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The call-by-value λ-calculus: a Semantic Investigation

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This paper is about a categorical approach for modeling the pure (i.e., without constants) call-by-value λ-calculus, defined by Plotkin as a restriction of the call-by-name λ-calculus. In particular, the properties a category **Cbv** must enjoy to describe a model of call-by-value λ-calculus are given. The category **Cbv** is general enough to catch models in Scott Domains and Coherence Spaces.

1. Introduction

The call-by-value λ-calculus is a restriction of the classical λ-calculus ($\lambda\beta$-calculus, for short), based on the notion of *value*. A value is a term which is either a variable or an abstraction. In particular, the call-by-value λ-calculus ($\lambda\beta_v$-calculus, for short) is obtained from the classical one by restricting the evaluation rule (the β-rule) to those redexes whose operand is a value. This leads to a *call-by-value parameter passing mechanism*, which is a feature present in many real programming languages. We recall that an evaluation is call-by-value if it evaluates a parameter before it is passed.

The call-by-value parameter passing, and the *lazy evaluation*, which evaluates the function bodies only after the parameters have been supplied, were both implemented in the SECD machine, defined in (Landin, 1964) for computing λ-terms. The call-by-value λ-calculus was introduced in (Plotkin, 1975) to define a paradigmatic language, modeling the behavior of SECD.

Here, we deal with the semantics of the *pure*, i.e., without constants, $\lambda\beta_v$-calculus.

Concerning the denotational semantics, a general definition of models for the $\lambda\beta_v$-calculus was in (Egidi et al., 1992), where Hindley-Longo's approach for defining the models for $\lambda\beta$-calculus (Hindley and Longo, 1980) is followed. Any model for the $\lambda\beta_v$-calculus is an applicative structure with an interpretation function that maps terms to

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elements of the applicative structure, such that the map satisfies some constraints, given an environment to interpret the free variables of the terms. The main difference between the original definition by Hindley-Longo and the one in (Egidi et al., 1992) is that the existence of a proper subset V of the carrier of the applicative structure is required. The set V serves to interpret all the values, and we call it the set of the *semantic values*. Such a definition is certainly intuitive. However, it does not help to build models of the $\lambda\beta_v$-calculus, for it does not characterize the properties that an applicative structure must enjoy in order to satisfy the constraints about the interpretation function.

The aim of this paper is to give a categorical description of models for the $\lambda\beta_v$-calculus, and to use it for building models in different mathematical structures.

Recall that the models of the $\lambda\beta$-calculus have a very nice categorical characterization: they are the *reflexive* objects of a *cartesian closed category* with *enough points*. We recall that an object A is reflexive if, and only if, $A \to A$ is a retract of it (notation: $A \triangleright A \to A$). Moreover, the condition of having *enough points* is a suitable notion of "concreteness" for categories. A categorical characterization of models for the $\lambda\beta_v$-calculus cannot be obtained by modifying or restricting the categorical definition for the $\lambda\beta$-calculus, just recalled. A counterexample is the model in (Egidi et al., 1992), built in the category of Scott Domains, and strict continuous functions. We refer to the model which is the initial solution of: $D \approx [D \to_\perp D]_\perp$, being $[D \to_\perp D]_\perp$ the lifted space of strict continuous functions. Indeed, the category of Scott Domains, and strict continuous functions is not cartesian closed.

We give a categorical description of models for the $\lambda\beta_v$-calculus, starting from logical considerations. Our logical intuition is that, while the $\lambda\beta$-calculus is related to the Intuitionistic Logic through the Curry-Howard Isomorphism, extended to the untyped case (with reflexive types), the $\lambda\beta_v$-calculus is related to the Intuitionistic Linear Logic, where the modality characterizes the values. It turns out that a suitable class of categories for interpreting the $\lambda\beta_v$-calculus is a restriction of the one defined in (Benton et al., 1990), where the interpretation of the multiplicative and exponential fragment of Intuitionistic Linear Logic is given. However, we need to endow the category in (Benton et al., 1990) with a suitable retraction, and to require it having *enough values*. The retraction is $\mathcal{D} \triangleright T(\mathcal{D} \Longrightarrow \mathcal{D})$, where $\mathcal{D}$ is the object representing the domain of interpretation, T is a suitable functor, and $\Longrightarrow$ represents the internalization of the morphisms in a monoidal closed category. The notion of "having *enough values*" is the natural restriction to the $\lambda\beta_v$-calculus of the notion of "having enough points" for the $\lambda\beta$-calculus. The meaning of this notion is that morphisms are different if, and only if, there is at least a value where they behave differently. We call **Cbv** this class of category, and, consequently, **Cbv**-models any model built in a category of this class.

This class of categories is general enough to catch models in different settings. We prove that every Scott Domain $\mathcal{D}$, solution of $D \triangleright [D \to_\perp D]_\perp$, and that every Coherence Domain $\mathcal{D}$, solution of $D \triangleright !(D \Longrightarrow D)$, is a **Cbv**-model. We have to say that the first who conjectured the Coherence Space $\mathcal{D}$ here above, to be a model of the $\lambda\beta_v$-calculus was Girard. However, this domain is also a model for $\lambda\beta$-calculus, and it was the leading idea in (Gonthier et al., 1992) for building an optimal reduction machine for β-reduction, translating $\lambda\beta$-calculus into untyped proof-nets. In this paper, we show also that, despite the intuition, a model for the $\lambda\beta$-calculus is not necessarily a model for the $\lambda\beta_v$-calculus (see Remark 5.1).

Moreover, we study the problem of modeling the call-by-value extensionality. Syntactically, the call-by-value extensionality is expressed by the η_v-rule, which is a restriction of the classical η-rule. We define a semantic notion of extensionality, suitably restricting the analogous notion for the $\lambda\beta$-calculus. Namely, a model for the $\lambda\beta_v$-calculus is extensional if the equality relation between its elements reflects their extensional functional behavior. However, the elements of the model are not seen as total functions. They are considered as partial functions, having the set of semantic values as domain. The unexpected consequence is that, unlike the $\lambda\beta$-calculus, a model of the $\lambda\beta_v$-calculus can be extensional without modeling the $\beta_v\eta_v$-equality. As evidence for this, we show that the Coherence Space which is the least solution of $D \approx !(D \Longrightarrow D)$ satisfies the $\beta_v\eta_v$-equality, while not being extensional. Roughly speaking, to model the $\beta_v\eta_v$-equality it is sufficient that only the elements of the models which are interpretation of valuable terms have an extensional behavior.

The class **Cbv** is not a complete characterization of the models for the $\lambda\beta_v$-calculus, at least with respect to those with an extensional theory. We prove that all **Cbv**-models, having a $\beta_v\eta_v$-theory, satisfy the equality $IM = M$, where I is the identity term $\lambda x.x$, and M is any term. This equality, which is correct with respect to the operational semantics of the $\lambda\beta_v$-calculus, does not belong to all $\beta\eta_v$-theories. For example, it is not in the term model induced by the $\beta_v\eta_v$-theory, and it is not in the model of (Honsell and Lenisa, 1993). The equality $IM = M$ reflects the substitution property of the Intuitionistic Linear Logic which we choose for modeling the typed version of the $\lambda\beta_v$-calculus.

We leave as an open problem to say whether the class of **Cbv**-models not having an extensional theory is complete or not.

Index. In Section 2 the $\lambda\beta_v$-calculus, and its notion of model are recalled. In Section 3, starting from some logical argumentation, the categorical structure needed for modeling the $\lambda\beta_v$-calculus is defined. This categorical structure is used in Section 4, and in Section 5 to define a categorical model for the $\lambda\beta_v$-calculus. Section 6 is about extensionality. Section 7 proves the incompleteness of the subclass of **Cbv**-models with an extensional theory. In Section 8 two instances of the categorical model are introduced. In Section 9, we discuss the relation between **Cbv**, and the models of the $\lambda\beta_v$-calculus, given in (Moggi, 1991). Finally, A recalls some of the categorical concepts used in the paper. However, we assume a basic knowledge about Category Theory, Scott Domains, and Coherence Spaces.

An earlier, and partial version of this paper was in (Pravato et al., 1995).

2. Modeling the call-by-value λ-calculus

The *call-by-value* lambda calculus, or $\lambda\beta_v$-calculus, is a restriction of the classical one, based on the concept of value. In particular, the restriction concerns the evaluation rule, namely the β-rule, which is replaced by the β_v-rule.

Definition 2.1. Let *Var* be a denumerable set of variables, ranged over by x, y, z. Let Λ be the set of pure untyped λ-terms M built out from the following grammar:

$$M ::= x \mid MM \mid \lambda x.M$$

We use M, N, P, Q for denoting terms. Terms of the form MN are called *application* while that of the form $\lambda x.M$ are called *abstraction*. The set of *syntactic values*, or simply values, is the set $\mathit{Val} \subset \Lambda$ defined as:

$$\mathit{Val} = \mathit{Var} \cup \{\lambda x.M \mid x \in \mathit{Var} \quad \text{and} \quad M \in \Lambda\}.$$

The call-by-value evaluation rule is the following reduction rule:

$$(\beta_v) \quad (\lambda x.M)N \to_v [N/x]M \qquad \text{if } N \in \mathit{Val}$$

where $[N/x]M$ denotes the substitution of N for every free occurrences of x in M, renaming bound variables in M to avoid variable clash. The reflexive, symmetric, transitive and contextual closure of $\to_v$, together with the possibility of renaming bound variables, lead an equivalence theory on terms of Λ. Formally, the *formal theory* $\lambda\beta_v$ is a set of rules for deriving formulas of the following shape:

$$M =_v N$$

where both M and N belong to Λ. The rules are:

$$\frac{}{M =_v M}(\rho) \qquad \frac{M =_v N}{N =_v M}(\sigma) \qquad \frac{M =_v N \quad N =_v P}{M =_v P}(\tau)$$

$$\frac{y \notin \mathcal{FV}(M)}{\lambda x.M =_v \lambda y.[y/x]M}(\alpha) \qquad \frac{N \in \mathit{Val}}{(\lambda x.M)N =_v [N/x]M}(\beta_v)$$

$$\frac{N =_v P}{MN =_v MP}(\mu) \qquad \frac{M =_v N}{MP =_v NP}(\nu) \qquad \frac{M =_v N}{\lambda x.M =_v \lambda x.N}(\xi)$$

where $\mathcal{FV}(M)$ is the set of the free variables of M.

Two terms M and N will be said β_v-*equal* if the formula $M =_v N$ will be derivable in the above system, writing:

$$\lambda\beta_v \vdash M =_v N.$$

Definition 2.2. A term $M \in \Lambda$ is *valuable* iff there exists $N \in \mathit{Val}$ such that:

$$\lambda\beta_v \vdash M =_v N.$$

Notice that, if we take *Val* to be Λ, then the β_v-reduction rule becomes the classical β-reduction rule, hence the theory $\lambda\beta_v$ becomes the usual theory $\lambda\beta$. That is, the classical lambda calculus can be viewed as a variant of the call-by-value lambda calculus, defining Λ as the set of values.

As far as extensionality is concerned, Plotkin pointed out that the η-rule ($\lambda x.Mx \to_\eta M$ if $x \notin \mathcal{FV}(M)$), which makes extensional the classical λ-calculus, is unsound for the $\lambda\beta_v$-calculus. The extensionality in $\lambda\beta_v$-calculus is realized by the restriction of the η-rule, recalled in the following definition.

Definition 2.3. The η_v-rule is defined as follows:

$$(\eta_v) \quad \lambda x.Mx \to_{\eta_v} M \qquad \text{if } M \in \mathit{Val} \text{ and } x \notin \mathcal{FV}(M).$$

Two terms M and N will be said $\beta_v\eta_v$-equal if the formula $M =_v N$ will be derivable in the system given in Definition 2.1, extended by the rule:

$$\frac{(M \in \mathit{Val}) \quad \text{and} \quad (x \notin \mathcal{FV}(M))}{\lambda x.(Mx) =_v M}(\eta_v)$$

writing:

$$\lambda\beta_v\eta_v \vdash M =_v N.$$

An operational semantics can be defined for $\lambda\beta_v$, inducing the following equivalence: given two terms M and N,

$$M \sim_v N \Leftrightarrow \begin{array}{l}(\text{ for all context } C[\,],\ \text{such that } C[M] \text{ and } C[N] \text{ are closed.}\\ C[M] \text{ reduces to a value } \Leftrightarrow C[N] \text{ reduces to a value }).\end{array}$$

This definition of operational semantics corresponds to the Leibniz principle for programs. Namely, a program (closed term) is characterized by its observational behavior, and so two subprograms (terms) will be equivalent if they can be replaced each for other in the same program without changing the global behavior. In a language without constants, like $\lambda\beta_v$, the simplest observational property is termination.

A model for the $\lambda\beta_v$-calculus will be a structure in which a term $M \in \Lambda$ is interpreted. This interpretation must satisfy two constraints. The first one is that two β_v-equal terms should have the same interpretation. The second one is that it must be *contextual closed*, namely, if two terms M and N have the same interpretation, then for every context $C[\cdot]$, $C[M]$ and $C[N]$ must have the same interpretation.

A general definition of a model for the $\lambda\beta_v$-calculus, following Hindley-Longo's approach for defining a lambda calculus model (Hindley and Longo, 1980), has been given in (Egidi et al., 1992). We recall here such a definition in a slightly different form:

Definition 2.4. Let S and V be two non empty sets such that $V \subset S$, and call V the set of *semantic values*. Let **Env** be the set of *environments*, where an environment is a map $\theta : \mathcal{X} \to V$, where $\mathcal{X} = \mathit{dom}(\theta)$ is a finite subset of *Var*.

i) A *pseudo-λ_v-structure* is an applicative structure $\mathcal{M} = \langle S, V, \bullet, \mathcal{I}\rangle$, where $\bullet : S \times S \to S$ and $\mathcal{I} : \mathbf{Env} \to \Lambda \to S$ is such that $\mathcal{I}\theta$ is defined only for terms with free variables in $\mathit{dom}(\theta)$, and satisfies the following conditions:

$(\mathit{var})\ \mathcal{I}\theta[x] = \theta(x)$,

$(\mathit{abs})\ \mathcal{I}\theta[\lambda x.M] \in V$,

$(\mathit{app})\ \mathcal{I}\theta[M\,N] = \mathcal{I}\theta[M] \bullet \mathcal{I}\theta[N]$,

$(\mathit{eval})\ \mathcal{I}\theta[\lambda x.M] \bullet d = \mathcal{I}\theta^d_x[M]$, for every $d \in V$,

(ceq) if $\forall x \in \mathcal{FV}(M).\, \theta(x) = \theta'(x)$ then $\mathcal{I}\theta[M] = \mathcal{I}\theta'[M]$,

$(\overline{\alpha})$ if $y \notin \mathcal{FV}(M)$ then $\mathcal{I}\theta[\lambda x.M] = \mathcal{I}\theta[\lambda y.[y/x]M]$,

where θ^d_x behaves as θ on every $y \not\equiv x$, while $\theta(x) = d$.

ii) A λ_v*-model* is a pseudo-λ_v-structure such that $\mathcal{I}$ satisfies also:

$(\overline{\xi})$ if $\forall d \in V. \mathcal{I}\theta^d_x[M] = \mathcal{I}\theta^d_x[N]$ then $\mathcal{I}\theta[\lambda x.M] = \mathcal{I}\theta[\lambda x.N]$.

iii) Let $M, N \in \Lambda$. An environment θ is *compatible* both with M and N iff $\mathcal{FV}(M) \cup \mathcal{FV}(N) \subseteq dom(\theta)$. Let $\mathcal{M} = \langle S, V, \bullet, \mathcal{I}\rangle$ be a λ_v-model. The formula $M =_v N$ is *valid in* $\mathcal{M}$, writing

$$\mathcal{M} \models M =_v N,$$

iff for every θ compatible both with M and N, $\mathcal{I}\theta[M] = \mathcal{I}\theta[N]$.

