cap \tau} \qquad (\& I)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \quad \Gamma \vdash_{\mathrm{NJr}} N : \tau}{\Gamma \vdash_{\mathrm{NJr}} (M, N) : \sigma \& \tau}$$

$$(\cap E^l)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \cap \tau}{\Gamma \vdash_{\mathrm{NJr}} M : \sigma} \qquad (\cap E^r)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \cap \tau}{\Gamma \vdash_{\mathrm{NJr}} M : \tau}$$

$$(\& E^l)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \& \tau}{\Gamma \vdash_{\mathrm{NJr}} \pi_l(M) : \sigma} \qquad (\& E^r)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \& \tau}{\Gamma \vdash_{\mathrm{NJr}} \pi_r(M) : \tau}$$

$$(\rightarrow I)\frac{\Gamma, x : \sigma \vdash_{\mathrm{NJr}} M : \tau}{\Gamma \vdash_{\mathrm{NJr}} \lambda x.M : \sigma \rightarrow \tau} \qquad (\rightarrow E)\frac{\Gamma \vdash_{\mathrm{NJr}} M : \sigma \rightarrow \tau \quad \Gamma \vdash_{\mathrm{NJr}} N : \sigma}{\Gamma \vdash_{\mathrm{NJr}} MN : \tau}$$

$\Pi : \Gamma \vdash_{\mathrm{NJr}} \sigma$ means that the deduction Π concludes by proving $\Gamma \vdash_{\mathrm{NJr}} \sigma$.

Intuitively, **NJr** identifies derivations of **NJ** which are *synchronous* with respect to the introduction and the elimination of the implication. In other words, $\cap$ merges sub-deductions where $\rightarrow$ is introduced or eliminated in the "same points".

Formally, let $e : \mathcal{F}_{\mathbf{NJr}} \longrightarrow \mathcal{F}_{\mathbf{NJ}}$ be defined as follows:

$$e(a) = a, \quad e(\sigma \rightarrow \tau) = e(\sigma) \rightarrow e(\tau) \quad e(\sigma \& \tau) = e(\sigma \cap \tau) = e(\sigma) \wedge e(\tau)$$

The function e can be extended to contexts in the obvious way:

$$e(x_1 : \sigma_1, \ldots, x_n : \sigma_n) = e(\sigma_1), \ldots, e(\sigma_n)$$

Moreover, let E be an erasure function from **NJr**-proofs to **NJ**-proofs, that erases all type information from **NJr**-proofs and collapses the introduction and elimination rules of ($\cap$) and (&) to the corresponding rules for ($\wedge$).

The following theorem shows the relation between **NJ** and **NJr**:

Theorem 1. *i) If $\Pi : \Gamma \vdash_{\mathrm{NJr}} M : \sigma$ then $E(\Pi) : e(\Gamma) \vdash_{\mathrm{NJ}} e(\sigma)$.*
ii) If $\Pi : \Gamma \vdash_{\mathrm{NJ}} \sigma$ then there is a proof $\Pi' : \Gamma' \vdash_{\mathrm{NJr}} M : \sigma'$ such that $E(\Pi) = \Pi, e(\Gamma') = \Gamma$ and $e(\sigma') = \sigma$.

Now we can define the Intersection Type Assignment System **IT** as a sub-system of **NJr** where only synchronous conjunction is used.

Definition 3 (IT).

i) The set $\mathcal{F}_{\boldsymbol{IT}}$ of types *of **IT** is the subset of $\mathcal{F}_{\boldsymbol{NJr}}$ generated by the grammar: $\sigma ::= a \mid \sigma \rightarrow \sigma \mid \sigma \cap \sigma$, where a belongs to a denumerable set of constants.*
*ii) The Intersection Type Assignment System **IT** proves statements of the shape: $\Gamma \vdash_{\mathrm{IT}} M : \sigma$ where M is a λ-term, Γ is an **IT**-context, assigning types to variables, and σ is a type. The rules of the system are the rules of **NJr** but $(\& I)$, $(\& E^l)$ and $(\& E^r)$.*
$\Pi : \Gamma \vdash_{\mathrm{IT}} M : \sigma$ means that the deduction Π concludes by proving $\Gamma \vdash_{\mathrm{IT}} \sigma$.

Note that also the Curry's type assignment for Λ_p can be seen as a sub-system of **NJr**, where only the asynchronous conjunction is used.

