A local criterion for polynomial time stratified computations

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Abstract. This work is a consequence of studying the (*un*)relatedness of the principles that allow to implicitly characterize the polynomial time functions (PTIME) under two perspectives. One perspective is predicative recursion, where we take Safe Recursion on Notation as representative. The other perspective is structural proof theory, whose representative can be Light Affine Logic (LAL). A way to make the two perspectives closer is to devise polynomial sound generalizations of LAL whose set of interesting proofs-as-programs is larger than the set LAL itself supplies. Such generalizations can be found in MS.
MS is a *Multimodal Stratified framework* that contains *subsystems* among which we can find, at least, LAL. Every subsystem is essentially determined by two sets. The first one is countable and finite, and supplies the modalities to form modal formulæ. The second set contains building rules to generate proof nets. We call MS *multimodal* because the set of modalities we can use in the types for the proof nets of a subsystem is arbitrary. MS is also *stratified*. This means that every box, associated to some modality, in a proof net of a subsystem can never be opened. So, inside MS, we preserve *stratification* which, we recall, is the main structural proof theoretic principle that makes LAL a polynomial time sound deductive system. MS is expressive enough to contain LAL and Elementary Affine Logic (EAL), which is PTIME-*un*sound. We supply a set of syntactic constraints on the rules that identifies the PTIME-maximal subsystems of MS, i.e. the PTIME-sound subsystems that contain the largest possible number of rules. It follows a syntactic condition that discriminates among PTIME-sound and PTIME-*un*sound subsystems of MS: a subsystem is PTIME-sound if its rules are among the rules of some PTIME-maximal subsystem. All our proofs widely use the techniques Context Semantics supplies, and in particular the geometrical configuration that we call *dangerous spindle*: a subsystem is polytime if and only if its rules cannot build dangerous spindles.

1 Introduction

This work fits the theoretical side of Implicit Computational Complexity (ICC). Our primary goal is looking for the systems that can replace the question marks (1), (2), and (3) in Fig.1. In there, SRN is Safe Recursion on Notation [1], namely a polynomial time

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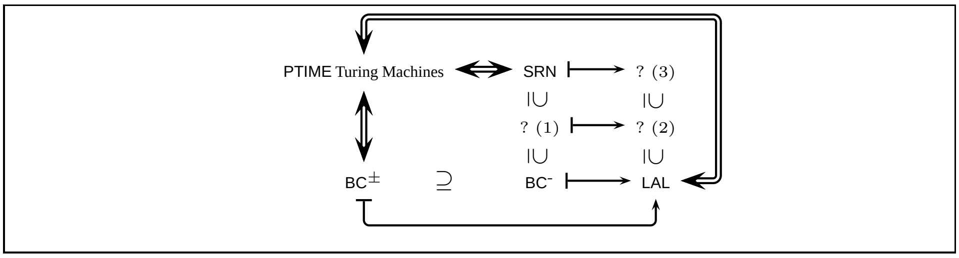


Fig. 1. Relations between SRN and LAL.

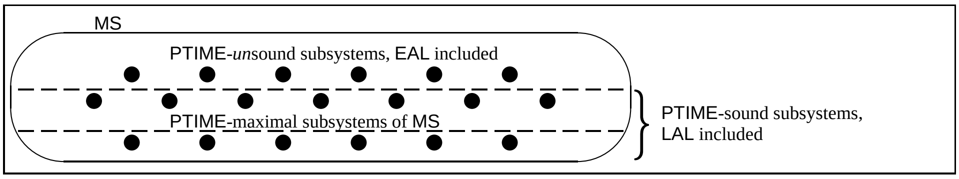
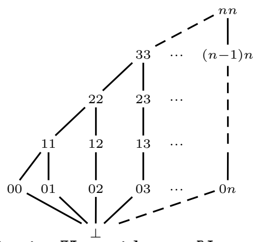


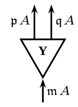
Fig. 2. PTIME-*un*sound, PTIME-sound, and PTIME-maximal subsystems of MS.