Some remarks are in order about the previous definition. The subset V of S provides a semantic account of the syntactic values. So, the environments map variables to V, as variables are values. Moreover, since every abstraction is a value too, we need *abs*. Condition *app* exploits the binary operation over S for modeling the application. Condition *eval* is necessary for modeling the substitution mechanism of values for variables. *C*ontext *eq*uality (*ceq*) states an obviousness: the interpretation of a term depends only on its free variables. Condition $\overline{\alpha}$ is the semantic counter part of the α-conversion.

Definition 2.5. A model $\mathcal{M}$ of $\lambda\beta_v$ is *adequate* with respect to the operational semantics $\sim_v$ if and only if:

$$\forall M, N. \mathcal{M} \models M = N \Rightarrow M \sim_v N$$

Note that the definition of model we gave does not include adequacy, i.e., non adequate models can satisfy the definition. But all the model we will show are adequate.

The problem of the semantic interpretation of $\beta_v\eta_v$-equality, and so the definition of extensional $\lambda\beta_v$-model, will be discussed in Section 6.

Remark 2.1. The conditions on $\mathcal{I}$ given in Definition 2.4 *i)* do not give a definition of the interpretation function $\mathcal{I}$ by standard induction, because of condition (*abs*). So, condition $(\overline{\xi})$ is necessary to make the interpretation contextually closed.

3. The Cbv category

In this section we define a class of categories to model the $\lambda\beta_v$-calculus. We follow (Scott, 1975): the *untyped* lambda calculus can be considered as the "limit" for the *typed* lambda calculus. So, first, we consider a full typed version of the $\lambda\beta_v$-calculus, and we use the logic behind it to define a category that interprets the language. Then, we extend such a category in order to capture the meaning of the whole untyped language. The idea is to interpret a lambda term M with a free variables set $\mathcal{FV}(M) = \{x_1, \ldots, x_n\}$, by a judgment $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ proved in the type system we want to start from. Any judgment becomes a morphism of the category from $A_1 \odot \cdots \odot A_n$ to A, being every A_i an object, and $\odot$ a suitable bifunctor.

The logic behind the usual lambda calculus is the Intuitionistic Logic: the terms of (simply) typed lambda calculus can be viewed as natural deduction proofs in such a logic.

The β-equality is modeled by the substitution property of derivations. From all this, it follows that the models of *untyped* lambda calculus are cartesian closed categories, namely the models of Intuitionistic Logic, extended with a reflexive object. Our starting point is the observation that the β_v-equality is a restriction of the β-equality. If we want to model it in terms of the substitution property of a natural deduction, we need a logic where the substitution property holds only partially.

Let us focus on the type assignment in Figure 1. It is a restriction of the natural deduction for full Intuitionistic Linear Logic studied in (Ronchi della Rocca and Roversi, 1997). Its judgments have the form:

$$T\Gamma, \Delta \vdash M : A \ .$$

The symbol A is a type and is generated by the grammar:

$$A, B ::= \alpha, \alpha_1, \alpha_2, \ldots \mid T(A \Longrightarrow B) \ , \tag{1}$$

being $\alpha, \alpha_1, \alpha_2, \ldots$ type variables. By $T\Gamma$ we mean a, possibly empty, set of *modal* assumptions $x_1 : TA_1, \ldots, x_n : TA_n$. On the contrary, Δ is a, possibly empty, set of *non modal* assumptions $x_{n+1} : B_1, \ldots, x_{n+m} : B_m$, *i.e.*, every $B_i \not\equiv TC$, for any C. Finally, M is a term in the language $T\Lambda$, generated by the grammar:

$$M, N ::= x \mid T(\lambda x : A.M) \mid \mathtt{d}(M)N \ ,$$

where x ranges over a countable set of variables. In particular, we call TV the set $\{x, T(\lambda x : A.M) \mid A \text{ is a type}\}$ of *values* on $T\Lambda$. The system in Figure 1 gives types to this language, and a restricted substitution property holds for it:

Property 3.1.

i) If $T\Gamma, x : TA, \Delta \vdash M : B$ and $T\Gamma, \emptyset \vdash N : TA$, then $T\Gamma, \Delta \vdash M[N/x] : B$.
ii) If $T\Gamma, \Delta_1, x : A \vdash M : B$ and $T\Gamma, \Delta_2 \vdash N : A$, where A is non modal, then $T\Gamma, \Delta_1, \Delta_2 \vdash M[N/x] : B$.

Thanks to Property 3.1, we can define a rewriting system $\rightarrow_T$ on $T\Lambda$:

$$\begin{aligned} (\mathtt{d}(T(\lambda x : TA.M)))N &\rightarrow_T M[N/x] \text{ if and only if } N \\ &\qquad \text{reduces to some } P \in TV \\ &\qquad \text{by one or more steps of } \rightarrow_T \end{aligned} \tag{2}$$

$$(\mathtt{d}(T(\lambda x : A.M))N \rightarrow_T M[N/x] \text{ being } A \text{ non modal} \tag{3}$$

The definition of $\rightarrow_T$ formalizes the substitution property of the system in Figure 1 at the level of the terms of $T\Lambda$. To verify this, it is enough to check that $\Gamma, \emptyset \vdash M : TA$ implies that M reduces to some value N after some steps of $\rightarrow_T$, namely: $M \rightarrow_T^* N$, where $N \in TV$. Observe that clause (2) recalls β_v-reduction: in $T\Lambda$ the values are the terms with modal type. Moreover, clause (3) tells us that, in $T\Lambda$, we can replace an arbitrary term N for a variable x, if x has non modal type , *i.e.*, if x will never be neither duplicated nor erased during a reduction of M by means of $\rightarrow_T$. Finally, $T\Lambda$ is a sub-system of the one introduced in (Ronchi della Rocca and Roversi, 1997) which was strongly normalizing. So is $T\Lambda$ with respect to $\rightarrow_T$.

$$\frac{}{T\Gamma, x : A \vdash x : A}(\text{Id})$$

$$\frac{T\Gamma, x : A \vdash M : B}{T\Gamma \vdash T(\lambda x : A.M) : T(A \Longrightarrow B)}(\Longrightarrow I)$$

$$\frac{T\Gamma, \Delta_1 \vdash M : T(A \Longrightarrow B) \quad T\Gamma, \Delta_2 \vdash N : A}{T\Gamma, \Delta_1, \Delta_2 \vdash \mathtt{d}(M)N : B}(\Longrightarrow E)$$

Fig. 1. The typed language $T\Lambda$

$$\begin{array}{rcl} \alpha^\diamond & \mapsto & T\alpha \\ (\sigma \to \tau)^\diamond & \mapsto & T(\sigma^\diamond \Longrightarrow \tau^\diamond) \\ & & \\ x^\diamond & \mapsto & x \\ (\lambda x : \sigma.M)^\diamond & \mapsto & T(\lambda x : \sigma^\diamond.M^\diamond) \\ (MN)^\diamond & \mapsto & \mathtt{d}(M^\diamond)N^\diamond \end{array}$$

Fig. 2. The map from *typed* β_v-calculus to $T\Lambda$

Let see, now, how the rewriting system $\to_T$ allows to simulate the computations of the *typed* $\lambda\beta_v$-calculus, where, by *typed* $\lambda\beta_v$-calculus we mean the simply typed λ-calculus on which β_v-equality is used. To make this simulation explicit it is enough to introduce the (overloaded) function $(.)^\diamond$ in Figure 2. The function $(.)^\diamond$ goes from the types and the terms of typed $\lambda\beta_v$-calculus to the types and the terms of $T\Lambda$. Let σ, τ range over simple types. Let say that, in the *typed* $\lambda\beta_v$-calculus, a variable x is *linear* in M if and only if x is free in M, and x occurs once in every M' such that $M \to_v^* M'$. Observe that the *typed* $\lambda\beta_v$-calculus is strongly normalizing. So, given a variable x, and a term M of *typed* $\lambda\beta_v$-calculus, it is decidable to say whether x is linear in M, or not. We have:

Property 3.2. Let M, N be terms of *typed* $\lambda\beta_v$-calculus.

i) If $(\lambda x : \sigma.M)N \to_v M[N/x]$, then $((\lambda x : \sigma.M)N)^\diamond \to_T (M[N/x])^\diamond$.
ii) If N is not a value of $\lambda\beta_v$-calculus, and x is not linear in M, then $((\lambda x : \sigma.M)N)^\diamond$ is not a redex.
iii) If x is linear in M, then $((\lambda x : \sigma.M)N)^\diamond \to_T (M[N/x])^\diamond$.

Point *i)* of Property 3.2 says that the β_v-reduction of *typed* $\lambda\beta_v$-calculus can be simulated by $\to_T$ of $T\Lambda$. Point *ii)* says that $T\Lambda$ is not enough to model the full call-by-name lambda calculus. Point *iii)* says that the system $\to_T$ contains something more than the system $\to_v$. Indeed, point *iii)* holds because $\to_T$ describes the substitution property of a fragment of Intuitionistic Linear Logic, where substituting any term for a variable is always legal if the variable is linear (see Property 3.1). So $T\Lambda$ can be used as meta-language for studying the semantics of *typed* $\lambda\beta_v$-calculus.

To interpret $T\Lambda$ it is enough to observe that it is typed by a multiplicative and exponential fragment of Intuitionistic Linear Logic if we think of replacing $\multimap$, and ! for $\Longrightarrow$, and T, respectively. Models of such a fragment were introduced in (Benton et al.,

1990) and are symmetric monoidal closed categories, endowed with a monoidal comonad (T, δ, ϵ), such that:

- for every co-free T-coalgebra (TA, δ_A), there are two monoidal natural transformations Dup_A, and E_A which form a commutative comonoid and are coalgebra morphisms,
- for every $f : (TA, \delta_A) \to (TB, \delta_B)$, if f is a coalgebra between co-free coalgebras, then it is also a comonoid morphisms.

In principle, we could require less structure in our model for $T\Lambda$ than the one here above, as the logic encoded by $T\Lambda$ is structurally much simpler than the logic modeled in (Benton et al., 1990). However, we stick to the above class of categories because, as we shall see in the conclusions, we want exploit other results built on such a class.

Now, let extend the system $\to_v$ on *typed* $\lambda\beta_v$-calculus with the rule:

$$(\lambda x : \sigma.M)N \to_l M[N/x] \text{ if and only if } x \text{ is linear in } M \ ,$$

and observe that this extension is still correct with respect to the operational semantics, introduced in Section 2. By Property 3.2, all models of $T\Lambda$ are also models of $\to_v \cup \to_l$, if we use $T\Lambda$ as a meta-language to compile the extension of *typed* $\lambda\beta_v$-calculus here above, using function $(.)^\diamond$ in Figure 2. Moreover, every model of $T\Lambda$ is a model of the η_v-rule.

Now that we know the class of categories for interpreting the type system in Figure 1, so $T\Lambda$, and, hence, the *typed* $\lambda\beta_v$-calculus, we "degenerate" such a class to the untyped case, following the usual pattern to give models to call-by-name lambda calculus. First, we restrict the language of types in (1), by generating it from a single constant D:

$$A, B ::= D \mid T(A \Longrightarrow B) \ .$$

Second, we consider this new language of types up to the congruence:

$$D = T(D \Longrightarrow D) \ . \tag{4}$$

This congruence is analogous to $D = D \to D$, used by Scott on call-by-name lambda calculus, to assign to every of its terms the type D. Note that the congruence $D = D \to D$ can be obtained from (4) by Girard's translation: $(D \to D)^* = T(D^* \Longrightarrow D^*)$, originally given in (Girard, 1987) to translate intuitionistic formulas in intuitionistic linear formulas. So, the class of categories we need to interpret *untyped* $\lambda\beta_v$-calculus, using the *untyped* version of $T\Lambda$, restricts to the following definition of **Cbv** category:

Definition 3.1. **Cbv** is a *call-by-value category* if it is symmetric monoidal closed, being $\odot$ its monoidal product, and $\Longrightarrow$ its *Hom*-sets internalization, such that:

- **Cbv** has a monoidal comonad (T, δ, ϵ),
- for every co-free T-coalgebra (TA, δ_A) of **Cbv**, there are two monoidal natural transformations $Dup_A : TA \dot{\to} TA \odot TA$, and $E_A : TA \dot{\to} I$ which form a commutative comonoid and are coalgebra morphisms,
- for every morphism $f : (TA, \delta_A) \to (TB, \delta_B)$ of **Cbv**, if f is a coalgebra between co-free coalgebras, then it is also a comonoid morphisms,

- **Cbv** has a *model object* $\mathcal{D}$, which has $T(\mathcal{D} \Longrightarrow \mathcal{D})$ as a retract (written $\mathcal{D} \rhd T(\mathcal{D} \Longrightarrow \mathcal{D})$). We mean that there exist $F : \mathcal{D} \to T(\mathcal{D} \Longrightarrow \mathcal{D})$ and $G : T(\mathcal{D} \Longrightarrow \mathcal{D}) \to \mathcal{D}$, such that $F \circ G = id_{T(\mathcal{D} \Longrightarrow \mathcal{D})}$.

In particular, we shall denote the object $(\mathcal{D} \Longrightarrow \mathcal{D})$ by $\mathcal{V}$.

Clearly, moving from a *typed* to an *untyped* $\lambda\beta_v$-calculus, the definition of $\to_l$ becomes undecidable. However, in Section 6, we shall see how to take the behavior of $\to_l$ in account, at a pure semantic level.

Remark 3.1. A particular choice of the monoidal functor T of Definition 3.1 is the identity functor. In this case every object of the category induces a commutative comonoid and it is easy to check that the category contains a cartesian closed category with a retraction $\mathcal{D} \rhd (\mathcal{D} \Longrightarrow \mathcal{D})$. Hence, we have a pseudo-structure, or a *combinatory algebra*, for call-by-name lambda calculus. This is not surprising: every formula provable in the theory of $\lambda\beta_v$-calculus is also provable in the theory of call-by-name lambda calculus. This implies that a model of call-by-name lambda calculus is a particular case of a $\lambda\beta_v$-model. This is the semantic counterpart to the following: syntactically, the call-by-name lambda calculus can be viewed as a $\lambda\beta_v$-calculus where the set of values coincides with the set of all the terms of the calculus.

Remark 3.2. A discussion about models of *typed* $\lambda\beta_v$-calculus, based on translations into a linear calculus is in (Benton and Wadler, 1996). There the linear calculus used as a meta-language to give a meaning to *typed* $\lambda\beta_v$-calculus is the one in (Benton et al., 1990). The discussion is developed by translating *typed* $\lambda\beta_v$-calculus into the linear calculus by using the, so called, call-by-value translation of intuitionistic formulas to intuitionistic linear formulas. It's clear that we use a different meta-language than in (Benton and Wadler, 1996). This choice is motivated by our interest to *typed* $\lambda\beta_v$-calculus just as "bridge" to get to the *untyped* one. Indeed, *untyped* $\lambda\beta_v$-calculus can be obtained from $T\Lambda$, which is typed, by applying a standard *erasure* function for ruling out the types of the terms in $T\Lambda$. Namely, we do what it is usually done on *typed* call-by-name lambda calculus to get its *untyped* version.

4. The categorical pseudo-λ_v-structure

In this section we will prove that every **Cbv** category induces a pseudo-λ_v-structure. First, let us introduce some useful notation. In the sequel we refer simply to "the category", in place of "one category belonging to the class **Cbv**".

Notation 4.1.