2.1 An Example

The difference between synchronous and asynchronous conjunction, being related to a meta-condition on the form of proofs, cannot be expressed inside **NJ**. The following example can be useful for better understanding this point.
Let $\sigma = ((\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha) \& (\alpha \rightarrow \alpha)$ and let us consider the following derivation:

$$\dfrac{\Pi_2'' : \vdash_{\mathrm{NJr}} \lambda x.\pi_1(x)\pi_2(x) : \sigma \rightarrow \alpha \rightarrow \alpha \quad \Pi_1'' : \vdash_{\mathrm{NJr}} (\lambda x.x, \lambda x.x) : \sigma}{\vdash_{\mathrm{NJr}} (\lambda x.\pi_1(x)\pi_2(x))(\lambda x.x, \lambda x.x) : \alpha \rightarrow \alpha}\ (\rightarrow E)$$

where Π_1'' is:

$$\dfrac{\dfrac{\dfrac{}{x : \alpha \rightarrow \alpha \vdash_{\mathrm{NJr}} x : \alpha \rightarrow \alpha}\ (A)}{\vdash_{\mathrm{NJr}} \lambda x.x : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha}\ (\rightarrow I) \quad \dfrac{\dfrac{}{x : \alpha \vdash_{\mathrm{NJr}} x : \alpha}\ (A)}{\vdash_{\mathrm{NJr}} \lambda x.x : \alpha \rightarrow \alpha}\ (\rightarrow I)}{\vdash_{\mathrm{NJr}} (\lambda x.x, \lambda x.x) : ((\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha) \& (\alpha \rightarrow \alpha)}\ (\& I)$$

and Π_2'' is:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\sigma \vdash_{\mathrm{NJr}} x:\sigma}\,(A)}{x:\sigma \vdash_{\mathrm{NJr}} \pi_1(x):(\alpha\to\alpha)\to\alpha\to\alpha}\,(\&E^l) \qquad \dfrac{\dfrac{}{x:\sigma \vdash_{\mathrm{NJr}} x:\sigma}\,(A)}{x:\sigma \vdash_{\mathrm{NJr}} \pi_2(x):\alpha\to\alpha}\,(\&E^r)}{x:\sigma \vdash_{\mathrm{NJr}} \pi_1(x)\pi_2(x):\alpha\to\alpha}\,(\to E)}{\vdash_{\mathrm{NJr}} \lambda x.\pi_1(x)\pi_2(x):\sigma\to\alpha\to\alpha}\,(\to I)$$

In the derivation Π_1'', the conjunction & has been introduced. But since the two terms typing the premises are syntactically the same, we could replace it by $\cap$, so obtaining the following derivation, where$\tau = ((\alpha\to\alpha)\to\alpha\to\alpha)\cap(\alpha\to\alpha)$:

$$\dfrac{\Pi_2' :\vdash_{\mathrm{NJr}} (\lambda x.xx):\tau\to\alpha\to\alpha \quad \Pi_1' :\vdash_{\mathrm{NJr}} \lambda x.x:\tau}{\vdash_{\mathrm{NJr}} (\lambda x.xx)(\lambda x.x):\alpha\to\alpha}\,(\to E)$$

where Π_1' is:

$$\dfrac{\dfrac{\dfrac{}{x:\alpha\to\alpha \vdash_{\mathrm{NJr}} x:\alpha\to\alpha}\,(A)}{\vdash_{\mathrm{NJr}} \lambda x.x:(\alpha\to\alpha)\to\alpha\to\alpha}\,(\to I) \qquad \dfrac{\dfrac{}{x:\alpha \vdash_{\mathrm{NJr}} x:\alpha}\,(A)}{\vdash_{\mathrm{NJr}} \lambda x.x:\alpha\to\alpha}\,(\to I)}{\vdash_{\mathrm{NJr}} \lambda x.x:((\alpha\to\alpha)\to\alpha\to\alpha)\cap(\alpha\to\alpha)}\,(\cap I)$$

and Π_2' is:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\tau \vdash_{\mathrm{NJr}} x:\tau}\,(A)}{x:\tau \vdash_{\mathrm{NJr}} x:(\alpha\to\alpha)\to\alpha\to\alpha}\,(\cap E^l) \qquad \dfrac{\dfrac{}{x:\tau \vdash_{\mathrm{NJr}} x:\tau}\,(A)}{x:\tau \vdash_{\mathrm{NJr}} x:\alpha\to\alpha}\,(\cap E^r)}{x:\tau \vdash_{\mathrm{NJr}} xx:\alpha\to\alpha}\,(\to E)}{\vdash_{\mathrm{NJr}} \lambda x.xx:\tau\to\alpha\to\alpha}\,(\to I)$$