(PTIME) sound and complete restriction of Primitive Recursion; LAL is Light Affine Logic [2], a deductive system derived from Linear Logic (LL, [3]), based on proof nets, which is PTIME sound and complete under the proofs-as-programs analogy. As shown in [4], there exists a subalgebra BC^- of SRN that compositionally embeds into LAL. However, it is not possible to extend the same embedding to the whole SRN. As far as we know, any reasonable replacement of the question marks (1), (2), and (3) in Fig. 1 is still unknown. The results in [5] and [6] justify the obstructions to the extension of the embedding in [4] that would replace LAL for (2), or (3). Indeed, [5] shows the *strong* PTIME-soundness of LAL, while [6] shows that SRN is just *weakly* PTIME-sound, once we see it as a term rewriting system. Since the strong PTIME-sound programs are, intuitively speaking, far less than the weak ones, the misalignment looks evident. So, a way to fill the gaps in Fig. 1 is looking for an extension of LAL. This is not impossible since LAL does not supply the largest set of programs we can write under the structural proof theoretic constraints that LAL itself relies on. To better explain this, we need some steps.

Recalling LAL. The proof nets of LAL inherits the !-boxes from LL. Every !-box identifies a specific region of a proof net that can be duplicated by the cut elimination. *In* LAL *each box that can be duplicated, called* !*-box, depends on at most a single assumption*. Besides the duplicable !-boxes, LAL has $\S$-boxes, which cannot be duplicated. $\S$-boxes implement the *weak dereliction* principle $!A \multimap \S A$. Namely, every $\S$-box allows to access the content of a !-box, preserving its border. Accessing the proof net inside a !-box is useful to program iterations through Church numerals, for example. Since also the $\S$-boxes can never be "opened", *the proof nets of* LAL *are stratified*. "Stratification" means that every node u of every proof net of LAL either gets erased by the cut elimination, or the number of nested boxes the residuals of u are contained in keeps being constant when the cut elimination proceeds.



(a) Carrier $\mathbb{X}_n$, with $n \in \mathbb{N}$.



(b) Contraction nodes, with $\mathfrak{m}, \mathfrak{p}, \mathfrak{q} \in \mathbb{X}_n$ such that $\mathfrak{p}, \mathfrak{q}$ are incomparable and $\mathfrak{m}$ is their least upper bound.

Fig. 3. Carrier and Contraction nodes of $\mathsf{Linear}_{\mathbb{X}_n}$.

PTIME-sound and PTIME-*un*sound extensions of LAL. Once recalled LAL, we propose some extensions of it that preserve stratification. We start extending LAL to $\mathsf{LAL}_{\not\chi}$, whose !-boxes, by definition, can depend on more than one assumption. $\mathsf{LAL}_{\not\chi}$ is PTIME-*un*sound, because it contains EAL. Recall that EAL is the affine version of ELL [7], which soundly and completely characterizes the class of Elementary Functions. As a second try, we might think to extend LAL to $\mathsf{LAL}_{\nabla\S}$ by adding a Contraction $\S A \multimap (\S A \otimes \S A)$. Once again, $\mathsf{LAL}_{\nabla\S}$ is PTIME-*un*sound, because EAL easily embeds into it, translating ! of EAL with $\S$ of $\mathsf{LAL}_{\nabla\S}$. Finally, we extend LAL to $\mathsf{LAL}_{\flat}$ adding new modal rules, identical to the ones for !, but relative to a modal operator $\flat$. $\mathsf{LAL}_{\flat}$ is PTIME-sound as it trivially embeds into LAL. The PTIME-completeness of $\mathsf{LAL}_{\flat}$ should be evident since it contains LAL. Moreover, it is not possible to prove neither $!A \multimap \flat A$, nor $\flat A \multimap !A$. In principle, the new rules could make $\mathsf{LAL}_{\flat}$ *more expressive* than LAL. The reason is that the new rules, *in combination with the original modal rules of* LAL, can be used to represent structures inside SRN, otherwise not representable in LAL. Thus, (i) LAL *can be* PTIME-*soundly extended* to another system by adding some rules, and (ii) *the new rules may allow to write new interesting programs*. This potentially candidates $\mathsf{LAL}_{\flat}$ to replace (2) or (3) in Fig.1 because $\mathsf{LAL}_{\flat}$ might have programs missing in LAL that allow to simulate terms not in BC^-. In fact, the situation is a bit more complex, but we shall get back to this in the final section.