- Let $A_1, \ldots, A_n$ be either morphisms or objects of the category. Thanks to the associativity of $\odot$ and the Coherence Theorem (A), $A_1 \odot \cdots \odot A_n$ is an abbreviation for $A_1 \odot (A_2 \odot \cdots (A_{n-1} \odot A_n) \cdots)$ or else for $(\cdots (A_1 \odot A_2) \odot A_3) \cdots \odot A_n$, modulo isomorphisms.
- Let A be either a morphism or an object of the category. By A^n we denote the tensor product $A \odot \cdots \odot A$, n times, where $A^0 = I$ if A is an object, and $A^0 = id_I$ if A is a morphism.

- For all $A_1,\ldots,A_n \in Obj_{\mathbf{Cbv}}$, let $m_{A_1,\ldots,A_n}$ $(n>2)$, be the generalization of $m_{A,B}$ inductively defined as $m_{A_1,\ldots,A_n} = m_{A_1,A_2\odot\cdots\odot A_n} \circ (id_{TA_1} \odot m_{A_2,\ldots,A_n})$, implicitly exploiting the associativity of $\odot$. We define: $m_{nA} : (TA)^n \to T(A^n)$ as

$$\begin{array}{rcl} m_{0A} &=& m_I \\ m_{1A} &=& id_{TA} \\ m_{nA} &=& m_{A,\ldots,A} \quad \text{for } n>1. \end{array}$$

  and $\mathsf{m}_n : (I^n) \to T(I^n)$ as:

$$\begin{array}{rcl} \mathsf{m}_0 = \mathsf{m}_1 &=& m_I \\ \mathsf{m}_n &=& m_{nI} \circ m_I^n \quad \text{for } n>1. \end{array}$$

We now introduce some morphisms useful for defining the interpretation of a term in a concise way. Let notice that for interpreting the terms of $\lambda\beta_v$-calculus we must be able to duplicate environments and to project arguments. In the next definition, the structure of the comonoids in **Cbv** helps us in the definition of projections and duplications.

For all $A_1,\ldots,A_n \in Obj_{\mathbf{Cbv}}$ and for every permutation σ of the sequence $1,\ldots,n$, we call $Exc^{A_1\odot\cdots\odot A_n}_{A_{\sigma(1)}\odot\cdots\odot A_{\sigma(n)}}$ the natural isomorphism between $A_1 \odot \cdots \odot A_n$ and $A_{\sigma(1)} \odot \cdots \odot A_{\sigma(n)}$. The isomorphism is defined using the symmetry isomorphism γ on **Cbv**.

Definition 4.1. Let $A_1,\ldots,A_n \in Obj_{\mathbf{Cbv}}$.
Duplications: Let $A = TA_1 \odot \cdots \odot TA_n$. We define *duplication* $\Delta_A : A \to A \odot A$ as

$$Exc^{A'}_{A\odot A} \circ (Dup_{A_1} \odot \cdots \odot Dup_{A_n}),$$

where $A' = TA_1 \odot TA_1 \odot \cdots \odot TA_n \odot TA_n$. In particular, $\Delta_I : I \to I \odot I$ is defined as $\Delta_I = \lambda_I^{-1} = \rho_I^{-1}$.
Projections: For every $1 \le i \le n$, we define the *projection* $\pi^i_{A_1\odot\cdots\odot A_n} : TA_1 \odot \cdots \odot TA_n \to TA_i$ as

$$iso \circ (E_{A_1} \odot \cdots \odot E_{A_{i-1}} \odot id_{A_i} \odot E_{A_{i+1}} \odot \cdots \odot E_{A_n}),$$

where *iso* stands for the natural isomorphism between $I \odot \cdots \odot I \odot A_i \odot I \odot \cdots \odot I$ and A_i built out of λ and ρ.

We now define the interpretation of the terms of $\lambda\beta_v$-calculus in **Cbv** with a model object $\mathcal{D}$, following (Asperti and Longo, 1991), and (Koymans, 1982). Therefore, we interpret a term M, having free variables $\{x_1,\ldots,x_n\}$, as a morphism from $\mathcal{D}^n$ to $\mathcal{D}$.

Definition 4.2. Let $M \in \Lambda$ such that $\mathcal{FV}(M) \subseteq \{x_1,\ldots,x_n\}$. Let $\mathcal{C}(\mathcal{D})$ be a **Cbv** category with $\mathcal{D}$ as a model object. The *interpretation function* $[\![.]\!]^{\mathcal{C}(\mathcal{D})}$ such that $[\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \in \mathbf{Hom}(\mathcal{D}^n,\mathcal{D})$ is defined by induction on M as follows (remember that $\mathcal{V}$ denotes $(\mathcal{D} \Longrightarrow \mathcal{D})$):

$$[\![x_1,\ldots,x_n \vdash x_i]\!]^{\mathcal{C}(\mathcal{D})} = G \circ \pi^i_{\mathcal{V}^n} \circ F^n, \tag{5}$$

$$[\![x_1,\ldots,x_n \vdash MN]\!]^{\mathcal{C}(\mathcal{D})} = \tag{6}$$
$$ev_{\mathcal{D},\mathcal{D}} \circ ((\epsilon_{\mathcal{V}} \circ F \circ [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \odot [\![x_1,\ldots,x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n,$$

$$[\![x_1,\ldots,x_n \vdash \lambda x.M]\!]^{\mathcal{C}(\mathcal{D})} = \qquad (7)$$
$$G \circ T(\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1,\ldots,x_n,x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ G^n) \circ s_n,$$

where

$r_n = (G^n \odot G^n) \circ \Delta_{(T\mathcal{V})^n} \circ F^n$,
$s_n = m_{n\,T\mathcal{V}} \circ \delta^n_{\mathcal{V}} \circ F^n$.

Clause (5) defines a projection of the i-th variable in the sequence $x_1,\ldots,x_n$. The interpretation of MN, defined by clause (6), is usual. It exploits the monoidal closure, namely, $ev_{\mathcal{D},\mathcal{D}}$ is used for applying the interpretation of M to the one of N. In particular, $\epsilon_{\mathcal{V}}$ extracts the functional behavior of the interpretation of M. Moreover, r duplicates the environment so that it can be given to both the interpretation of M and N. Clause (7) interprets $\lambda x.M$ using the monoidal functor of the comonad T. In this way, the morphism interpreting an abstraction can be both erased and duplicated. Morphism s_n merely serves for correctly composing the interpretation.

Remark 4.1. In clause (7), if $n = 0$, we take $\Lambda_{I,\mathcal{D},\mathcal{D}}([\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ \lambda_{\mathcal{D}}) : I \to \mathcal{V}$, because $[\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \in \mathbf{Hom}(\mathcal{D},\mathcal{D}) \approx \mathbf{Hom}(I \odot \mathcal{D},\mathcal{D}) \ni [\![x \vdash M]\!]^{\mathcal{D}} \circ \lambda_{\mathcal{D}}$.

Now, we are ready to show the following:

Theorem 4.1. Let $\mathcal{C}(\mathcal{D})$ be a **Cbv** category. Then, $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ is a pseudo-λ_v-structure.

The proof of Theorem 4.1 consists in checking that the construction of $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$, as in Definition 4.3 here below, yields what we want. Those interested to the whole proof can find it in Subsection 4.1.

Definition 4.3. Let $\mathcal{C}(\mathcal{D})$ be denoting a **Cbv** category with a model object $\mathcal{D}$. The **Cbv**-model $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ built on $\mathcal{C}(\mathcal{D})$ is

$$\mathcal{M}^{\mathcal{C}(\mathcal{D})} = \left\langle S^{\mathcal{C}(\mathcal{D})}, V^{\mathcal{C}(\mathcal{D})}, \bullet^{\mathcal{C}(\mathcal{D})}, \mathcal{I}^{\mathcal{C}(\mathcal{D})} \right\rangle \quad ,$$

where:

- $S^{\mathcal{C}(\mathcal{D})} = \mathbf{Hom}(I,\mathcal{D})$ (Notice that $\mathbf{Hom}(I,\mathcal{D}) \approx \mathbf{Hom}(I^n,\mathcal{D})$ for all $n \geq 1$.)
- $V^{\mathcal{C}(\mathcal{D})} = \{f \mid f \in \mathbf{Hom}(I,\mathcal{D}) \text{ and } \exists h \in \mathbf{Hom}(I,\mathcal{V}).f = G \circ Th \circ m_I\}$,
- $f \bullet^{\mathcal{C}(\mathcal{D})} g = ev_{\mathcal{D},\mathcal{D}} \circ ((\epsilon_{\mathcal{V}} \circ F \circ f) \odot g)$, for every pair of morphisms $f, g \in \mathbf{Hom}(I,\mathcal{D})$,
- $\mathcal{I}^{\mathcal{C}(\mathcal{D})}\theta[M] = [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n))$, where $\mathcal{FV}(M) \subseteq dom(\theta) = \{x_1,\ldots,x_n\}$. We call every $\theta(x_i)$ *environment component*. Since θ maps variables to values, every environment component $\theta(x_i)$ is of the form $G \circ Th_i \circ m_I$ for some h_i. As consequence of the definition of the set of semantic values $V^{\mathcal{C}(\mathcal{D})}$, an interpretation $[\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}$ is a value iff $\mathcal{I}^{\mathcal{C}(\mathcal{D})}\theta[M] = G \circ Th \circ \mathsf{m}_n$ for each θ and for some $h : I^n \to \mathcal{V}$.

4.1. *From a **Cbv** category to a pseudo-λ_v-structure: details*

This part has mainly a technical content. It is devoted to show formally that every **Cbv** category induces a pseudo-λ_v-structure.

Before developing the proof, we need a couple of lemmas.

Lemma 4.1. Let M be a term such that $\mathcal{FV}(M) \subseteq \{x_1,\ldots,x_n\}$. Let $1 \leq i \leq n-1$ and $x_{n+1} \notin \mathcal{FV}(M)$. The following equations hold:
(exchange) $[\![x_1,\ldots,x_i,x_{i+1},\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} =$
$= [\![x_1,\ldots,x_{i+1},x_i,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (id_{\mathcal{D}}^{i-1} \odot \gamma_{\mathcal{D},\mathcal{D}} \odot id_{\mathcal{D}}^{n-i-1})$.
(weakening) $[\![x_1,\ldots,x_n,x_{n+1} \vdash M]\!]^{\mathcal{C}(\mathcal{D})} =$
$= [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^n} \circ (id_{\mathcal{D}}^n \odot (E_{\mathcal{V}} \circ F))$.

Proof. Exchange can be proved by induction on M substantially using the naturality of γ.

Much work must be done for proving *weakening*. We proceed by induction on M.
Let $M = x_i$, for $1 \leq i \leq n-1$.

$$\begin{aligned}
&[\![x_1,\ldots,x_n,x_{n+1} \vdash x_i]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = G \circ \pi^i_{\mathcal{V}^{n+1}} \circ F^{n+1} \\
&\quad \text{(by naturality of } \rho) \\
&\quad = G \circ \pi^i_{\mathcal{V}^n} \circ \rho_{(T\mathcal{V})^n} \circ (id^n_{T\mathcal{V}} \odot E_{\mathcal{V}}) \circ F^{n+1} \\
&\quad = G \circ \pi^i_{\mathcal{V}^n} \circ \rho_{(T\mathcal{V})^n} \circ (F^n \odot (E \circ F)) \\
&\quad \text{(by naturality of } \rho) \\
&\quad = G \circ \pi^i_{\mathcal{V}^n} \circ F^n \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)) \\
&\quad = [\![x_1,\ldots,x_n \vdash x_i]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E_{\mathcal{V}} \circ F)).
\end{aligned}$$

Let $M = PQ$.

$$\begin{aligned}
&[\![x_1,\ldots,x_n,x_{n+1} \vdash PQ]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = ev \circ ((\epsilon \circ F \circ [\![x_1,\ldots,x_n,x_{n+1} \vdash P]\!]^{\mathcal{C}(\mathcal{D})}) \odot \\
&\qquad\qquad [\![x_1,\ldots,x_n,x_{n+1} \vdash Q]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_{n+1} \\
&\quad \text{(by inductive hypothesis)} \\
&\quad = ev \circ ((\epsilon \circ F \circ [\![x_1,\ldots,x_n \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F))) \odot \\
&\qquad\qquad ([\![x_1,\ldots,x_n \vdash Q]\!]^{\mathcal{C}(\mathcal{D})} \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)))) \circ r_{n+1} \\
&\quad = ev \circ ((\epsilon \circ F \circ [\![x_1,\ldots,x_n \vdash P]\!]^{\mathcal{C}(\mathcal{D})}) \odot \\
&\qquad\qquad [\![x_1,\ldots,x_n \vdash Q]\!]^{\mathcal{C}(\mathcal{D})}) \circ (\rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)))^2 \circ r_{n+1},
\end{aligned}$$

to conclude, it is sufficient to show:

$$(\rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)))^2 \circ r_{n+1} = r_n \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)).$$

Without loss of generality, we proceed for $n = 1$:

$$\begin{aligned}
&(\rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)))^2 \circ r_2 = \\
&\quad = (\rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)))^2 \circ (id_{T\mathcal{V}} \odot \gamma_{T\mathcal{V},T\mathcal{V}} \odot id_{T\mathcal{V}}) \circ Dup^2 \circ F^2 \\
&\quad \text{(by naturality of } \gamma)
\end{aligned}$$

$= \rho^2_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot \gamma_{\mathcal{D},I} \odot id_{\mathcal{D}}) \circ (G^2 \odot E^2) \circ Dup^2 \circ F^2$
(by the comonoid and naturality of λ^{-1})
$= \rho^2_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot \gamma_{\mathcal{D},I} \odot id_{\mathcal{D}}) \circ (G^2 \odot \lambda_I^{-1}) \circ (Dup \odot E) \circ F^2$
(by naturality of ρ and definition of γ)
$= G^2 \circ \rho_{(T\mathcal{V})^2} \circ (Dup \odot E) \circ F^2$
(by naturality of ρ)
$= \rho_{\mathcal{D}^2} \circ (G^2 \odot id_I) \circ (Dup \odot E) \circ F^2$
$= \rho_{\mathcal{D}^2} \circ ((G^2 \circ Dup \odot F) \odot (E \circ F))$
$= \rho_{\mathcal{D}^2} \circ (r_1 \odot (E \circ F))$
(by naturality of ρ)
$= r_1 \circ \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F))$.

Let $M = \lambda x.P$.