Both the previous derivations correspond, in the sense of Theorem 1, to the following derivation in **NJ**:

$$\dfrac{\Pi_2 :\vdash_{\mathrm{NJ}} \rho\to\alpha\to\alpha \quad \Pi_1 :\vdash_{\mathrm{NJ}} \rho}{\vdash_{\mathrm{NJ}} \alpha\to\alpha}\,(\to E)$$

where: $\rho = e(\sigma) = e(\tau) = ((\alpha\to\alpha)\to\alpha\to\alpha)\wedge(\alpha\to\alpha)$ and Π_1, Π_2 are, respectively:

$$\dfrac{\dfrac{\alpha\to\alpha \vdash_{\mathrm{NJ}} \alpha\to\alpha}{\vdash_{\mathrm{NJ}} (\alpha\to\alpha)\to\alpha\to\alpha}\,(\to I) \qquad \dfrac{\alpha \vdash_{\mathrm{NJ}} \alpha}{\vdash_{\mathrm{NJ}} \alpha\to\alpha}\,(\to I)}{\vdash_{\mathrm{NJ}} ((\alpha\to\alpha)\to\alpha\to\alpha)\wedge(\alpha\to\alpha)}\,(\wedge I)$$

and

$$\dfrac{\dfrac{\dfrac{\sigma \vdash_{\mathrm{NJ}} \sigma}{\sigma \vdash_{\mathrm{NJ}} (\alpha\to\alpha)\to\alpha\to\alpha}\,(\wedge E^l) \qquad \dfrac{\sigma \vdash_{\mathrm{NJ}} \sigma}{\sigma \vdash_{\mathrm{NJ}} \alpha\to\alpha}\,(\wedge E^r)}{\sigma \vdash_{\mathrm{NJ}} \alpha\to\alpha}\,(\to E)}{\vdash_{\mathrm{NJ}} \sigma\to\alpha\to\alpha}\,(\to I)$$

3 The logical system ISL

The logical system **ISL**, that will be presented below, internalizes at the logical level the two different behaviors of the conjunction, the synchronous and asynchronous one. The main features for searching this result is the notion of *molecule*, which is, roughly speaking, a collection of judgements of **NJ** that can be proved by proofs of the same shape.

Definition 4. *i)* Formulae *of **ISL** are formulae of **NJr**. Contexts are finite sequences of such formulae, ranged over by Δ, Γ.*

ii) An atom *is a pair $(\Gamma; \alpha)$, where Γ (the context) is a finite sequence of formulas. $\mathcal{A}, \mathcal{B}$ will range over atoms.*

iii) A finite multiset of atoms, such that the contexts in all atoms have the same cardinality is called a molecule. *$[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ denotes a molecule consisting of the atoms $\mathcal{A}_1, \ldots, \mathcal{A}_n$. $\mathcal{M}, \mathcal{N}$ will range over molecules. $\cup$ is the multiset union.*

*iv) **ISL** derives molecules by the following rules:*

$$\frac{}{[(\alpha_i; \alpha_i) \mid 1 \leq i \leq r]}\ (A) \qquad \frac{\mathcal{M} \cup \mathcal{N}}{\mathcal{M}}\ (P)$$

$$\frac{[(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i, \alpha_i; \beta_i) \mid 1 \leq i \leq r]}\ (W) \qquad \frac{[(\Gamma_1^i, \beta_i, \alpha_i, \Gamma_2^i; \sigma_i) \mid 1 \leq i \leq r]}{[(\Gamma_1^i, \alpha_i, \beta_i, \Gamma_2^i; \sigma_i) \mid 1 \leq i \leq r]}\ (X)$$

$$\frac{[(\Gamma_i, \alpha_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\ (\to I)$$

$$\frac{[(\Gamma_i; \alpha_i \to \beta_i) \mid 1 \leq i \leq r] \quad [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}\ (\to E)$$

$$\frac{[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r] \quad [(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i \& \beta_i) \mid 1 \leq i \leq r]}\ (\& I)$$

$$\frac{[(\Gamma_i; \alpha_i \& \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]}\ (\& E_L) \qquad \frac{[(\Gamma_i; \alpha_i \& \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}\ (\& E_R)$$

$$\frac{\mathcal{M} \cup [(\Gamma; \alpha), (\Gamma; \beta)]}{\mathcal{M} \cup [(\Gamma; \alpha \cap \beta)]}\ (\cap I)$$

$$\frac{\mathcal{M} \cup [(\Gamma; \alpha \cap \beta)]}{\mathcal{M} \cup [(\Gamma; \alpha)]}\ (\cap E_L) \qquad \frac{\mathcal{M} \cup [(\Gamma; \alpha \cap \beta)]}{\mathcal{M} \cup [(\Gamma; \beta)]}\ (\cap E_R)$$

*v) $\vdash_{\mathbf{ISL}} \mathcal{M}$ denotes the existence of an **ISL** deduction rooted at $\mathcal{M}$.*

Note that a molecule captures exactly the behavior of proofs being isomorphic, in the sense that the proof of each atom belonging to a molecule is built in the same way step by step, starting from the leafs[2]. In this sense, we say that elements of a molecule have a *synchronous* behavior.

[2] This is the main reason of considering contexts inside atoms as lists and not sets (see Subsection 4.1 for more details on technical accounts on designing the system).

Two different molecules (hence having independent or *asynchronous* behavior) can be merged only through the introduction of an conjunction or an elimination of an implication.