The stratified and multimodal framework MS (Section 2). Abstracting away from the experiments on LAL led to this work. Fig. 2 visualizes what we mean. MS was first introduced in [8]. Here we make the definition more essential, while extending it so that the (sub)systems of proof nets MS contains can use unconstrained Weakening. MS is a *Multimodal* and *Stratified* framework. MS is a class of triples $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$. The first element $\mathbb{X}$, we call carrier, is an *arbitrary* countable and finite set of elements we use as modalities inside a language $\mathcal{F}_{\mathbb{X}}$ of formulæ. The second element $\mathcal{B}_{\mathbb{X}}$ is a set of *building rules*, ideally partitioned into *linear* and *modal* ones. The *linear* building rules define the proof nets of Multiplicative and Second Order Fragment (MLL_2) of LL. The *exponential* building rules in $\mathcal{B}_{\mathbb{X}}$ of any given $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$ define: (i) which modal formulæ of $\mathcal{F}_{\mathbb{X}}$, that label the premises of a proof net, can be contracted into a single premise, and (ii) which modal formulæ are associated to the conclusion of a box we can build around a proof net. At this point, to keep things intuitive, we can think

a *subsystem* $\mathcal{P}$ of MS is every triple that satisfies an essential requirement to use the proof nets $\mathbf{PN}(\mathcal{P})$ that $\mathcal{B}_{\mathbb{X}}$ can generate as a rewriting system. This amounts to require that $\mathcal{R}_{\mathbb{X}}$ is the largest rewriting relation in $\mathbf{PN}(\mathcal{P}) \times \mathbf{PN}(\mathcal{P})$.
An example of a whole class of subsystems of MS is Linear_n, with $n \in \mathbb{N}$, already defined in [8]. Linear_n allows to remark the freedom we have when choosing a carrier set. Fig. 3(b) shows the partial order that we can take as $\mathbb{X}_n$ to define Linear_n, while Fig. 3(a) shows a subset of the Contraction nodes induced by $\mathbb{X}_n$. Remarkably, (i) Linear_n has a linear normalization cost, like MLL_2, but (ii) it can represent the Church numerals as much long as n, together with the basic operations on them, so strictly extending the expressiveness of MLL_2.

The local criterion (Section 3-4). Its statement relies on the notion of PTIME-maximal subsystem $\mathcal{P}$ of MS. Specifically, $\mathcal{P}$ is PTIME-maximal if it is PTIME-sound, and any of its extensions becomes PTIME-*un*sound. Our criterion says that a given $\mathcal{P}$ is PTIME-maximal by listing a set of sufficient and necessary conditions on the syntax of the building rules in $\mathcal{P}$. As a corollary, any given $\mathcal{P}'$ is PTIME-sound if its rules are among those ones of a PTIME-maximal subsystems of MS. To conclude, *spindle* (Section 3, Fig. 12(a)) is the technical notion we base our criterion on. A spindle is a conceptual abstraction of the general quantitative analysis tools that Context Semantics (CS) [9] supplies. Intuitively, if a subsystem $\mathcal{P}$ allows to concatenate arbitrary long *chains of spindles* (Fig. 12(b)), then it is PTIME-*un*sound, namely some of its rules cannot by instance of any PTIME-maximal subsystem of MS.

Acknowledgments. We like to thank the anonymous referees whose comments allowed to better justify the main aspects of this work, we hope.

2 The framework MS

We define MS by extending, and cleaning up, the definition in [8]. Our current MS generates subsystems with unconstrained weakening, like LAL, to easy programming.

The Formulæ. Let $\mathbb{X}$ be an alphabet of *modalities*, ranged over by $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \ldots$, and $\mathcal{V}$ be a countable set of propositional variables, ranged over by $x, y, w, \ldots$. The set $\mathcal{F}_{\mathbb{X}}$ of *formulæ*, generated with F as start symbol, is:

$$F ::= L \mid E \qquad E ::= \mathfrak{m}F \qquad L ::= x \mid F \otimes F \mid F \multimap F \mid \forall x.F$$

E generates *modal* formulæ, L *linear* (*non-modal*) ones. A, B, C range over formulæ of $\mathcal{F}_{\mathbb{X}}$. Γ, Δ, Φ range over, possibly empty, multisets of formulæ. $A\left[{}^{B}/_{y}\right]$ is the substitution of B for y in A. The *number of modalities* of $\mathcal{F}_{\mathbb{X}}$ is the cardinality of $\mathbb{X}$.