$[\![x_1,\ldots,x_n,x_{n+1} \vdash \lambda x.P]\!]^{\mathcal{C}(\mathcal{D})} =$
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x_{n+1},x \vdash P]\!]^{\mathcal{C}(\mathcal{D})}) \circ G^{n+1}) \circ s_{n+1}$
(using *exchange*)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x,x_{n+1} \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ (id^n_{\mathcal{D}} \odot \gamma_{\mathcal{D},\mathcal{D}})) \circ G^{n+1}) \circ s_{n+1}$
(by inductive hypothesis)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^{n+1}} \circ (id^{n+1}_{\mathcal{D}} \odot (E \circ F)) \circ$
$(id^n_{\mathcal{D}} \odot \gamma_{\mathcal{D},\mathcal{D}})) \circ G^{n+1}) \circ s_{n+1}$
(by naturality of γ)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^{n+1}} \circ$
$(id^n_{\mathcal{D}} \odot (\gamma_{I,\mathcal{D}} \circ ((E \circ F) \odot id_{\mathcal{D}}))\circ) \circ G^{n+1}) \circ s_{n+1}$
(since $\rho_{\mathcal{D}} \circ \gamma_{I,\mathcal{D}} = \lambda_{\mathcal{D}}$ and $\rho_{\mathcal{D}^{n+1}} = id_{\mathcal{D}^n} \odot \rho_{\mathcal{D}}$)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})}$
$\circ (id_{\mathcal{D}^n} \odot (\lambda_{\mathcal{D}} \circ ((E \circ F) \odot id_{\mathcal{D}})))) \circ G^{n+1}) \circ s_{n+1}$
(let us suppose $n > 0$. If $n = 0$ the proof is simpler and uses Remark 4.1)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^{n+1}} \circ$
$(id^{n-1}_{\mathcal{D}} \odot \rho_{\mathcal{D}}) \circ (id^n_{\mathcal{D}} \odot (E \circ F) \odot id_{\mathcal{D}})) \circ G^{n+1}) \circ s_{n+1}$
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^{n+1}} \circ$
$(id^{n-1}_{\mathcal{D}} \odot (\rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F))) \odot id_{\mathcal{D}})) \circ G^{n+1}) \circ s_{n+1}$
(by naturality of Λ and functoriality of T)
$= G \circ T(\Lambda([\![x_1,\ldots,x_n,x \vdash P]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}^{n+1}})) \circ$
$T(\rho_{\mathcal{D}^n} \circ (id_{\mathcal{D}^n} \odot (E \circ F)) \circ G^{n+1}) \circ s_{n+1}$,

to conclude it is sufficient to prove that:

$$T(\rho_{\mathcal{D}^n} \circ (id_{\mathcal{D}^n} \odot (E \circ F)) \circ G^{n+1}) \circ s_{n+1} = T(G^n) \circ s_n \circ \rho_{\mathcal{D}^n} \circ (id_{\mathcal{D}^n} \odot (E \circ F)).$$

Step by step:

$$T(\rho_{\mathcal{D}^n} \circ (id_{\mathcal{D}^n} \odot (E \circ F)) \circ G^{n+1}) \circ s_{n+1} =$$
$$= T(\rho_{\mathcal{D}^n}) \circ T(G^n \odot E) \circ m_{(n+1)T\mathcal{V}} \circ \delta^{n+1} \circ F^{n+1}$$
(by naturality of $m_{\Box,\diamond}$)
$$= T(\rho_{\mathcal{D}^n}) \circ m_{\mathcal{D}^n,I} \circ (m_{n\mathcal{D}} \odot id_{TI}) \circ ((TG)^n \odot TE) \circ \delta^{n+1} \circ F^{n+1}$$
$$= T(\rho_{\mathcal{D}^n}) \circ m_{\mathcal{D}^n,I} \circ (m_{n\mathcal{D}} \odot id_{TI}) \circ ((TG \circ \delta)^n \odot (TE \circ \delta)) \circ F^{n+1}$$
(since E_A is an element of T-$\mathbf{coalg}_{\mathbf{Cbv}}((TA, \delta_A), (I, m_I)))$
$$= T(\rho_{\mathcal{D}^n}) \circ m_{\mathcal{D}^n,I} \circ (m_{n\mathcal{D}} \odot id_{TI}) \circ ((TG \circ \delta)^n \odot (m_I \circ E)) \circ F^{n+1}$$
$$= T(\rho_{\mathcal{D}^n}) \circ m_{\mathcal{D}^n,I} \circ (m_{n\mathcal{D}} \odot id_{TI}) \circ ((m_{n\mathcal{D}} \circ (TG \circ \delta)^n) \odot E) \circ F^{n+1}$$
(by monoidality of T)
$$= \rho_{T(\mathcal{D}^n)} \circ ((m_{n\mathcal{D}} \circ (TG \circ \delta)^n) \odot E) \circ F^{n+1}$$
(by naturality of ρ)
$$= m_{n\mathcal{D}} \circ (TG \circ \delta)^n) \circ F^n \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F))$$
(by naturality of $m_{\Box,\diamond}$)
$$= T(G^n) \circ m_{nT\mathcal{V}} \circ \delta^n \circ F^n \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F))$$
$$= T(G^n) \circ s_n \circ \rho_{\mathcal{D}^n} \circ (id^n_{\mathcal{D}} \odot (E \circ F)).$$

□

Lemma 4.2. Let r_n be as in Definition 4.2. The interpretation of the application of a lambda-abstraction $(\lambda x.M)$ to a generic term N can have one of the forms:
for $n > 0$: $[\![x_1, \dots, x_n \vdash (\lambda x.M)N]\!]^{\mathcal{C}(\mathcal{D})} =$
$= [\![x_1, \dots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (id^n_{\mathcal{D}} \odot [\![x_1, \dots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$,
for $n = 0$: $[\![\vdash (\lambda x.M)N]\!]^{\mathcal{C}(\mathcal{D})} = [\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ [\![\vdash N]\!]^{\mathcal{C}(\mathcal{D})}$.

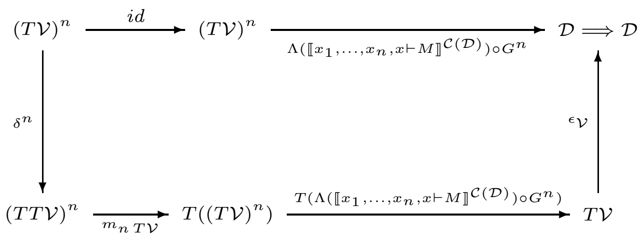
Proof. We proceed step by step. For $n > 0$:

$$[\![x_1, \dots, x_n \vdash (\lambda x.M)N]\!]^{\mathcal{C}(\mathcal{D})} =$$
$$= ev \circ ((\epsilon \circ F \circ [\![x_1, \dots, x_n \vdash \lambda x.M]\!]^{\mathcal{C}(\mathcal{D})}) \odot [\![x_1, \dots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$$
(by Diagram 8 below)
$$= ev \circ ((\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1, \dots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ G^n \circ F^n) \odot [\![x_1, \dots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$$
$$= ev \circ (\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1, \dots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \odot id) \circ ((G^n \circ F^n) \odot [\![x_1, \dots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$$
(by naturality of Λ)
$$= [\![x_1, \dots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ ((G^n \circ F^n) \odot [\![x_1, \dots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$$

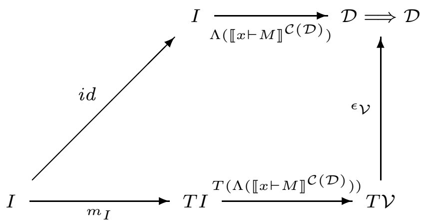
(collapsing $G^n \circ F^n$ in r_n, exploiting $F \circ G = id$)
$= [\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (id^n \odot [\![x_1, \ldots, x_n \vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$

For $n = 0$ we have:

$$
\begin{aligned}
&[\![\vdash (\lambda x.M)N]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = ev \circ ((\epsilon \circ F \circ [\![\vdash \lambda x.M]\!]^{\mathcal{C}(\mathcal{D})}) \odot [\![\vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ \lambda_I^{-1} \\
&\quad \text{(by Diagram 9 below)} \\
&\quad = ev \circ ((\Lambda_{I,\mathcal{D},\mathcal{D}}([\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ \lambda_{\mathcal{D}})) \odot [\![\vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ \lambda_I^{-1} \\
&\quad = [\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ \lambda_{\mathcal{D}} \circ (id_I \odot [\![\vdash N]\!]^{\mathcal{C}(\mathcal{D})}) \circ \lambda_I^{-1} \\
&\quad \text{(by naturality of } \lambda) \\
&\quad = [\![x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ [\![\vdash N]\!]^{\mathcal{C}(\mathcal{D})}.
\end{aligned}
$$

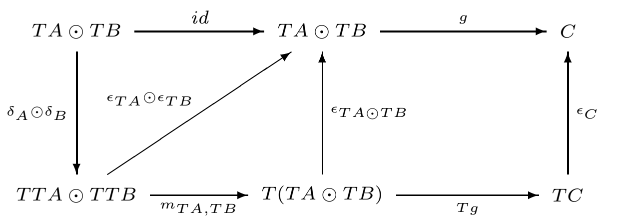


(8)

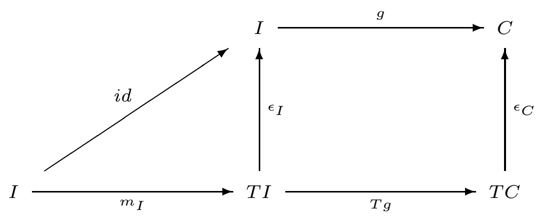


(9)

Diagrams 8 and 9 commute because they are essentially instances of Diagrams 10 and 11 below, that can be proved to commute using the comonad and both the naturality and the monoidality of ϵ.



(10)



(11)

□

Finally, the proof of Theorem 4.1:

Proof. We shall prove that $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ satisfies the first part of Definition 2.4. We skip the index $\mathcal{C}(\mathcal{D})$ for sake of clarity. To prove condition *var* we use the definition of π and both the naturality and the monoidality of E. Condition *app* comes from the definition of $\bullet$ essentially using the naturality and the monoidality of *Dup*. Condition *eval* is proved as follows. Let $\mathcal{FV}(\lambda x.M) = \{x_1, \ldots, x_n\}$, and denote every $\theta(x_i)$ $(i = 1, \ldots, n)$ by θ_i. Let also $d \in V$. Step by step, we have that:

$$
\begin{aligned}
&\mathcal{I}\theta[\lambda x.M] \bullet d = \\
&\quad = ev \circ ((\epsilon \circ F \circ [\![x_1, \ldots, x_n \vdash \lambda x.M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta_1 \odot \cdots \odot \theta_n)) \odot d) \\
&\quad \text{(by Diagram 8 in the proof of Lemma 4.2} \\
&\quad\quad \text{and exploiting the fact that } \theta_i\text{'s have form } G \circ \cdots) \\
&\quad = ev \circ ((\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ (\theta_1 \odot \cdots \odot \theta_n)) \odot d) \\
&\quad = ev \circ (\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \odot id) \circ ((\theta_1 \odot \cdots \odot \theta_n) \odot d) \\
&\quad = [\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta_1 \odot \cdots \odot \theta_n \odot d) = \mathcal{I}\theta^d_x[M].
\end{aligned}
$$

Conditions *ceq* follows from Lemma 4.1, while condition $\overline{\alpha}$ is easily satisfied. To show condition *abs*, assuming $\theta_i = G \circ Th_i \circ m_I (1 \leq i \leq n)$, and

$$f = \Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}([\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ G^n :$$

$$
\begin{aligned}
&\mathcal{I}\theta[\lambda x.M] = \\
&\quad = G \circ Tf \circ m_{nT\mathcal{V}} \circ \delta^n_{\mathcal{V}} \circ F^n \circ (\theta_1 \odot \cdots \odot \theta_n) \\
&\quad = G \circ Tf \circ m_{nT\mathcal{V}} \circ ((\delta_{\mathcal{V}} \circ Th_1 \circ m_I) \odot \cdots \odot (\delta_{\mathcal{V}} \circ Th_n \circ m_I)) \\
&\quad \text{(by Naturality of } \delta) \\
&\quad = G \circ Tf \circ m_{nT\mathcal{V}} \circ ((TTh_1 \circ \delta_I \circ m_I) \odot \cdots \odot (TTh_n \circ \delta_I \circ m_I)) \\
&\quad \text{(by monoidality of } \delta) \\
&\quad = G \circ Tf \circ m_{nT\mathcal{V}} \circ ((TTh_1 \circ Tm_I \circ m_I) \odot \cdots \odot (TTh_n \circ Tm_I \circ m_I)) \\
&\quad = G \circ Tf \circ m_{nT\mathcal{V}} \circ (T(Th_1 \circ m_I) \odot \cdots \odot T(Th_n \circ m_I)) \circ m^n_I \\
&\quad \text{(by Naturality of } m_{A,B}) \\
&\quad = G \circ Tf \circ T((Th_1 \circ m_I) \odot \cdots \odot (Th_n \circ m_I)) \circ m_{nI} \circ m^n_I \\
&\quad \text{(by definition of } \mathsf{m}) \\
&\quad = G \circ T(f \circ ((Th_1 \circ m_I) \odot \cdots \odot (Th_n \circ m_I))) \circ \mathsf{m}_n,
\end{aligned}
$$

hence, we have the form of a value. □

5. The categorical λ_v-model

It is well known that a cartesian closed category with a reflexive object, which is a pseudo-λ-structure, is a λ-model, *i.e.*, a model for the *untyped* call-by-name lambda calculus, if it has *enough points* (Koymans, 1982). We prove that a similar condition is required to have a model of $\lambda\beta_v$-calculus. Namely, a **Cbv** category, satisfies also condition $\overline{\xi}$ in Definition 2.4 if it has *enough values.* This means that two morphisms in the model object $\mathcal{D}$ of a pseudo-λ_v-structure are different only if they have a different behavior on, at least, one *value.* More compactly:

Definition. A **Cbv** category $\mathcal{C}(\mathcal{D})$ has enough values if, and only if,

$$\forall f, g : \mathcal{D} \to \mathcal{D}.\exists h \in \mathbf{HOM}(I, \mathcal{D} \Longrightarrow \mathcal{D})(f \neq g \Rightarrow f \circ (G \circ Th \circ m_I) \neq g \circ (G \circ Th \circ m_I)) \ .$$

This property is the natural restriction of "having enough points" to the case where the β-rule is restricted to arguments which are only values.

In fact, for proving the theorem here below we need a more general form of the definition here above, because we manage morphisms from a tensor product of $\mathcal{D}$ to $\mathcal{D}$.

Definition 5.1. A **Cbv** category has enough values if, and only if:

$$\forall n \geq 1.\forall i \leq n.\forall f, g : I^i \odot \mathcal{D} \odot I^{(n-i-1)} \to \mathcal{D}.\exists h \in \mathbf{HOM}(I, \mathcal{D} \Longrightarrow \mathcal{D}).$$
$$(f \neq g \Rightarrow f \circ id_I^i \odot (G \circ Th \circ m_I) \odot id_I^{(n-i-1)} \neq g \circ id_I^i \odot (G \circ Th \circ m_I) \odot id_I^{(n-i-1)}).$$

Theorem 5.1. Let $\mathcal{C}(\mathcal{D})$ be a **Cbv** category with enough values. The pseudo-λ_v-structure $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ (as defined in Definition 4.3) is a λ_v-model.