Making a parallel with the so called *hypersequents*, this means that the intersection is an *internal* connective, while the conjunction and implication are *external*. More about the relationship between molecules and hypersequents (in fact, with hyperformulae since **ISL** is in natural deduction style) can be seen in Section 6.

Example 1. Let τ denote $\alpha \to \alpha$, ρ denote $(\tau \to \tau)\&\tau$, and σ denote $(\tau \to \tau)\cap\tau$. The deductions of the Section 2.1 can be developed inside **ISL**, without the need of λ-terms:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[(\tau;\tau),(\alpha;\alpha)]}(A)}{[(\emptyset;\tau\to\tau),(\emptyset;\tau)]}(\to I)}{[(\emptyset;\sigma)]}(\cap I) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{[(\sigma;\sigma)]}(A)}{[(\sigma;\tau\to\tau)]}(\cap E_L) \quad \dfrac{\dfrac{}{[(\sigma;\sigma)]}(A)}{[(\sigma;\tau)]}(\cap E_R)}{[(\sigma;\tau)]}(\to E)}{[(\emptyset;\sigma\to\tau)]}(\to I)}{[(\emptyset;\tau)]}(\to E)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[(\rho;\rho)]}(A)}{[(\rho;\tau\to\tau)]}(\&E_L) \quad \dfrac{\dfrac{}{[(\rho;\rho)]}(A)}{[(\rho;\tau)]}(\&E_R)}{[(\rho;\tau)]}(\to E)}{[(\emptyset;\rho\to\tau)]}(\to I) \quad \dfrac{\dfrac{\dfrac{}{[(\tau;\tau)]}(A)}{[(\emptyset;\tau\to\tau)]}(\to I) \quad \dfrac{\dfrac{}{[(\alpha;\alpha)]}(A)}{[(\emptyset;\tau)]}(\to I)}{[(\emptyset;\rho)]}(\&I)}{[(\emptyset;\tau)]}(\to E)$$

4 ISL, NJ and IT

This section states the formal correspondence between **ISL**, **NJ** and **IT**. This is done decorating proofs of **ISL** by means of λ-terms. The decoration is similar to the one described in [13] and it is inspired in the so called “Curry-Howard isomorphism”: every deduction Π is associated to a λ-term in order to keep track of some structural properties of Π.

Note that this decoration is not standard: the λ-term associated to Π is *untyped*, and does not encode the *whole* structure of Π, but only the order of occurrences of the rules for the implication and conjunction.

Definition 5. *i) Let $\Gamma \equiv \beta_1, ..., \beta_n$ be a context. A decoration of Γ, with respect to a sequence of different variables $x_1, ..., x_n$, is $(\Gamma)^{x_1,\dots,x_n} \equiv x_1 : \beta_1, ..., x_n : \beta_n$. s denotes a sequence of variables, different from each other.*

ii) Every Π proving the molecule $\mathcal{M} = [(\Gamma_i;\beta_i) \mid 1 \le i \le r]$ can be decorated giving rise to a type assignment proving $M_s : (\mathcal{M})_s$, where $(\mathcal{M})_s \equiv [((\Gamma_i)^s;\beta_i) \mid 1 \le i \le r]$, and M_s is a λ-term. The decoration procedure is inductively defined in Figure 4.

*iii) $\vdash^*_{\mathbf{ISL}} M : (\mathcal{M})_s$ denotes the existence of a decorated proof of **ISL** rooted at $M : (\mathcal{M})_s$.*

– $\Pi : \dfrac{}{[(\alpha_i;\alpha_i) \mid 1 \le i \le m]}\ (A) \Rightarrow \dfrac{}{M_x \equiv x : [(x:\alpha_i;\alpha_i) \mid 1 \le i \le m]}\ (A^*)$;

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\beta_i) \mid 1 \le i \le r]}{[(\Gamma_i,\alpha_i;\beta_i) \mid 1 \le i \le r]}\ (W) \Rightarrow$

$$\frac{M_s(\Pi_1) : [((\Gamma_i)^s;\beta_i) \mid 1 \le i \le r] \quad x \notin dom(\Gamma)^s}{M_{s,x}(\Pi) \equiv M_s(\Pi_1) : [((\Gamma_i)^s, x:\alpha_i;\beta_i) \mid 1 \le i \le r]}\ (W^*);$$

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_1^i,\beta_i,\alpha_i,\Gamma_2^i;\sigma_i) \mid 1 \le i \le r]}{[(\Gamma_1^i,\alpha_i,\beta_i,\Gamma_2^i;\sigma_i) \mid 1 \le i \le r]}\ (X) \Rightarrow$

$$\frac{M_{s_1,y,x,s_2}(\Pi_1) : [((\Gamma_1^i)^{s_1}, y:\beta_i, x:\alpha_i, (\Gamma_2^i)^{s_2};\sigma_i) \mid 1 \le i \le r]}{M_{s_1,x,y,s_2}(\Pi) \equiv M_{s_1,y,x,s_2}(\Pi_1) : [((\Gamma_1^i)^{s_1}, x:\alpha_i, y:\beta_i, (\Gamma_2^i)^{s_2};\sigma_i) \mid 1 \le i \le r]}\ (X^*);$$