The framework. MS is a class of triples $(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$, for every countable set $\mathbb{X}$. An element $\mathcal{B}_{\mathbb{X}}$ is a subset of the *building rules* in Fig. 4, typed with formulæ of $\mathcal{F}_{\mathbb{X}}$. The nodes (i), (o) just show inputs and output, respectively. The other nodes are standard ones. Both **Promotion** and **Contraction** are *modal* as opposed to the *linear* remaining ones. The second component in $\mathcal{R}_{\mathbb{X}}$ is a subset of the *rewriting rules* in Fig. 5, 6, 7, and 8, typed with formulæ of $\mathcal{F}_{\mathbb{X}}$. The name of every rewriting rule recalls the nodes it involves. Fig. 5 defines the *linear rewriting rules* which, essentially, just rewires a proof net. Fig. 6 contains the *modal rewriting rules*, those ones that, once instantiated, may cause exponential blow up. Fig. 7 and 8 describe *garbage collection*.

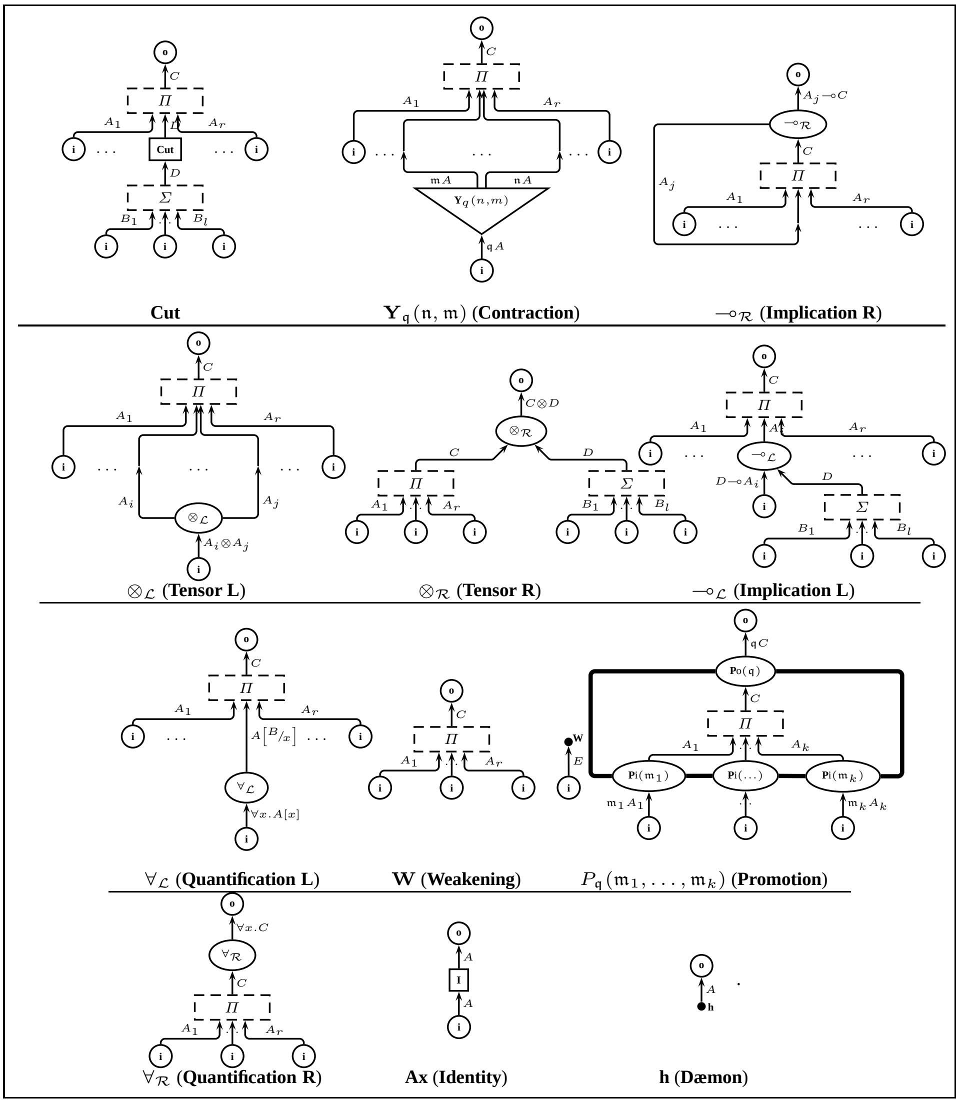


Fig. 4. Building rules, with short and long names.