Proof. We must prove Condition $\overline{\xi}$ of Definition 2.4. Let M and N be two terms. If $\forall d \in V.\mathcal{I}\theta_x^d[M] = \mathcal{I}\theta_x^d[N]$, this means that, using the notation introduced in Definition 4.3 and in the proof of Theorem 4.1,
$[\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot d) =$
$[\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}) \circ (id_I \odot \cdots \odot id_I \odot d) =$
$[\![x_1, \ldots, x_n, x \vdash N]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}) \circ (id_I \odot \cdots \odot id_I \odot d).$
Since $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ has enough values, we have
$[\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}) =$
$[\![x_1, \ldots, x_n, x \vdash N]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}).$ (1)
The thesis, $\mathcal{I}\theta[\lambda x.M] = \mathcal{I}\theta[\lambda x.N]$, holds in the following way:
$\mathcal{I}\theta[\lambda x.M] = G \circ T(\Lambda([\![x_1, \ldots, x_n, x \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}))) \circ \mathsf{m}_k,$
$\mathcal{I}\theta[\lambda x.N] = G \circ T(\Lambda([\![x_1, \ldots, x_n, x \vdash N]\!]^{\mathcal{C}(\mathcal{D})} \circ (\theta(x_1) \odot \cdots \odot \theta(x_n) \odot id_{\mathcal{D}}))) \circ \mathsf{m}_k,$
using the same steps in the proof of Theorem 4.1 and the naturality of $\Lambda_{\mathcal{D}^n,\mathcal{D},\mathcal{D}}$, i.e. $\Lambda(g \circ (h \odot id)) = \Lambda(g) \circ h$. Then we exploit (1). □

Remark 5.1. Remark 3.1 highlights that a cartesian closed category is an instance of **Cbv**. This implies that every pseudo-structure for the call-by-name λ-calculus is a pseudo-λ_v-structure. However, this does not imply the contrary, namely, that every model of the call-by-name λ-calculus is a λ_v-model as well. This because the condition of having enough values is stronger than the requirement of having enough points. This reflects the fact that condition $\overline{\xi}$ in Definition 2.4 is stronger than the corresponding condition defining a model for the call-by-name λ-calculus. For example, let $\mathcal{D}$ be the Scott Domain which is the least solution of:

$$D \rhd (D \to D) \oplus \{\bot, \top\} \ ,$$

where $\oplus$ is the smash sum, $\perp$ is smaller than $\top$, and $D \to D$ is the domain of the continuous functions from D to D. The domain $\mathcal{D}$ is a cartesian closed category with enough points, and can be used as a model for the call-by-name λ-calculus. On the contrary, it has not enough values to interpret the $\lambda\beta_v$-calculus, using the domain $\mathcal{V} = \mathcal{D} \to \mathcal{D}$ as the natural choice to represent the set of the semantic values. Indeed, the two points f and g of $\mathcal{D}$, representing the two step functions:

$$\lambda x \in \mathcal{D}.\mathtt{if}\ x = \top\ \mathtt{then}\ d_1\ \mathtt{else}\ d'\ ,$$

and

$$\lambda x \in \mathcal{D}.\mathtt{if}\ x = \top\ \mathtt{then}\ d_2\ \mathtt{else}\ d'\ ,$$

respectively, being d_1 and d_2 incomparable in $\mathcal{D}$, are different, but equal on every value of $\mathcal{V}$. Of course, this does not say that $\mathcal{D}$ can not yield a λ_v-model. Indeed, it can be the case that all the step functions like f and g can never be in the interpretation of any terms of $\lambda\beta_v$-calculus. However, this can only be checked with an *ad hoc* study of the interpretation.

6. Extensionality

The notion of *extensionality* in a given semantics is relative to the extensional behavior of the applicative structure. If an applicative structure $\langle D, \bullet, \mathcal{I}\rangle$ is a model for the classical lambda calculus, then the extensionality, syntactically corresponding to the η-equality, can be expressed in the usual way: for all $d_1, d_2 \in D$, if for all $d_3 \in D$ we have $d_1 \bullet d_3 = d_2 \bullet d_3$, then $d_1 = d_2$. Recall that the extensional models for the $\lambda\beta$-calculus are all and only models of the $\beta\eta$-equality. In a call-by-value setting, instead, the extensionality is a property concerning the behavior of a "function" w.r.t. the values. Namely:

Definition 6.1. A pseudo-λ_v-structure $\langle S, V, \bullet, \mathcal{I}\rangle$ is *extensional* iff:

$$\forall d_1, d_2 \in S.((\forall v \in V.\, d_1 \bullet v = d_2 \bullet v) \Longrightarrow d_1 = d_2).$$

Since the extensionality of a pseudo-λ_v-structure implies condition $\overline{\xi}$ of Definition 2.4, we have:

Proposition 6.1. Every extensional pseudo-λ_v-structure is an extensional λ_v-model. □

Definition 6.2. A λ_v-model $\mathcal{M} = \langle S, V, \bullet, \mathcal{I}\rangle$ is a $\lambda\eta_v$-model if for every pair of terms M, N,

$$\lambda\beta_v\eta_v \vdash M =_v N \quad\Longrightarrow\quad \mathcal{M} \models M =_v N.$$

An obvious result is that every extensional λ_v-model is a $\lambda\eta_v$-model. Differently from what happens for models of the classical lambda calculus, the opposite is not always true: there are non-extensional $\lambda\eta_v$-models, as we will see in Example 8.1.

We now prove that the categorical λ_v-model $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ of the previous section, where the retraction $\mathcal{D} \triangleright T(\mathcal{D} \Longrightarrow \mathcal{D})$ is an isomorphism, namely, $G \circ F = id_{\mathcal{D}}$, is, in fact, a $\lambda\eta_v$-model.

Theorem 6.1. A categorical λ_v-model $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ such that $\mathcal{D} \approx T(\mathcal{D} \Longrightarrow \mathcal{D})$, is a $\lambda\eta_v$-model.

Proof. It is sufficient to prove that $\mathcal{I}\theta[\lambda x.yx] = \mathcal{I}\theta[y]$, for any θ,. Applying the interpretation function: $[\![y \vdash \lambda x.yx]\!]^{\mathcal{C}(\mathcal{D})} = G \circ T(\Lambda_{\mathcal{D},\mathcal{D},\mathcal{D}}([\![y, x \vdash yx]\!]^{\mathcal{C}(\mathcal{D})}) \circ G) \circ \delta \circ F$. Let us see the form of $[\![y, x \vdash yx]\!]^{\mathcal{C}(\mathcal{D})}$. By Lemma 4.1,

$$\begin{aligned}
&[\![y, x \vdash y]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = [\![y \vdash y]\!]^{\mathcal{C}(\mathcal{D})} \circ \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)) \\
&\quad = G \circ F \circ \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)) \\
&\quad = \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)).
\end{aligned}$$

$$\begin{aligned}
&[\![y, x \vdash x]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = [\![x, y \vdash x]\!]^{\mathcal{C}(\mathcal{D})} \circ \gamma_{\mathcal{D},\mathcal{D}} \\
&\quad = \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E \circ F)) \circ \gamma_{\mathcal{D},\mathcal{D}} \\
&\quad \text{(by naturality of } \gamma) \\
&\quad = \rho_{\mathcal{D}} \circ \gamma_{I,\mathcal{D}} \circ ((E \circ F) \odot id_{\mathcal{D}}) \\
&\quad = \lambda_{\mathcal{D}} \circ ((E \circ F) \odot id_{\mathcal{D}}),
\end{aligned}$$

hence, we have

$$\begin{aligned}
&[\![y, x \vdash yx]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F \circ \rho_{\mathcal{D}} \circ (id_{\mathcal{D}} \odot (E_{\mathcal{V}} \circ F))) \odot (\lambda_{\mathcal{D}} \circ ((E_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}))) \circ r_2 \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}) \circ (\rho_{\mathcal{D}} \odot \lambda_{\mathcal{D}}) \circ (id_{\mathcal{D}} \odot (E \circ F)^2 \odot id_{\mathcal{D}}) \circ \\
&\qquad\qquad (G^2 \odot G^2) \circ (id_{T\mathcal{V}} \odot \gamma_{T\mathcal{V},T\mathcal{V}} \odot id_{T\mathcal{V}}) \circ (Dup \odot Dup) \circ F^2 \\
&\quad \text{(by naturality of } \gamma) \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}) \circ (\rho_{\mathcal{D}} \odot \lambda_{\mathcal{D}}) \circ (G \odot id_I^2 \odot G) \circ \\
&\qquad\qquad (id_{T\mathcal{V}} \odot \gamma_{I,I} \odot id_{T\mathcal{V}}) \circ (Dup \odot Dup) \circ F^2 \\
&\quad \text{(by the comonoid and the fact that } \gamma_{I,I} = id_{I \odot I}) \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}) \circ (\rho_{\mathcal{D}} \odot \lambda_{\mathcal{D}}) \circ (G \odot id_I^2 \odot G) \circ (\rho_{T\mathcal{V}}^{-1} \odot \lambda_{T\mathcal{V}}^{-1}) \circ F^2 \\
&\quad \text{(by naturality of } \rho \text{ and } \lambda) \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}) \circ (\rho_{\mathcal{D}} \odot \lambda_{\mathcal{D}}) \circ (\rho_{\mathcal{D}}^{-1} \odot \lambda_{\mathcal{D}}^{-1}) \circ G^2 \circ F^2 \\
&\quad = ev \circ ((\epsilon_{\mathcal{V}} \circ F) \odot id_{\mathcal{D}}).
\end{aligned}$$

Substituting:

$$\begin{aligned}
&[\![y \vdash \lambda x.yx]\!]^{\mathcal{C}(\mathcal{D})} = \\
&\quad = G \circ T(\Lambda_{\mathcal{D},\mathcal{D},\mathcal{D}}(ev \circ ((\epsilon \circ F) \odot id)) \circ G) \circ \delta \circ F \\
&\qquad \text{(by naturality of } \Lambda) \\
&\quad = G \circ T(\epsilon \circ F \circ G) \circ \delta \circ F \\
&\quad = G \circ T(\epsilon) \circ \delta \circ F
\end{aligned}$$

(by the comonad)

$= G \circ F = [\![y \vdash y]\!]^{\mathcal{C}(\mathcal{D})}$.

□

7. Incompleteness of the Cbv models

Recall that the class of models of the $\lambda\beta_v$-calculus we defined has been obtained by starting from the definition of model for its typed version. We said in Section 3 that every model for the typed $\lambda\beta_v$-calculus is closed under the congruence induced by (the typed version of) the β_v-rule, of the η_v-rule, and of the $\rightarrow_l$. This fact has the following consequence at the untyped level. Let us consider the term $\lambda x.x$ which is a term trivially linear, according to Property 3.2. Here below, Proposition 7.1 tells us that $\lambda x.x$ is interpreted as the identity in every model of the $\lambda\beta_v\eta_v$-calculus, no matter what its arguments are:

Proposition 7.1. Let $\mathbf{I} \equiv \lambda z.z$ and let $M \in \Lambda$ a generic term (may be not valuable) such that $\mathcal{FV}(M) \subseteq \{x_1,\ldots,x_n\}$. In the categorical $\lambda\eta_v$-model $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$,

$$[\![x_1,\ldots,x_n \vdash \mathbf{I}M]\!]^{\mathcal{C}(\mathcal{D})} = [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}.$$

Proof. Step by step:

$[\![x_1,\ldots,x_n \vdash \mathbf{I}M]\!]^{\mathcal{C}(\mathcal{D})} =$

(by Lemma 4.2)

$= [\![x_1,\ldots,x_n, z \vdash z]\!]^{\mathcal{C}(\mathcal{D})} \circ (id^n_{\mathcal{D}} \odot [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ r_n$

$= G \circ \pi^{n+1}_{\mathcal{V}^{n+1}} \circ F^{n+1} \circ (id^n_{\mathcal{D}} \odot [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}) \circ$
$\circ (G^n \odot G^n) \circ \Delta_{(T\mathcal{V})^n} \circ F^n$

$= G \circ iso \circ ((E \circ F)^n \odot (F \circ [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})})) \circ$
$\circ (G^n \odot G^n) \circ \Delta_{(T\mathcal{V})^n} \circ F^n$

$= G \circ iso \circ (id^n_I \odot (F \circ [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})})) \circ (E^n \odot G^n) \circ \Delta_{(T\mathcal{V})^n} \circ F^n$

(Exploiting the technique used in the proof of Theorem 6.1
and the coherence theorem)

$= G \circ F \circ [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})} \circ G^n \circ F^n$

(Since $\mathcal{D} \approx T\mathcal{V}$)

$= [\![x_1,\ldots,x_n \vdash M]\!]^{\mathcal{C}(\mathcal{D})}$.

□

Notice that the preceding property is correct with the operational semantics induced by SECD machine, as $IM \sim_v M$, for any M, and has the following important theorem as its corollary:

Theorem 7.1. The class of **Cbv** models satisfying $\beta_v\eta_v$-equality is incomplete with respect to the class of $\beta_v\eta_v$-theories.

An example of a model of $\beta_v\eta_v$-equality, for which Proposition 7.1 does not hold is the model defined in (Honsell and Lenisa, 1993). It is based on the *-Coherence Spaces, which are a variant of Girard's Coherence Spaces. In such a model, $\mathbf{I}M \neq M$, if M is not valuable.

8. Instances of Cbv

The definition of our categorical $\lambda\eta_v$-model has some models of $\lambda\beta_v\eta_v$-calculus as its instances .

8.1. *An instance of **Cbv** in Scott Domains*

In this subsection we prove that every model of $\lambda\beta_v$-calculus, belonging to the class defined in (Dezani-Ciancaglini et al., 1986), is a categorical $\lambda\eta_v$-model.

Let **CPOS** be the category such that:

- the objects are the *complete partial orders* (cpo) or *Scott domains*,
- the morphisms are the *strict continuous functions*, namely those continuous functions that always take the bottom element of the source object to the bottom element of the target object.

Let D_1, D_2 be two cpos. $(D_1 \to_\perp D_2)$ is the cpo of the strict continuous functions from D_1 to D_2 ordered point wise. We denote with $\perp_D$ the bottom element of a cpo D. The bottom element of $(D_1 \to_\perp D_2)$ is the function constantly equal to $\perp_{D_2}$. Moreover, with $D_\perp$ we denote the cpo (the *lifted* of D) obtained from D adding a new bottom element $\perp$.

Lemma 8.1. Let $\mathbf{CPOS}^{\mathcal{D}}$ be the category **CPOS** equipped with a retraction $\mathcal{D} \rhd (\mathcal{D} \to_\perp \mathcal{D})_\perp$. The category $\mathbf{CPOS}^{\mathcal{D}}$ is a **Cbv** category.

Proof.

- $\odot$ is the *smash product*:

$$D_1 \odot D_2 = \{\langle d_1, d_2\rangle \mid d_1 \in D_1, d_2 \in D_2, d_1 \neq \perp_{D_1}, d_2 \neq \perp_{D_2}\} \cup \{\perp_{D_1 \odot D_2}\} \ ,$$

  with unit $I = \{\perp, \mathtt{1}\}$, being $\perp$ smaller than $\mathtt{1}$, and for any $f : D_1 \to D_2$, and $g : D_3 \to D_4$:

$$f \odot g(d) = \begin{cases} \langle f(d_1), g(d_3)\rangle & \text{if } d = \langle d_1, d_3\rangle \text{ and } f(d_1) \neq \perp_{D_2}, f(d_3) \neq \perp_{D_4} \\ \perp & \text{if } d = \langle d_1, d_3\rangle \text{ and } f(d_1) = \perp_{D_2} \text{ or } f(d_3) = \perp_{D_4} \\ \perp & \text{if } d = \perp \end{cases}$$

- $\Longrightarrow$ is the *strict continuous functions* functor $\to_\perp$,
- T is the *lifting* monoidal functor $(.)_\perp$, namely:
  - $TD = D_\perp$,
  - for any $f : D_1 \to D_2$, the morphism $Tf = f_\perp : D_{1\perp} \to D_{2\perp}$ is $f_\perp(d) = f(d)$ if $d \in D_1$, while $f_\perp(\perp) = \perp$,
- $\epsilon_D : D_\perp \to D$ is $\epsilon_D(d) = d$ if $d \in D$, while $\epsilon_D(\perp) = \perp_D$,

- $\delta_D : D_\perp \to D_{\perp\perp'}$ is $\delta_D(d) = d$ if $d \in D$, while $\delta_D(\perp) = \perp'$,
- $m_{D_1,D_2} : D_{1\perp} \odot D_{2\perp} \to (D_1 \odot D_2)_\perp$ and $m_I : I \to I_\perp$ are:

$$m_{D_1,D_2}(d) = \begin{cases} \langle d_1, d_2\rangle & \text{if } d = \langle d_1, d_2\rangle \text{ and } d_1 \neq \perp_{D_1}, d_2 \neq \perp_{D_2} \\ \perp & \text{if } d = \langle d_1, d_2\rangle \text{ and } d_1 = \perp_{D_1} \text{ or } d_2 = \perp_{D_2} \\ \perp & \text{if } d = \perp \end{cases}$$

$$m_I(\mathtt{1}) = \mathtt{1} \qquad m_I(\perp_I) = \perp,$$

- $E_D : D_\perp \to I$ is $E_D(d) = \mathtt{1}$ if $d \in D$, while $E_D(\perp) = \perp_I$,
- $Dup_D : D_\perp \to D_\perp \odot D_\perp$ is $Dup_D(d) = \langle d, d\rangle$ if $d \in D$, while $Dup_D(\perp) = \perp_{D_\perp \odot D_\perp}$,
- $ev_{D_1,D_2} : (D_1 \Longrightarrow D_2) \odot D_1 \to D_2$ is $ev_{D_1,D_2}(\langle f, d_1\rangle) = f(d_1)$, while $ev_{D_1,D_2}(\perp_{(D_1 \to_\perp D_2)\odot D_1}) = \perp_{D_2}$,
- $\Lambda : \mathbf{Hom}((D_1 \Longrightarrow D_2), D_3) \to \mathbf{Hom}((D_1 \odot D_2), D_3)$ is such that $\Lambda(f)(\perp_{D_1}) = \perp_{(D_2 \to_\perp D_3)}$, while $\Lambda(f)(d_1)(d_2) = f(\langle d_1, d_2\rangle)$ if $d_2 \neq \perp_{D_2}$, and $\Lambda(f)(d_1)(\perp_{D_2}) = \perp_{D_3}$.