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_i,\alpha_i;\beta_i) \mid 1 \le i \le r]}{[(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \le i \le r]}\ (\to I) \Rightarrow$

$$\frac{M_{s,x}(\Pi_1) : [((\Gamma_i)^s, x:\alpha_i;\beta_i) \mid 1 \le i \le r]}{M_s(\Pi) \equiv \lambda x.M_{s,x}(\Pi_1) : [((\Gamma_i)^s;\alpha_i \to \beta_i) \mid 1 \le i \le r]}\ (\to I^*);$$

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha_i \to \beta_i) \mid 1 \le i \le r] \quad \Pi_2 : [(\Gamma_i;\alpha_i) \mid 1 \le i \le r]}{[(\Gamma_i;\beta_i) \mid 1 \le i \le r]} (\to E) \Rightarrow$

$$\frac{M_1 : [((\Gamma_i)^s;\alpha_i \to \beta_i) \mid 1 \le i \le r] \quad M_2 : [((\Gamma_i)^s;\alpha_i) \mid 1 \le i \le r]}{M_s(\Pi) \equiv M_1 M_2 : [((\Gamma_i)^s;\beta_i) \mid 1 \le i \le r]} (\to E^*),$$

where $M_1 \equiv M_s(\Pi_1), M_2 \equiv M_s(\Pi_2)$;

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha_i) \mid 1 \le i \le r] \quad \Pi_2 : [(\Gamma_i;\beta_i) \mid 1 \le i \le r]}{[(\Gamma_i;\alpha_i \& \beta_i) \mid 1 \le i \le r]}\ (\& I) \Rightarrow$

$$\frac{M_s(\Pi_1) : [((\Gamma_i)^s;\alpha_i) \mid 1 \le i \le r] \quad M_s(\Pi_2) : [((\Gamma_i)^s;\beta_i) \mid 1 \le i \le r]}{M_s(\Pi) \equiv (M_s(\Pi_1), M_s(\Pi_2)) : [((\Gamma_i)^s;\alpha_i \& \beta_i) \mid 1 \le i \le r]}\ (\& I^*);$$

– $\Pi : \dfrac{\Pi_1 : [(\Gamma_i;\alpha_i^l \& \alpha_i^r) \mid 1 \le i \le r]}{[(\Gamma_i;\alpha_i) \mid 1 \le i \le r]}\ (\& E_X) \Rightarrow$

$$\frac{M_s(\Pi_1) : [((\Gamma_i)^s;\alpha_i^l \& \alpha_i^r) \mid 1 \le i \le r]}{Ms(\Pi) = \pi_t(M_s(\Pi_1)) : [((\Gamma_i)^s;\alpha_i) \mid 1 \le i \le r]}\ (\& E_X^*)$$

where $X \in \{L, R\}$, and, if $X = L$ then $t = l$ else $t = r$;

– $\Pi : \dfrac{\Pi_1 : \mathcal{M}_1}{\mathcal{M}_2}\ (R) \Rightarrow \dfrac{M_s(\Pi_1) : (\mathcal{M}_1)^s}{M_s(\Pi) = M_s(\Pi_1) : (\mathcal{M}_2)^s}\ (R^*)$

where $R \in \{(\cap I), (\cap E_L), (\cap E_R), (P)\}$;

Fig. 1. The decoration of ISL.

The following theorem is straightforward:

Theorem 2. $\vdash_{\mathbf{ISL}} \mathcal{M}$ *implies that there is a term decoration* M *such that* $\vdash^*_{\mathbf{ISL}} M : (\mathcal{M})_s$, *for some* s .

Observe that the decoration is defined in a parametric way with respect to the sequence of variables s.

The next theorem, together with Theorem 1, proves that **ISL** is as powerful as **NJ**. Moreover, it puts into evidence the fact that a molecule represents a set of *synchronous* proofs of **NJ**. The proofs of this and the next theorem are by induction on derivations.

Theorem 3 (ISL and NJ). *Let* $\mathcal{M} = [(\Gamma_1; \alpha_1), \ldots, (\Gamma_m; \alpha_m)]$. *Then* $\vdash^*_{\mathbf{ISL}} M : (\mathcal{M})_s$ *if and only if* $(\Gamma_i)^s \vdash_{\mathrm{NJr}} M : \alpha_i$ *for all* $1 \leq i \leq m$.