Subsystems. A triple $\mathcal{P} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$ in MS is a *subsystem* (of MS) whenever:

1. $\mathcal{B}_{\mathbb{X}}$ contains all the instances of the linear building rules;
2. $\mathcal{B}_{\mathbb{X}}$ contains every instance of **Promotion** $P_{\mathfrak{n}}()$, that generates *closed* $\mathfrak{n}$-*boxes*;
3. $\mathcal{B}_{\mathbb{X}}$ is *downward closed*. Namely, for every $P_{\mathfrak{q}}(\mathfrak{m}_0, \ldots, \mathfrak{m}_k)$ in $\mathcal{B}_{\mathbb{X}}$, $P_{\mathfrak{q}}(\mathfrak{n}_0, \ldots, \mathfrak{n}_l)$ belongs to $\mathcal{B}_{\mathbb{X}}$ as well, for every $\{\mathfrak{n}_0, \ldots, \mathfrak{n}_l\} \subseteq \{\mathfrak{m}_0, \ldots, \mathfrak{m}_k\}$;
4. If we denote by $\mathbf{PN}(\mathcal{P})$ the set of proof nets that $\mathcal{B}_{\mathbb{X}}$ inductively generates, using **Identity** and **Dæmon** as base cases, then $\mathcal{R}_{\mathbb{X}}$ is the *largest* rewriting relation inside $\mathbf{PN}(\mathcal{P}) \times \mathbf{PN}(\mathcal{P})$.

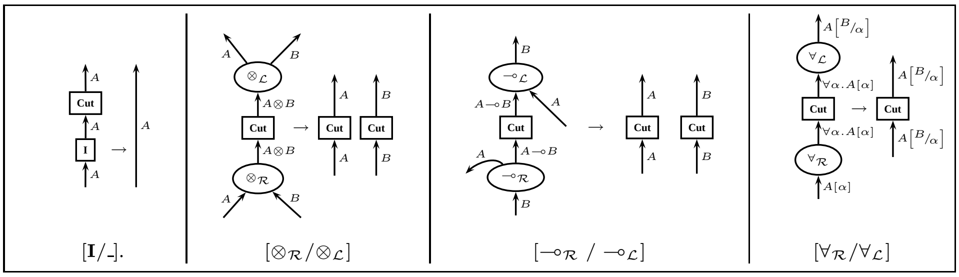


Fig. 5. *Linear* rewriting rules. $[\forall_{\mathcal{R}}/\forall_{\mathcal{L}}]$ substitutes B for α as usual.