If, in addition, there is an object $\mathcal{D}$ having $(\mathcal{D} \to_\perp \mathcal{D})_\perp$ as a retract, then it is routine to prove that all the diagrams for having a **Cbv** category commute. □

At this point we can use the **Cbv** category just introduced for defining a pseudo-λ_v-structure as in Definition 4.3. Let us see how the set of semantic values $V^{\mathbf{CPOS}^{\mathcal{D}}}$ is defined. Starting from a strict continuous function $h : I \to (\mathcal{D} \to_\perp \mathcal{D})$, we have that this function relates $\mathtt{1} \in I$ to an element $d \in (\mathcal{D} \to_\perp \mathcal{D})$. Furthermore, $h_\perp : I_\perp \to (\mathcal{D} \to_\perp \mathcal{D})_\perp$ is different from the function constantly equal to $\perp$, since $h_\perp(\mathtt{1}) = d \neq \perp$. (Let notice that d can be $\perp_{(\mathcal{D}\to_\perp\mathcal{D})}$, that is different from $\perp$.) Hence, the function $h_\perp \circ m_I$ picks an element $d \in (\mathcal{D} \to_\perp \mathcal{D})_\perp$, different from $\perp$, out. Finally, from $F \circ G = id_{((\mathcal{D}\to_\perp\mathcal{D})_\perp)}$, we have that a semantic value $G \circ h_\perp \circ m_I \neq \perp_{\mathcal{D}}$. The following remark outlines this point:

Remark 8.1. Let the pseudo-λ_v-structure $\mathcal{M}^{\mathbf{CPOS}^{\mathcal{D}}} = \langle S, V, \bullet, \mathcal{I}\rangle$ be given (see Lemma 8.1). The morphisms $v : I \to \mathcal{D}$ of V are such that:

$$\begin{aligned} v(\mathtt{1}) &= d \neq \perp_{\mathcal{D}} \\ v(\perp_I) &= \perp_{\mathcal{D}}. \end{aligned}$$

Theorem 8.1. Let $\mathcal{D}$ be a *Scott domain* such that $\mathcal{D} \approx (\mathcal{D} \to_\perp \mathcal{D})_\perp$. Then $\mathcal{D}$ gives a categorical $\lambda\eta_v$-model.

Proof. By Lemma 8.1, and Definition 4.3 we know how to define a pseudo-λ_v-structure $\mathcal{M}^{\mathbf{CPOS}^{\mathcal{D}}} = \langle D, V, \bullet, \mathcal{I}\rangle$. Moreover, $\mathcal{M}^{\mathbf{CPOS}^{\mathcal{D}}}$ has enough values. Let us take two strict and different continuous functions $f, g : \mathcal{D} \to \mathcal{D}$. Now, f, g both strict, and different, implies the existence of $\perp_{\mathcal{D}} \neq \overline{d} \in \mathcal{D}$ such that $f(\overline{d}) \neq g(\overline{d})$. G is an isomorphism, hence $\overline{d} = G(\overline{e})$, where $\overline{e} \neq \perp_{(\mathcal{D}\to_\perp\mathcal{D})_\perp}$. So, by Remark 8.1, $\overline{e}$ can be written as $\overline{e} = (h_\perp \circ m_I)(\mathtt{1})$. Hence, $f \circ G \circ h_\perp \circ m_I \neq g \circ G \circ h_\perp \circ m_I$, for some h.

By Theorem 4.1 $\mathcal{M}^{\mathbf{CPOS}^{\mathcal{D}}}$ is a λ_v-model, and, thanks to $\mathcal{D} \approx (\mathcal{D} \to_\perp \mathcal{D})_\perp$, by Theorem 6.1, $\mathcal{M}^{\mathbf{CPOS}^{\mathcal{D}}}$ is also a $\lambda\eta_v$-model. □

In (Egidi et al., 1992) the initial solution to $D \approx (D \to_\perp D)_\perp$ in the category of Scott domains is extensively studied.

8.2. *An instance of **Cbv** in Coherence Spaces*

In this subsection we show an instance of **Cbv** in coherence spaces. which was first presented in (Pravato et al., 1995).

8.2.1. *Coherence Spaces.* In this subsection we recall the notions of *coherence space* and *linear function* together with some of their basic constructions.

Definition 8.1. Let $|A|$ be a set of elements called *atoms*. Let $\mathtt{c}_{|A|} \in |A| \times |A|$ be a symmetric and reflexive relation, called *compatibility relation*. Given $|A|$ and $\mathtt{c}_{|A|}$, a *coherence space* A is the set of all sets of compatible atoms in $|A|$, namely, $A \subseteq \mathcal{P}(|A|)$ and $\alpha \in A \Leftrightarrow \forall a, b \in \alpha.\mathtt{c}_{|A|}(a, b)$.

Let observe that if A is a coherence space, then $\emptyset \in A$.

Definition 8.2. Let A and B be two coherence spaces.

i) A function $f : A \to B$ is *continuous* iff:
- f is monotonic, namely:
for every $\alpha, \alpha' \in A$, if $\alpha \subseteq \alpha'$, then $f(\alpha) \subseteq f(\alpha')$,
- if $(\alpha_i)_{i \in I}$ is a directed family in A, then:
$f(\bigcup_{i \in I} \alpha_i) = \bigcup_{i \in I} f(\alpha_i)$.

ii) A continuous function $f : A \to B$ is *stable* iff:
for every $\alpha, \alpha' \in A$, if $\alpha \cup \alpha' \in A$, then $f(\alpha \cap \alpha') = f(\alpha) \cap f(\alpha')$.

iii) A stable function $f : A \to B$ is *linear* iff f preserves arbitrary unions, namely:
$f(\bigcup_{\alpha \in \mathcal{A}} \alpha) = \bigcup_{\alpha \in \mathcal{A}} f(\alpha)$ for every $\mathcal{A} \subseteq A$.

Let notice that every linear function is strict, in the sense that $f(\emptyset) = \emptyset$.

Let A, B be two coherence spaces. $(A \to_s B)$ denotes the coherence space of the stable functions from A to B ordered by Berry's order, namely given two stable functions $f, g \in (A \to_s B)$:

$$f \leq g \quad \text{iff} \quad \forall \alpha, \alpha' \in A.(\alpha \subseteq \alpha' \Rightarrow f(\alpha) = g(\alpha) \cap f(\alpha')).$$

$(A \multimap B)$ denotes the coherence space of linear functions from A to B ordered like stable functions.

Notation 8.1. Let A be a coherence space. Atoms of A will be ranged over by $a, b, \ldots$, while elements of A (i.e. sets of atoms) will be ranged over by $\alpha, \beta, \ldots$. With $\mathtt{1}$ we denote the unique atom of I, namely $I = \{\emptyset, \{\mathtt{1}\}\}$. $\mathtt{c}(a, a')$ means that a and a' are compatible. Let $\otimes$, !, and $\multimap$ be the functors over coherence domains such that:

- $|A \otimes B| = \{[a, b] \mid a \in |A| \text{ and } b \in |B|\}$, where $\mathtt{c}([a, b], [a', b'])$ iff both $\mathtt{c}(a, a')$ and $\mathtt{c}(b, b')$.
- $|!A| = \{d \mid d \text{ is a finite element of } A\}$, where, if $d, d' \in |!A|$, then $\mathtt{c}(d, d')$ iff $d \cup d' \in A$.
- the functor $\multimap$ builds the coherence domains of linear functions where $|A \multimap B| = \{(a, b) \mid a \in |A| \text{ and } b \in |B|\}$, where $\mathtt{c}((a, b), (a', b'))$ iff $\mathtt{c}(a, a')$ implies both $\mathtt{c}(b, b')$ and if $b = b'$, then $a = a'$.

The elements of $A \multimap B$ are linear traces of linear functions from A to B. If $f : A \to B$ is a linear function, then its *linear trace* is denoted by $ltr(f)$ and it is used as follows: $f(\{a_i \mid i \in I\}) = \{b_i \mid (a_i, b_i) \in \ ltr(f)\}$.

8.2.2. *The Linear Instance of **Cbv**.* Let **Lin** be the category such that:

- the objects are all *coherence spaces*,
- the morphisms are all *linear functions*.

Let $\odot$, $\Longrightarrow$, and T of Definition 3.1 be $\otimes$, $\multimap$, and !, respectively, as defined in the previous Subsection 8.2.1.

Lemma 8.2. Let $\mathbf{Lin}^{\mathcal{D}}$ be the category **Lin** equipped with the retraction $\mathcal{D} \triangleright !(\mathcal{D} \multimap \mathcal{D})$. The category $\mathbf{Lin}^{\mathcal{D}}$ is a **Cbv** category.

Proof. We have a **Cbv** category if we use the following definitions:

- The linear traces of the monoidal closure are

$$ltr(\Lambda_{A,B,C}) = \{(\,([a,b],c)\,,\,(a,(b,c))\,) \mid ([a,b],c) \in |(A \otimes B) \multimap C|\}$$

$$ltr(ev_{B,C}) = \{([(b,c),b],c) \mid (b,c) \in |B \multimap C|\}$$

- If f is a linear function from A to B, then

$$\begin{aligned} ltr(!f) \quad = \quad & \{(\{a_{i_1}, \ldots, a_{i_k}\}, \{b_{i_1}, \ldots, b_{i_k}\}) \mid (a_{i_j}, b_{i_j}) \in ltr(f) \\ & \text{and } \{a_{i_1}, \ldots, a_{i_k}\}, \{b_{i_1}, \ldots, b_{i_k}\} \text{ are finite (perhaps empty)} \\ & \text{sets of compatible elements}\}. \end{aligned}$$

- The linear traces of the natural transformations for the comonad are

$$ltr(\delta_A) = \{(\bigcup \Omega, \Omega) \mid \Omega \in |!!A|\}, \qquad ltr(\epsilon_A) = \{(\{\alpha\}, \alpha) \mid \alpha \in |A|\}.$$

- The linear traces of the morphisms making ! monoidal are

$$ltr(m_I) = \{(\mathtt{1}, \emptyset), (\mathtt{1}, \{\mathtt{1}\})\}$$

$$\begin{aligned} ltr(m_{A_1,\ldots,A_n}) = \quad & \{([\alpha_1, \ldots, \alpha_n], \{[a_1^1, \ldots, a_n^1], \ldots, [a_1^k, \ldots, a_n^k]\}) \mid \\ & a_i^j \in \alpha_i, b_i^j \in \beta_i\}. \end{aligned}$$

  Remember that m_{nA} and m_n are defined starting from $m_{A_1,\ldots,A_n}$.
- The linear traces of the morphisms giving the comonoid are

$$ltr(E_A) = \{(\emptyset, \mathtt{1})\}, \qquad ltr(Dup_A) = \{(\alpha, [\alpha_1, \alpha_2]) \mid \alpha_1 \cup \alpha_2 = \alpha \in |!A|\}.$$

If, in addition, there is an object $\mathcal{D}$ having $!(\mathcal{D} \multimap \mathcal{D})$ as a retract, then it is routine to prove that all the diagrams for having a **Cbv** (introduced in Definition 3.1) commute. □

The **Cbv** category $\mathbf{Lin}^{\mathcal{D}}$ yields pseudo-λ_v-structure $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}} = \langle S, V, \bullet, \mathcal{I} \rangle$ (Definition 4.3). Let us see what the set V contains. We start from the following linear function $h : I \to (\mathcal{D} \multimap \mathcal{D})$ as an example:

$$ltr(h) = \{(\mathtt{1}, (d_1, d_2)), (\mathtt{1}, (e_1, e_2))\}\ .$$

In this case we have that h relates $\{\mathtt{1}\} \in I$ to the element $\{(d_1, d_2)), (e_1, e_2)\} \in (\mathcal{D} \multimap \mathcal{D})$. Furthermore, from:

$$ltr(!h) = \{(\emptyset, \emptyset), (\{\mathtt{1}\}, \{(d_1, d_2)\}), (\{\mathtt{1}\}, \{(e_1, e_2)\}), (\{\mathtt{1}\}, \{(d_1, d_2), (e_1, e_2)\})\},$$

and

$$ltr(!h \circ m_I) = \{(\mathtt{1}, \emptyset), (\mathtt{1}, \{(d_1, d_2)\}), (\mathtt{1}, \{(e_1, e_2)\}), (\mathtt{1}, \{(d_1, d_2), (e_1, e_2)\})\},$$

we have:

$$(!h \circ m_I)(\{\mathtt{1}\}) = \{\emptyset, \{(d_1, d_2)\}, \{(e_1, e_2)\}, \{(d_1, d_2), (e_1, e_2)\}\}.$$

Finally, from $F \circ G = id_{!(\mathcal{D} \multimap \mathcal{D})}$, and the linearity of G:

$$G(\{\emptyset, \{(d_1, d_2)\}, \{(e_1, e_2)\}, \{(d_1, d_2), (e_1, e_2)\}\}) = \{\emptyset, d', d'', d'''\}$$

for some d', d'', d''' such that, $d''' = d' \cup d''$. Generalizing this discussion, we get the following:

Remark 8.2. Let $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}} = \langle S, V, \bullet, \mathcal{I} \rangle$ be given (Lemma 8.2). The morphisms $v : I \to \mathcal{D}$ of V, are such that:

$$\begin{array}{rcl} v(\{\mathtt{1}\}) & = & \mathcal{P}(d) \\ v(\emptyset) & = & \emptyset, \end{array}$$

where $\mathcal{P}(d)$ is the power set of a given atom d of $\mathcal{D}$, and G is the embedding function of Definition 3.1.

Theorem 8.2. Let $\mathcal{D}$ be coherence space such that $\mathcal{D} \approx !(\mathcal{D} \multimap \mathcal{D})$. Then $\mathcal{D}$ gives a categorical $\lambda\eta_v$-model.