ISL can be proposed as the logic for **IT**, thanks to the following theorem:

Theorem 4 (ISL and IT).

i) Let $\mathcal{M} = [(\Gamma_1; \alpha_1), \ldots, (\Gamma_m; \alpha_m)]$, *where* α_i *and all types in* Γ_i *belong to* $\mathcal{F}_{\boldsymbol{IT}}$, *and let* $\vdash^*_{\mathbf{ISL}} M : (\mathcal{M})_s$ *with* $M \in \Lambda$, *for some* s . *Then* $(\Gamma_i)^s \vdash_{\mathrm{IT}} M : \alpha_i$.

ii) $\Gamma \vdash_{\mathrm{IT}} M : \alpha$ *implies* $\vdash^*_{\mathbf{ISL}} M : [(\Gamma; \alpha)]$.

Corollary 1. ***IT*** *collects the synchronous behavior of* ***NJ***.

4.1 The role of structural rules

There are in the literature a lot of intersection types assignment systems. Here clearly we want to consider a "minimal" version, in the sense that only the rules dealing with the two connectives $\rightarrow$ and $\cap$ occur (while there are systems with various kinds of subtyping and eta-rules) and also there is not a universal type. The reason for this choice is clear, being this a foundational investigation, and being these extra features not motivated from a logical point of view. But also in this minimal version **IT** is usually presented in a different style. I.e., contexts are *sets* of pairs $\{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$, and the three rules (A),(W),(X) are replaced by:

$$(A)\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{\mathrm{NJ}} x : \sigma}$$

The two formulations are equivalent. But the design of **ISL**, and consequently a logical account of **IT**, need explicit structural rules.

In fact, let **ISL**' be defined from **ISL** by considering contexts as sets and by replacing the rules (A) and (W) by the axiom:

$$\frac{}{[(\Gamma_i \cup \{\alpha_i\}; \alpha_i) \mid 1 \leq i \leq r]}\ (A')$$

Then the following molecules could be proved:

$$[(\{\alpha \cap \beta \rightarrow \gamma\}; \alpha \rightarrow \beta \rightarrow \gamma)] \text{ and } [(\{\alpha \rightarrow \beta \rightarrow \gamma\}; \alpha \cap \beta \rightarrow \gamma)]$$

hence collapsing $\cap$ to $\wedge$ (see also Section 5). This shows that implicit weakening can't be used in the definition of **ISL**.

On the other hand, let **ISL**" be defined from **ISL** by using contexts as sets (instead of sequences) but maintaining the explicit weakening rule (thus still having a linear axiom). Then it would be possible to derive:

$$\cfrac{\cfrac{\cfrac{[(\{\alpha\};\alpha),(\{\beta\};\beta)]}{[(\{\alpha,\beta\};\alpha),(\{\beta\};\beta)]}\,(W)}{[(\{\alpha,\beta\};\alpha),(\{\alpha,\beta\};\beta)]}\,(W)}{[(\{\alpha,\beta\};\alpha\cap\beta)]}\,(\cap I)$$

The proof above does not correspond to any derivation of **IT**. Indeed, assume the two atoms $(\{\alpha,\beta\};\alpha)$ and $(\{\alpha,\beta\};\beta)$ represent the two judgments $x:\alpha, y:\beta \vdash_{\mathrm{IT}} x:\alpha$ and $x:\alpha, y:\beta \vdash_{\mathrm{IT}} y:\beta$. They have the same context, being, however, labelled by different terms. So $\cap$ cannot be introduced.

Hence, in order to capture correctly the behavior of the intersection connective, we need *both* contexts as sequences and explicit structural rules.

5 Properties of ISL

This section proves that **ISL** enjoys properties expected for logical systems, like strong normalization and sub-formula. Both proofs follow the method described in [13] showing that, in fact, these properties are inherited from **NJ**. We also discuss the behavior of the two **ISL** conjunctions with respect to the implication.

Strong normalization

We start by noting that the rule (P) can be eliminated, and that the weakening can be moved up on the proof, being applied just after the axioms.

Lemma 1. *Let Π be a derivation for $\mathcal{M}$. Then there is a derivation Π' of $\mathcal{M}$ without any occurrences of the rule (P) and all applications of the weakening rule follow the axioms.*

Definition 6. *A derivation is said to be in* pre-normal form *if it doesn't have any occurrences of the rule (P) and all applications of the weakening rule follow the axioms.*

Definition 7. *Let Π be a derivation.*

i. A $\cap$-redex of Π is one of the sequences:

$$\cfrac{\cfrac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}\,(\cap I)}{\mathcal{M}\cup[(\Gamma;\alpha)]}\,(\cap E_L) \qquad\qquad \cfrac{\cfrac{\mathcal{M}\cup[(\Gamma;\alpha),(\Gamma;\beta)]}{\mathcal{M}\cup[(\Gamma;\alpha\cap\beta)]}\,(\cap I)}{\mathcal{M}\cup[(\Gamma;\beta)]}\,(\cap E_R)$$

ii. A &-redex of Π is the sequence:

$$\cfrac{\cfrac{[(\Gamma_i;\alpha_i)\mid 1\le i\le r] \qquad [(\Gamma_i;\beta_i)\mid 1\le i\le r]}{[(\Gamma_i;\alpha_i\&\beta_i)\mid 1\le i\le r]}\,(\&I)}{[(\Gamma_i;\alpha_i)\mid 1\le i\le r]}\,(\&E_L)$$

or the similar sequence for the right case.

iii. A $\to$-redex of Π is the sequence:

$$\dfrac{\dfrac{[(\Gamma_i, \alpha_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\ (\to I) \qquad [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}\ (\to E)$$

It is easy to see that the exchange rule can be moved up when between redexes. This result, together with Lemma 1, shows that the structural rules don't interfere with the normalization process.