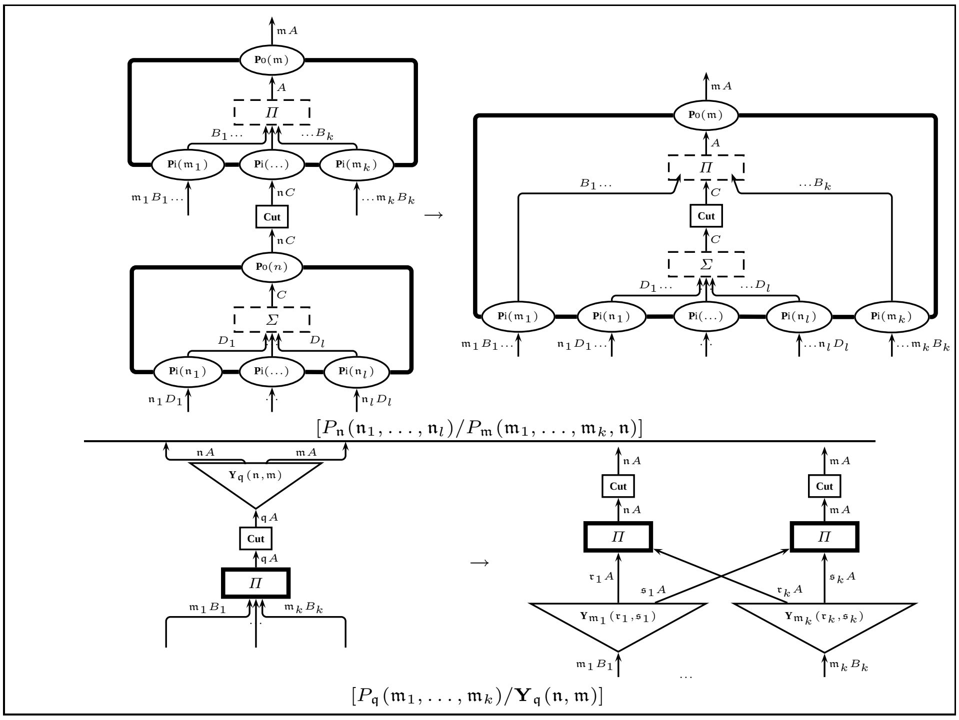


Fig. 6. Modal rewriting rules.

By abusing the notation, we write $\mathcal{P} \subseteq \mathsf{MS}$ to abbreviate that $\mathcal{P}$ is a subsystems of MS.

An example of subsystem. Let $M \geq 2$, and $\mathbb{X} = \{i! \mid i \leq M\} \cup \{i\S \mid i \leq M\}$. Then $\mathsf{sLAL} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$ is the subsystem of MS such that $\mathcal{B}_{\mathbb{X}}$ contains the modal building rules in Fig.9. The "s" in front of LAL stands for *sorted*. With $\mathfrak{n}^?$ we mean that the premise it represents can occur at most once. With $\mathfrak{n}^*$ we mean that the premise it represents can occur an unbounded, but finite, number of times. By the forthcoming

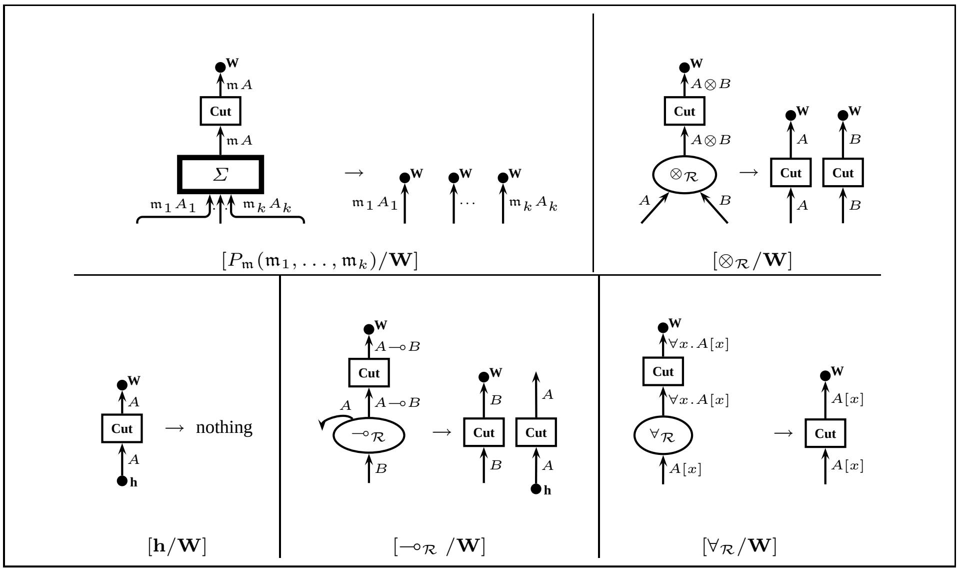


Fig. 7. Garbage collection rewritign rules, involving $\mathbf{W}$.