Proof. From Lemma 8.2 and Definition 4.3 we know how to build a pseudo-λ_v-structure $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}} = \langle D, V, \bullet, \mathcal{I} \rangle$. Moreover, $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}}$ has enough values. Let $f, g : \mathcal{D} \to \mathcal{D}$ be two linear functions such that $f \neq g$. This is equivalent to say that their traces are different. Hence, it must be $f(\{d\}) \neq g(\{d\})$, for at least one atom $d \in !(\mathcal{D} \multimap \mathcal{D}) \approx \mathcal{D}$. Notice that d is taken as atom of $!(\mathcal{D} \multimap \mathcal{D})$, and not as atom of $\mathcal{D}$, i.e., d is a finite trace of a linear function from $\mathcal{D}$ to $\mathcal{D}$. Now, let us consider the smaller atom $\overline{d} \in !(\mathcal{D} \multimap \mathcal{D})$ among those $d \in !(\mathcal{D} \multimap \mathcal{D})$ such that $f(\{d\}) \neq g(\{d\})$. Equivalently, $\overline{d}$ is one of the smallest traces of linear functions that allow to distinguish f and g. Now, by linearity:

$$f(\mathcal{P}(\overline{d})) = f(\{\overline{d}\}) \cup X$$

$$g(\mathcal{P}(\overline{d})) = g(\{\overline{d}\}) \cup Y \ \ ,$$

for some X and Y. Hence: $f(\mathcal{P}(\overline{d})) \neq g(\mathcal{P}(\overline{d}))$, or, equivalently, $f(v(\{\mathtt{1}\})) \neq g(v(\{\mathtt{1}\}))$, as $\mathcal{P}(\overline{d}) = v(\{\mathtt{1}\})$, by Remark 8.2. So, we have concluded the existence of a semantic value that takes f and g apart each to the other.

By Theorem 4.1, $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}}$ is a λ_v-model, and, thanks to $\mathcal{D} \approx !(\mathcal{D} \multimap \mathcal{D})$, by Theorem 6.1, $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}}$ is also a $\lambda\eta_v$-model. □

Remark 8.3. Let us note that, although $\mathcal{D} \approx !(\mathcal{D} \multimap \mathcal{D})$ gives a $\lambda\eta_v$-model, this model is not extensional, as the following example clarifies:

Example 8.1. Let $\mathcal{M}^{\mathbf{Lin}^{\mathcal{D}}} = \langle D, V, \bullet, \mathcal{I}\rangle$ be based on $\mathbf{Lin}^{\mathcal{D}}$. The binary operation $\bullet$, making $\langle D, \bullet\rangle$ an applicative structure, is defined as: $f \bullet g = ev_{\mathcal{D},\mathcal{D}} \circ ((\epsilon_{\mathcal{V}} \circ f) \otimes g)$, for every pair of morphisms $f, g \in \mathbf{Hom}(I, \mathcal{D})$. For simplicity, we have omitted F, as it is an isomorphism. Let consider the two morphisms $f_1, f_2 \in \mathbf{Hom}(I, \mathcal{D})$ with traces:

$$ltr(f_1) = \{(\mathtt{1}, \emptyset), (\mathtt{1}, \{(d_1, d_2)\}), (\mathtt{1}, \{(e_1, e_2)\}), (\mathtt{1}, \{(d_1, d_2), (e_1, e_2)\})\},$$

and

$$ltr(f_2) = \{(\mathtt{1}, \emptyset), (\mathtt{1}, \{(d_1, d_2)\}), (\mathtt{1}, \{(e_1, e_2)\})\},$$

where the atoms of $\mathcal{D}$ are identified with the atoms of $!(\mathcal{D} \multimap \mathcal{D})$. Both f_1 and f_2 have the same behavior, as consequence of the definition of $\epsilon_{\mathcal{V}}$: $f_1 \bullet g = f_2 \bullet g$, for every $g \in \mathbf{Hom}(I, \mathcal{D})$. However, $f_1 \neq f_2$ because they have different traces. □

We conclude this section giving an example of interpretation in $\mathbf{Lin}^{\mathcal{D}}$.

Example 8.2. Let $\omega = (\lambda x.xx)(\lambda x.xx)$. From Lemma 4.2 we have $[\![\vdash \omega]\!]^{\mathcal{C}(\mathcal{D})} = [\![x \vdash xx]\!]^{\mathcal{C}(\mathcal{D})} \circ [\![\vdash \lambda x.xx]\!]^{\mathcal{C}(\mathcal{D})}$. Since $[\![\vdash \lambda x.xx]\!]^{\mathcal{C}(\mathcal{D})} = !(\Lambda([\![x \vdash xx]\!]^{\mathcal{C}(\mathcal{D})})) \circ m_I$, let us see the form of $[\![x \vdash xx]\!]^{\mathcal{C}(\mathcal{D})}$.

$$\begin{aligned} &[\![x \vdash xx]\!]^{\mathcal{C}(\mathcal{D})} = \\ &\qquad = ev \circ ((\epsilon \circ [\![x \vdash x]\!]^{\mathcal{C}(\mathcal{D})} \otimes [\![x \vdash x]\!]^{\mathcal{C}(\mathcal{D})}) \circ Dup \\ &\qquad = ev \circ ((\epsilon \circ id) \otimes id) \circ Dup = ev \circ (\epsilon \otimes id) \circ Dup. \end{aligned}$$

Since the linear trace of $\epsilon \otimes id$ is of the form $\{([\{(a,b)\}, c], [(a,b), c]), \ldots\}$, we have $ltr(ev \circ (\epsilon \otimes id)) = \{([\{(a,b)\}, a], b), \ldots\}$, hence $ltr([\![x \vdash xx]\!]^{\mathcal{C}(\mathcal{D})}) = \{(\{(a,b)\} \cup a, b), \ldots\}$. Moreover, $ltr([\![\vdash \lambda x.xx]\!]^{\mathcal{C}(\mathcal{D})}) = \{(\mathtt{1}, \emptyset), (\mathtt{1}, \{(\{(a,b)\} \cup a, b), \ldots\}), \ldots\}$, with $\{(\{(a,b)\} \cup a, b), \ldots\}$ finite. This finiteness implies that $[\![\vdash \omega]\!]^{\mathcal{C}(\mathcal{D})} = \emptyset$, hence for every environment we cannot have a semantic value. □

9. Conclusions

This section is a summary of what we have done in this work and contains some relations with other models of $\lambda\beta_v$-calculus.

In this paper we mainly trace a relation between two hierarchies of structures that can be used to model $\lambda\beta_v$-calculus. A useful picture is in Figure 3. One is the set-theoretical hierarchy of pseudo-λ_v-structures (Definition 2.4), containing λ_v-models (Definition 2.4), $\lambda\eta_v$-models (Section 6), and extensional λ_v-models (Section 6). The other hierarchy distinguishes among **Cbv** categories with model object $\mathcal{D}$. A first distinction among **Cbv** categories rests on the property of having or not enough values . A second distinction among them relies on having either $\mathcal{D} \rhd T(\mathcal{D} \Longrightarrow \mathcal{D})$, or $\mathcal{D} \approx T(\mathcal{D} \Longrightarrow \mathcal{D})$ as a model object.

Given a **Cbv** category $\mathcal{C}(\mathcal{D})$, we show how to build a pseudo λ_v-structure $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ out of it (Definition 4.3). However, if **Cbv** has enough values, $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ is a λ_v-model.

Finally, we show that the two instances $\mathbf{CPOS}^{\mathcal{D}}$, with $\mathcal{D}$ least solution of $\mathcal{D} \approx (\mathcal{D} \rightarrow_{\perp} \mathcal{D})_{\perp}$, and $\mathbf{LIN}^{\mathcal{D}}$, with $\mathcal{D}$ least solution of $\mathcal{D} \approx !(\mathcal{D} \multimap \mathcal{D})$, of **Cbv** categories yields two

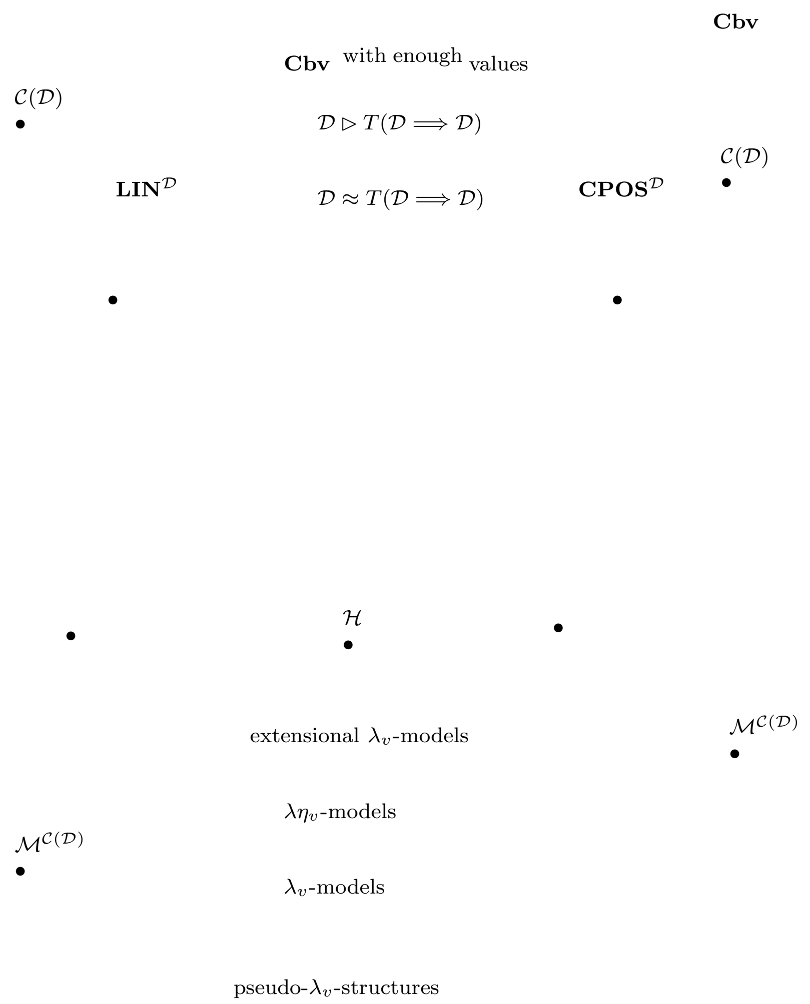


Fig. 3. Relation among the models dealt with in this paper: summary

λ_v-models (Section 8). In particular, the instance of $\mathbf{LIN}^{\mathcal{D}}$ is an example of $\lambda\eta_v$-model which is not an extensional λ_v-model (Example 8.1).

It is still an open problem the definition of the relation among the two hierarchies, going in the opposite direction. Just as an example, it is unknown how to extract a category in the class **Cbv** out of the model $\mathcal{H}$, introduced in (Egidi et al., 1992), and *fully abstract* with respect to the SECD operational semantics.

Finally, the relation among our class **Cbv** of categories and Moggi's categorical models (Moggi, 1991) for $\lambda\beta_v$-calculus. First, the existence of $\mathcal{D} \approx T(\mathcal{D} \Longrightarrow \mathcal{D})$ in **Cbv** induces a suitable cartesian closed category with both a commutative strong monad and an object to build a model *à la Moggi* for $\lambda\beta_v$-calculus. Second, the set of values in Moggi's model is isomorphic to the set of values in the λ_v-model, induced by **Cbv** itself. This is worth noticing because the set of values of Moggi's models is an object of the category he defines. On the contrary, **Cbv** has no object in it which elements can be thought of as values.

The relation between Moggi's and our approaches is obtained by "lifting" to the untyped case two results. The first is in (Benton, 1995), where a reformulation of the categorical models for intuitionistic linear logic we started from is introduced. The second result is in (Benton and Wadler, 1996): the categorical models of *typed* $\lambda\beta_v$-calculus, based both on the categorical models of intuitionistic linear logic, and on cartesian closed categories with a commutative comonad (Moggi, 1991), are essentially the same. Indeed, they correspond through an adjunction. A summary of the details for lifting this second point to the *untyped* case is in the following subsection.

9.1. *Relations among **Cbv** and Moggi's models: some details*

This section summarizes the main details to develop the relation between our definition of categorical models for $\lambda\beta_v$-calculus and *Moggi's approach*. The relation follows from the result that a category like **Cbv** induces a cartesian closed category with a commutative strong monad. An extended development of the details about this relation is in (Benton, 1995; Benton and Wadler, 1996).

The comonad (T, δ, ϵ) of **Cbv** gives rise to the Eilenberg-Moore category $\mathbf{Cbv}^T$, which objects are all T-coalgebras $(A, h_A : A \to TA)$, and all morphisms are T-coalgebras morphisms. Between **Cbv** and $\mathbf{Cbv}^T$ there exists an adjunction $\mathcal{F} \dashv \mathcal{U}$ where: $\mathcal{U}(A) = (TA, \delta_A)$, and $\mathcal{F}((A, h_A)) = A$, being, of course, $\mathcal{F} : \mathbf{Cbv}^T \to \mathbf{Cbv}$, and $\mathcal{U} : \mathbf{Cbv} \to \mathbf{Cbv}^T$.

The full sub-category $\mathcal{E}(\mathbf{Cbv}^T)$ of $\mathbf{Cbv}^T$, having as objects all the exponentiable coalgebras, is cartesian closed with the T-coalgebra (I, m_I) as terminal object. Moreover, $\mathcal{F} \dashv \mathcal{U}$ induces the strong monad $\mathcal{UF}$ on $\mathcal{E}(\mathbf{Cbv}^T)$, and, thanks to the closed structure of **Cbv**, it also yields the following isomorphism:

$$\begin{aligned}&\mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I), (T(A \Longrightarrow B), \delta_{A \Longrightarrow B}))\\ &\quad\approx\quad \mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I) \times (A, h_A), (TB, \delta_B))\end{aligned}$$

Now, assume $\mathcal{D} \approx T(\mathcal{D} \Longrightarrow \mathcal{D})$ in **Cbv**, which implies $T(\mathcal{D} \Longrightarrow \mathcal{D}) \approx T(T(\mathcal{D} \Longrightarrow \mathcal{D}) \Longrightarrow T(\mathcal{D} \Longrightarrow \mathcal{D}))$. From the naturality of δ, we get $(T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}) \approx (T(T(\mathcal{D} \Longrightarrow \mathcal{D}) \Longrightarrow T(\mathcal{D} \Longrightarrow \mathcal{D})), \delta_{T(\mathcal{D} \Longrightarrow \mathcal{D}) \Longrightarrow T(\mathcal{D} \Longrightarrow \mathcal{D})})$ in $\mathcal{E}(\mathbf{Cbv}^T)$, which, using the isomorphism between the Hom-sets of $\mathcal{E}(\mathbf{Cbv}^T)$, yields:

$$\begin{aligned}&\mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I), (T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}))\\ &\quad\approx\quad \mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I), (T(T(\mathcal{D} \Longrightarrow \mathcal{D}) \Longrightarrow T(\mathcal{D} \Longrightarrow \mathcal{D})), \delta_{T(\mathcal{D} \Longrightarrow \mathcal{D}) \Longrightarrow T(\mathcal{D} \Longrightarrow \mathcal{D})}))\\ &\quad\approx\quad \mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I) \times (T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}), (TT(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{T(\mathcal{D} \Longrightarrow \mathcal{D})}))\\ &\quad\approx\quad \mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I) \times (T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}), \mathcal{UF}(T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}))\end{aligned}$$

So, we can write:

$$(T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}) \approx (T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}) \to \mathcal{UF}(T(\mathcal{D} \Longrightarrow \mathcal{D}), \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}) \qquad (12)$$

being $\to$ the arrow of $\mathcal{E}(\mathbf{Cbv}^T)$.

Let us pose $\mathcal{R} = T(\mathcal{D} \Longrightarrow \mathcal{D})$ in (12). We get that (12) is the domain equation that Moggi requires to exists in $\mathcal{E}(\mathbf{Cbv}^T)$ with the strong monad $\mathcal{UF}$, in order to define a model of $\lambda\beta_v$-calculus.