Lemma 2. *Let $\diamond \in \{\to, \&, \cap\}$ and let SR be a certain number of occurrences of the rule (X). Then*

$$\dfrac{\dfrac{\dfrac{\mathcal{G}}{\mathcal{G}'}\ (\diamond I)}{\mathcal{G}''}\ (SR)}{\mathcal{G}'''}\ (\diamond E) \quad \textit{can be rewritten as} \quad \dfrac{\dfrac{\dfrac{\mathcal{G}}{\mathcal{G}^{iv}}\ (SR)}{\mathcal{G}^{v}}\ (\diamond I)}{\mathcal{G}'''}\ (\diamond E)$$

where $\mathcal{G}^{iv}$ and $\mathcal{G}^{v}$ are formed accordingly.

The following Lemma is proved by structural induction.

Lemma 3 (Substitution lemma). *Let Π_0 be a proof of $[(\Gamma_i, \alpha_i; \beta_i) \mid 1 \leq i \leq r]$ and Π_1 be a proof of $[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]$. Suppose that both Π_1 and Π_2 are in pre-normal form. Let $S(\Pi_0, \Pi_1)$ be the deductive structure obtained from Π_0 by substituting all axioms $[(\alpha_i; \alpha_i)]$ by Π_1 (using weakening and exchange rules where necessary in order to re-arrange contexts), and by eliminating all occurrences of weakening over α_i. Then $S(\Pi_1, \Pi_0) : [(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]$.*

Definition 8. *Let Π be a derivation in pre-normal form.*

i. A $\cap$-rewriting step on Π is:

$$\dfrac{\dfrac{\mathcal{M} \cup [(\Gamma; \alpha), (\Gamma; \beta)]}{\mathcal{M} \cup [(\Gamma; \alpha \cap \beta)]}\ (\cap I)}{\mathcal{M} \cup [(\Gamma; \alpha)]}\ (\cap E_L) \hookrightarrow \dfrac{\mathcal{M} \cup [(\Gamma; \alpha), (\Gamma; \beta)]}{\mathcal{M} \cup [(\Gamma; \alpha)]}\ (P)$$

The $(\cap E_R)$ case is analogous.

ii. A &-rewriting step on Π is:

$$\dfrac{\dfrac{\Pi_1 :\ [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r] \qquad \Pi_2 :\ [(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i \& \beta_i) \mid 1 \leq i \leq r]}\ (\& I)}{[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]}\ (\& E_L)$$

$$\hookrightarrow \Pi_1 :\ [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]$$

The $(\& E_R)$ case is analogous.

iii. A $\to$-rewriting step on Π is:

$$\dfrac{\dfrac{\Pi_0 :\ [(\Gamma_i, \alpha_i; \beta_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \alpha_i \to \beta_i) \mid 1 \leq i \leq r]}\ (\to I) \qquad \Pi_1 :\ [(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]}{[(\Gamma_i; \beta_i) \mid 1 \leq i \leq r]}\ (\to E)$$

$$\hookrightarrow S(\Pi_1, \Pi_0)$$

Theorem 5. ***ISL*** *is strongly normalizable.*

Proof Lemmas 1, 2 show that (P) can be eliminated and that the other structural rules can be moved up from redexes, so they don't play a significant role in the normalization process. Consider a sequence of normalization steps in **ISL**: $\Pi_1 \to \ldots \to \Pi_n$. By Theorem 3, there is a one-to-many correspondence between proofs in **ISL** and proofs in **NJ**. This means that the every redex of a derivation Π_i in the sequence above can be translated to redexes in **NJ** and the number of redexes of Π_i is bounded by the number of analogous redexes of any projection of Π in **NJ**. Hence the result follows from the fact that **NJ** has the property of strong normalization. ∎

Sub-formula property

Sub-formulae in **ISL** are defined as follows:

Definition 9 (Sub-formula). *Let α be a formula of **ISL**. Then:*

i. *α is a sub-formula of α.*
ii. *If $\beta \diamond \gamma$ is a sub-formula of α, then so are β and γ for $\diamond \in \{\&, \cap, \to\}$.*

Definition 10 (Sub-formula property). *Let Π be a **ISL** derivation of the molecule $[(\Gamma_i; \alpha_i) \mid 1 \leq i \leq r]$. Π enjoys the* sub-formula property*, written $\boldsymbol{sf}(\Pi)$, if every formula appearing in Π is a sub-formula of one of those occurring in $\Gamma_i \cup \{\alpha_i\}$.*