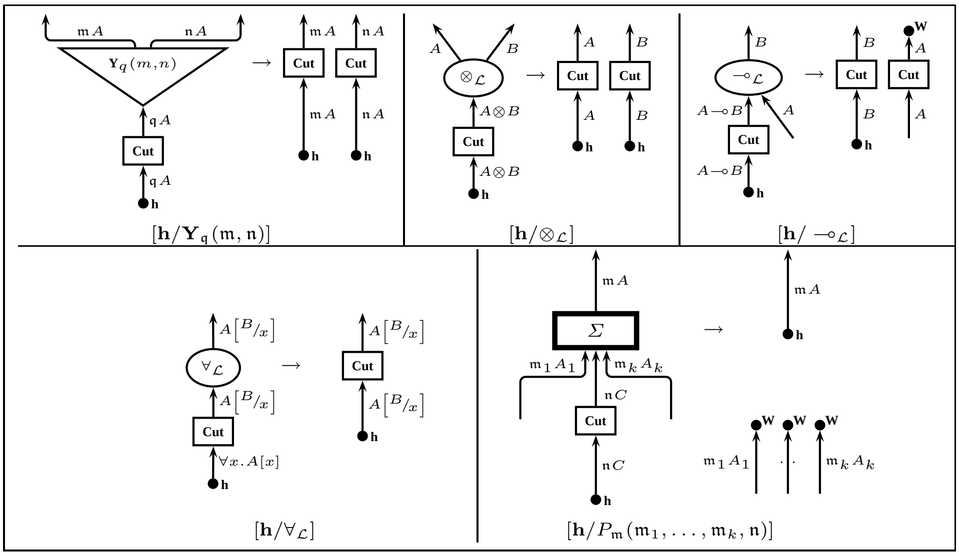


Fig. 8. Garbage collection rewriting rules, involving $\mathbf{h}$.

Proposition 1, sLAL is PTIME-sound. Notice that sLAL strictly extends LAL. We shall get back to the relevance of sLAL in Section 5.

Remarks on standard computational properties of subsystems. A subsystem $\mathcal{P}$ does not necessarily enjoy standard computational properties. For example, in a given $\mathcal{P}$ a

$$P_{i!}(i!^{?}, j_1\S^*, \dots, j_m\S^*) \text{ for every } j_1, \dots, j_m < i \leq M$$
$$P_{i\S}(i!^*, i\S^*, j_1\S^*, \dots, j_m\S^*) \text{ for every } j_1, \dots, j_m < i \leq M$$
$$\mathbf{Y}_{i!}(i!, i!) \text{ for every } i \leq M$$

Fig. 9. The modal rules of sLAL.

full normalization may fail because some building rule is missing. Analogously, the Church-Rosser property may not hold, because, for example, $[P_{\mathfrak{q}}(\mathfrak{r})/\mathbf{Y}_{\mathfrak{q}}(\mathfrak{n},\mathfrak{m})]$ generates a non-confluent critical pair. This might be considered a drawback of the "wild" freedom in the definition of the instances of $[P_{\mathfrak{q}}(\mathfrak{r})/\mathbf{Y}_{\mathfrak{q}}(\mathfrak{n},\mathfrak{m})]$. We believe such a freedom necessary to have a chance to find some replacement of the question marks in Fig. 1. Instead, every $\mathcal{P}$ is strongly normalizing since $\mathbf{PN}(\mathcal{P})$ embeds into the proof nets of EAL by collapsing all its modalities into the single one of EAL. So, standard results imply that $\mathcal{P}$ is Church-Rosser if it is locally confluent.

Notations. Let Π be a proof net of a given $\mathcal{P} \subseteq$ MS. The set of its nodes is V_Π, and E_Π the set of edges. Moreover, B_Π is the set of *Box-out* nodes, in natural bijection with the set of the *boxes*. A box, corresponding to some instance of $P_{\mathfrak{q}}(\mathfrak{m}_1, \dots, \mathfrak{m}_k)$, has one *conclusion* of type $\mathfrak{q}C$ and k *assumptions* of type respectively $\mathfrak{m}_1 A_1, \dots, \mathfrak{m}_k A_k$, for some $C, A_1, \dots, A_k$. The *depth*, or *level*, $\partial(u)$ of $u \in V_\Pi \cup E_\Pi$ is the greatest number of nested boxes containing u. The *depth* $\partial(\Pi)$ *of* Π is the greatest $\partial(u)$ with $u \in V_\Pi$. The *size* $|\Pi|$ counts the number of the nodes in Π. We notice that *Box-in/out* nodes do not contribute to the size of the proof net inside the box they delimit.