The next question is about the relation between $\mathcal{R}$ and and the set $V^{\mathcal{C}(\mathcal{D})}$ of values in the λ_v-model $\mathcal{M}^{\mathcal{C}(\mathcal{D})}$ of Definition 4.3. The answer is:

$$\mathcal{R} \approx V^{\mathcal{C}(\mathcal{D})} \ .$$

On one side, if $(G \circ Th \circ m_I)$ belongs to $V^{\mathcal{C}(\mathcal{D})}$, for any $h \in \mathbf{HOM}_{\mathbf{Cbv}}(I, \mathcal{D})$, then $(Th \circ m_I) \in \mathbf{HOM}_{\mathcal{E}(\mathbf{Cbv}^T)}((I, m_I), (V^{\mathcal{C}(\mathcal{D})}, \delta_{\mathcal{D} \Longrightarrow \mathcal{D}}))$ follows from naturality and monoidality of δ. We are interested to the contrary as well. So, we are interested to know if, for every coalgebra morphism $f : I \to \mathcal{V}$, there exists $\hat{f}$ such that $T\hat{f} \circ m_I = f$, and if $\hat{f}$ is unique. The answer is: yes, defining $\hat{f} \equiv \epsilon_{\mathcal{D} \Longrightarrow \mathcal{D}} \circ f$. The equation $T\hat{f} \circ m_I = f$ follows from the naturality and the monoidality of ϵ. For the unicity, its enough to assume the existence of $g \neq \epsilon_{\mathcal{D} \Longrightarrow \mathcal{D}} \circ f$ such that $Tg \circ m_I = f$, and we are done.

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Appendix A. Categorical tools

This section recalls standard categorical notions that can be found in usual reference books about the subject, like (Mac Lane, 1971), and that we use to introduce the categorical λ_v-model.

A *symmetric monoidal category* is a category $\mathbf{C}$ with a bifunctor $\odot : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object I, and, for any $A, B, C \in Obj_{\mathbf{C}}$, the natural isomorphisms:

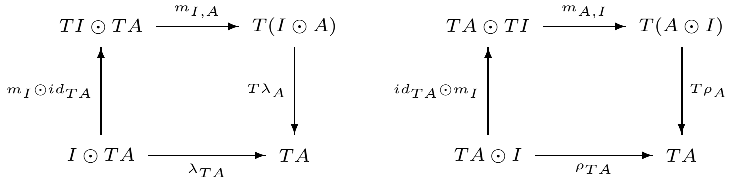
$$\begin{gathered}\alpha_{A,B,C} : A \odot (B \odot C) \dot{\to} (A \odot B) \odot C \\ \lambda_A : I \odot A \dot{\to} A \qquad \rho_A : A \odot I \dot{\to} A \\ \gamma_{A,B} : A \odot B \dot{\to} B \odot A\end{gathered}$$

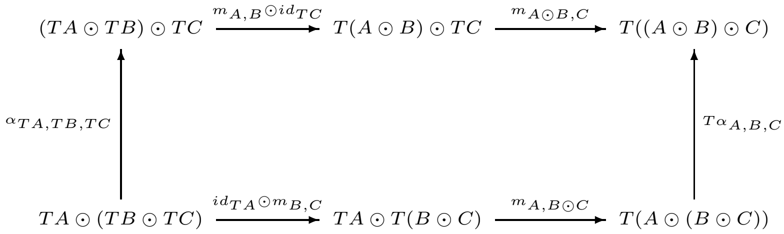
satisfying *coherence*. Namely:

$$\begin{gathered}\alpha_{(A\odot B),C,D} \circ \alpha_{A,B,(C\odot D)} = (\alpha_{A,B,C} \odot id_B) \circ \alpha^{-1}_{A,(B\odot C),D} \circ (id_A \odot \alpha_{B,C,D}) \\ (\rho_A \odot id_C) \circ \alpha_{A,I,C} = id_A \odot \lambda_C \qquad \lambda_I = \rho_I \\ \gamma_{A,B} \circ \gamma_{B,A} = id_{B\odot A} \qquad \rho_B = \lambda_B \circ \gamma_{B,I} \\ \alpha_{C,A,B} \circ \gamma_{(A\odot B),C} \circ \alpha_{A,B,C} = (\gamma_{A,C} \odot id_B) \circ \alpha_{A,C,B} \circ (id_A \odot \gamma_{B,C}) \ .\end{gathered}$$

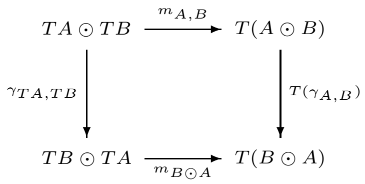
Recall also that, *Coherence Theorem* holds in a symmetric monoidal category, namely, two morphisms always coincide if they are built out of α, ρ, λ, id, composition, and $\gamma_{A,B}$, with $A \not\equiv B$.

Given a symmetric monoidal category $\mathbf{C}$, an endofunctor $T : \mathbf{C} \to \mathbf{C}$ is *monoidal* if, for every $A, B \in Obj_{\mathbf{C}}$, there are a natural transformation $m_{A,B} : TA \odot TB \dot{\to} T(A \odot B)$ and a map $m_I : I \to TI$ such that the following diagrams commute:

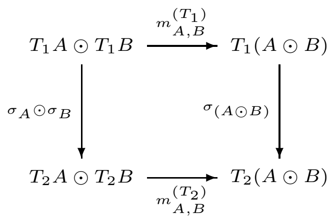


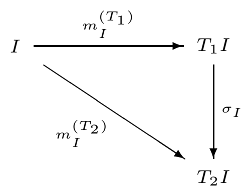


The monoidal functor T is *symmetric* if:



A natural transformation $\sigma : T_1 \dot{\to} T_2$ between two symmetric monoidal functors T_1 and T_2, is *symmetric monoidal* if the following diagrams commute:

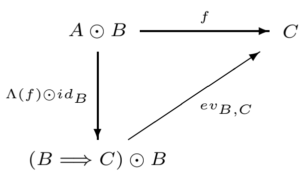


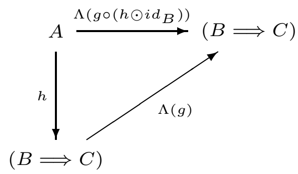


A symmetric monoidal category $\mathbf{C}$ is *closed* if, for every object B, there is a functor $B \Longrightarrow \_ : \mathbf{C} \to \mathbf{C}$, such that there exists an isomorphism

$$\Lambda_{A,B,C} : \mathbf{Hom_C}((A \odot B), C)) \to \mathbf{Hom_C}(A, (B \Longrightarrow C)) \;,$$

natural in A and C. Namely: for all $A, C \in Obj_{\mathbf{C}}$ there exists the *evaluation morphism* $ev_{B,C} : (B \Longrightarrow C) \odot B \to C$ such that, for all the morphisms $f : (A \odot B) \to C$, $h : A \to (B \Longrightarrow C)$, and $g : (B \Longrightarrow C) \odot B \to C$, there is a *unique* $\Lambda_{A,B,C}(f) : A \to (B \Longrightarrow C)$ such that:

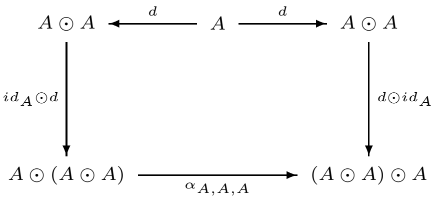


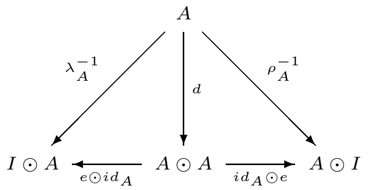


commute. Recall that in a symmetric monoidal closed category every object A is isomorphic to $(I \Longrightarrow A)$. Recall also that, by naturality:

$$\Lambda_{A,B,C}(ev_{B,C} \circ (h \odot id_B)) = h.$$

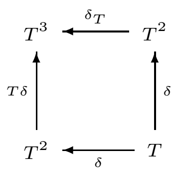
Given a monoidal category $\mathbf{C}$, a *comonoid* in $\mathbf{C}$ is a triple (A, d, e) where A is an object of **C**, and the morphisms $d : A \dot{\to} (A \odot A)$, and $e : A \dot{\to} I$ are such that:

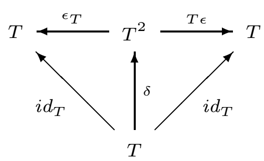




commute. The comonoid (A, d, e) on it is *commutative* if d commutes with γ, namely: $\gamma_{A,A} \circ d = d$.

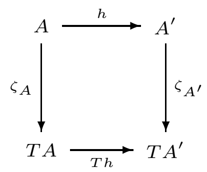
Given a category **C** (not necessarily monoidal), a *comonad* over **C** is a triple (T, δ, ϵ), where $T : \mathbf{C} \to \mathbf{C}$ is an endofunctor, and $\delta : T \dot{\to} T^2$ and $\epsilon : T \dot{\to} ID_{\mathbf{C}}$ are natural transformations, such that:



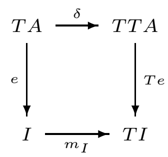


commute.

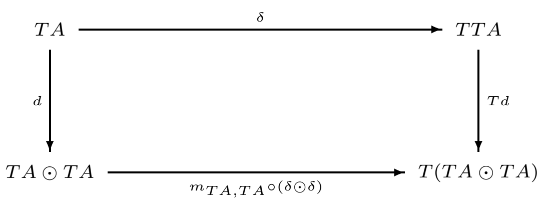
Given an endofunctor $T : \mathbf{C} \to \mathbf{C}$, the *category of T-coalgebras* has both T-coalgebras (A, ζ_A) as objects, being A an object of **C**, and $\zeta_A : A \to TA$, and the morphisms $h : A \to A'$ such that:



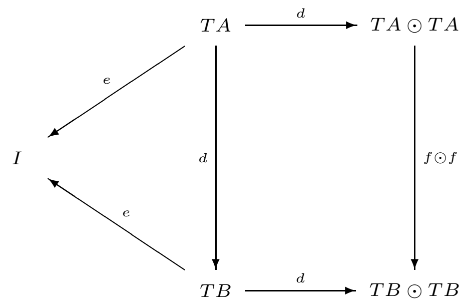
commutes, as arrows. The set $T\textbf{-coalg}_{\mathbf{C}}((A, \zeta_A), (A', \zeta_{A'}))$ denotes such morphisms h. So, let a symmetric monoidal category **C** be given with a comonad (T, δ, ϵ) such that T is monoidal, and (TA, d, e) is a comonoid. Then the natural transformation d belongs to $T\textbf{-coalg}_{\mathbf{C}}((TA, \delta), (TA \odot TA, m_{TA,TA} \circ (\delta \odot \delta)))$ if the diagram:



commutes. Analogously, e belongs to $T\textbf{-coalg}_{\mathbf{C}}((TA, \delta), (I, m_I))$ if the diagram:

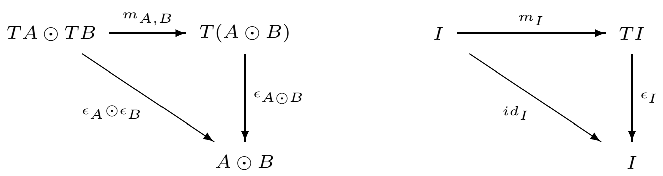


commutes. Moreover, let f belong to the set $T\textbf{-coalg}_{\mathbf{C}}((TA, \delta), (TB, \delta))$ of (free) coalgebras. The morphism f is a comonoid morphism from (A, d, e) to (B, d, e) if:

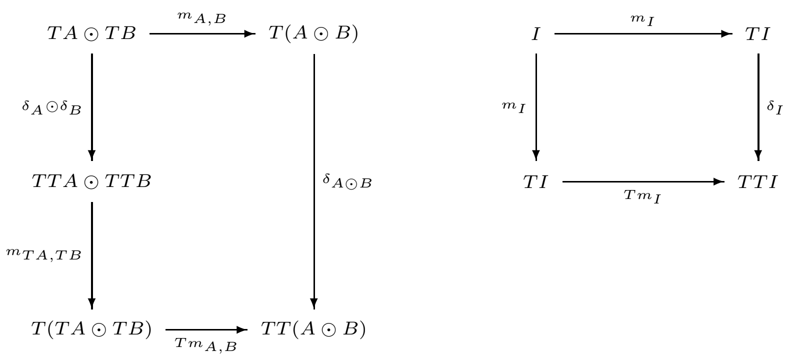


commutes.

Let $\mathbf{C}$ be a symmetric monoidal closed category with a comonad (T, δ, ϵ) on it. The natural transformation $\epsilon : T \dot{\to} ID_{\mathbf{C}}$ is monoidal if:

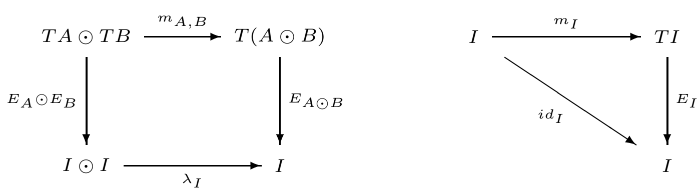


commute. The natural transformation $\delta : T \dot{\to} TT$ is monoidal if:

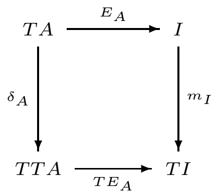


commute. Notice that $Tm_{A,B} \circ m_{TA,TB}$, and $Tm_I \circ m_I$ are, respectively, $m_{A,B}$, and m_I with respect to the monoidal functor TT.

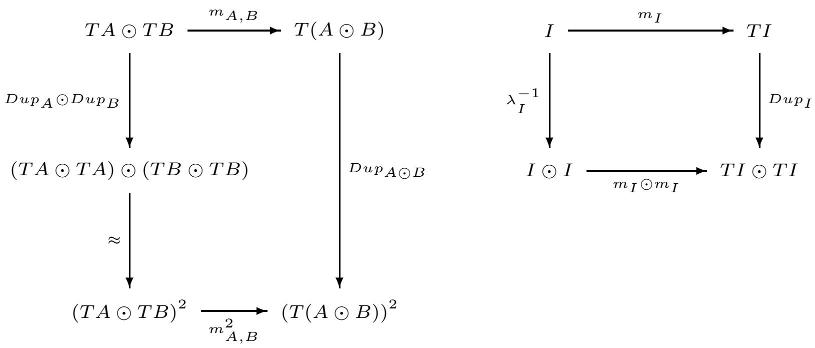
The natural transformation $E : T \dot{\to} K_I$ is monoidal if:



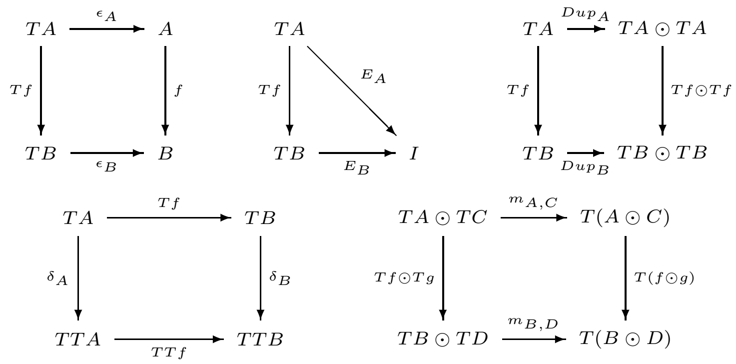
commute. Moreover, E is an element of $T\text{-}\mathbf{coalg}_{\mathbf{Cbv}}((TA, \delta_A), (I, m_I))$ if:



commutes. The natural transformation $Dup : T \dot{\to} T \odot T$ is monoidal if:



commute. Let notice that naturality of ϵ, E, Dup, δ and $m_{A,B}$ means that, for all $f : A \to B$ and $g : C \to D$:



commute.