Theorem 6 (Sub-formula property). *Let Π be a **ISL** proof in normal form. Then $\boldsymbol{sf}(\Pi)$.*

Proof The proof is an easy extension of the same property for **NJ**, given the relationship between **NJ** and **ISL** described by Theorem 3. ∎

The adjoint property

In **NJ**, the conjunction $(\wedge)$ is the adjoint of the implication, that is, the formulae:

$$\alpha \wedge \beta \to \gamma \text{ and } \alpha \to \beta \to \gamma$$

are provable equivalent. The question that arises then is if the conjunctions of **ISL** $(\&, \cap)$ also have this property.

It is easy to see that the answer is *yes* for the asynchronous conjunction: the molecules

$$[(\emptyset; \alpha\&\beta \to \gamma)] \text{ and } [(\emptyset; \alpha \to \beta \to \gamma)]$$

are provable equivalent in **ISL**.

However, the answer is *no* for the synchronous conjunction $(\cap)$. In fact, in the formula $\alpha \cap \beta \to \gamma$ it is implicit that α and β are dependent in the sense given by the Section 4, that is, these formulae are labelled by the same λ-term, while in the formula $\alpha \to \beta \to \gamma$, α and β are completely independent.

This same kind of behavior is observed, for example, in Linear Logic where the additive conjunction $(\&)$ is the adjoint of the implication, while the multiplicative one $(\otimes)$ is not.

6 Related work

The idea of studying the relationship between the intersection and intuitionistic conjunction connectives is not new. In fact, this kind of discussion started with Pottinger's observation [10] that $\cap$ does not correspond to the traditional conjunction (this was later formally proved by Hindley [7]). This subject was further motivated in [1, 2]. But still, the study of the behavior of these two connectives were always restricted to type assignment systems.

The first attempt of giving a logical foundation for **IT** appears in [14], where a new type inference system equivalent to **IT** was defined. This system, called $TA^*_\wedge$ avoids the traditional introduction rule for the intersection, and the logic $L_\wedge$ in a Hilbert-style axiom based formulation was proposed in such a way that combinators in the type assignment system can be associated to logical proofs. This approach is indeed very interesting, and it follows in many ways the ideas already in [10]. Still, the intersection type inference is investigated in the context of combinatory logic instead of λ-calculus and the presentation of the resultant logic is axiomatic. This work was further extended in order to support also union types [5].

In [3], hyperformulae were used in order to obtain the logic HP presented in standard natural deduction style, hence abandoning the axiomatic framework. Molecules are very much alike hyperformulae, the differences consisting on the fact that a a context inside an atom (sequent) is a list of formulae (and hence the ordering is crucial), the existence in HP of a distinguished formula ε (the empty formula) and explicit substitutions. [3] This makes the syntax of HP a little bit more complicated than the one presented here, but still easier to handle than *kits* appearing in [13] (see comment below). But is worthy to note that hyperformulae and molecules have the same computational interpretation: both can be seen as synchronous processes running in parallel.

However, in the logic HP, hyperformulae cannot interact. That means they just have internal rules. This is achieved in **ISL** by introducing the conjunction, that can merge processes. Hence in **ISL** has external rules as well.

Another approach on the logical foundation for **IT** is given in [13], where **IL** has been introduced. Roughly speaking, **ISL** can be viewed as **IL** enriched with conjunction. But, although inspired in this former work, the notation designed for **ISL** is completely different from that presented for **IL**, where kits (i.e., trees labelled by formulas) where used in order to keep track of the structure of proofs. The presence of trees introduces a beautiful geometry within the logical system, but at the same time it makes the definition of derivations a little bit harder to manipulate, in the sense that it is necessary the introduction of classes of equivalence between proofs in order to define valid derivations. It turns out that kits aren't really necessary: controlling the order of the leaves is enough in this case. That made it possible to choose a much simpler approach based on molecules, where we don't record the shape of proofs, but only group the isomorphic ones step by step.

[3] This permitted the implication to become an internal connective as well.

In any case, the logical systems proposed so far admit the presence of only one between intersection and conjunction, giving the idea that it was impossible to mix them in the same setting. The main contribution of this work is to present a logical system in natural deduction style in which conjunction and intersection can be represented and hence making it possible to characterize, at the proof theoretical level, the behavior of these two connectives. In this way, the intersection $\cap$ leaves the stigma of being a truly proof-functional connective (as described in [9]) in order to become a connective with synchronous behavior, contrasting with the asynchronous nature of the conjunction.

A logic for **IT** always gives, as sub-product, a typed version of Λ with intersection types, through a complete decoration of proofs [12]. But typed versions of **IT** can be obviously defined following a not logical approach: examples are in [8, 11, 15].

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