3 Polynomial time soundness

The goal of this section is an intermediate step to get the local criterion. Here we characterize the class PMS of *polynomial time sound* (PTIME-sound) subsystems of MS. MS is the one in Sec. 2, which, recall, generalizes, while cleaning it up, the one in [8] by adding unconstrained weakening, and the corresponding dæmon. This is why we shall briefly recall the main tools and concepts that allow to prove a sufficient structural condition for PTIME-soundness, relatively to subsystems of MS. Later, we prove the new PTIME-sound necessary condition.

3.1 Sufficient condition for PTIME-soundness

Context Semantics (CS) [9]. The basic tool to identify a sufficient structural condition on the subsystems of MS that implies PTIME-soundness is CS. We use a simplified version of CS, because the proof nets that any subsystem of MS can generate are free of both dereliction and digging.
CS identifies CS-paths to travel along any proof net of any subsystem. Every CS-path simulates the annihilation of pairs of nodes that, eventually, will interact thanks to the application of *r.steps*. The goal of using CS to analyze our proof nets is to count the number $R_\Pi(u)$ of *maximal* CS-*paths* which, traversing a proof net Π, go from any box root u of Π to either a weakening node, erasing it, or to the terminal node of Π or of

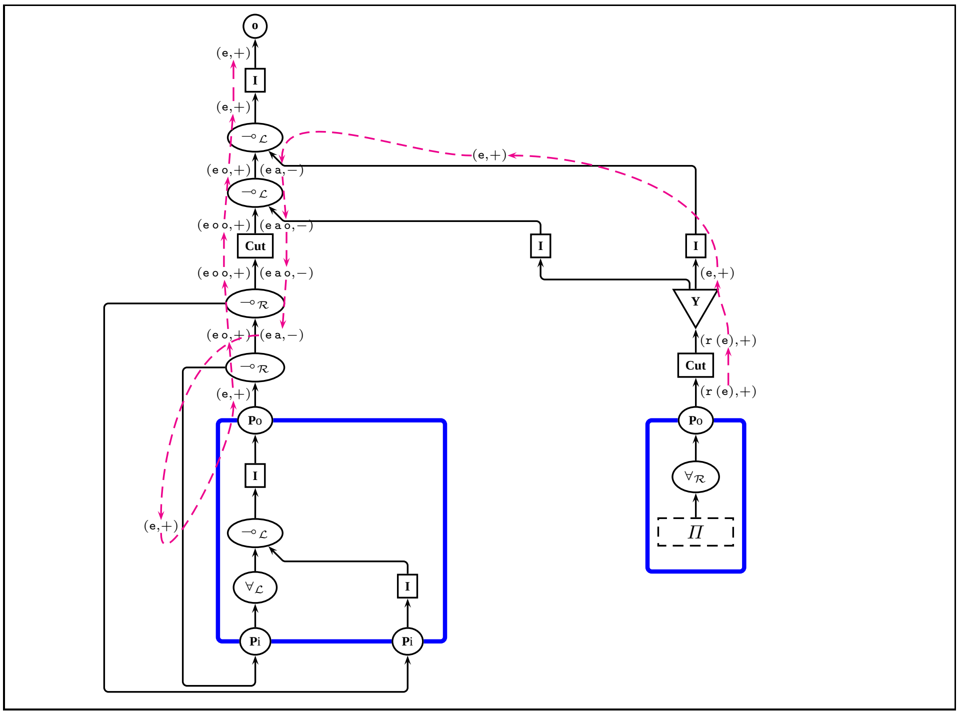


Fig. 10. An example of maximal path.

the proof net inside a box that contains u. The paths that we consider do never cross the border of a box. The value of $R_{\Pi}(u)$ counts the contraction nodes that, possibly, will duplicate the box rooted at u.

Figure 10 shows an example of maximal CS-path from the rightmost box to the conclusion of the proof net. It is built by interpreting every node as it was a kind of operator that manipulates the top element of a stack whose elements can contain symbols of a specific signature. For example, let us focus on the pair $(\mathtt{r}(\mathtt{e}), +)$ the CS-path in Figure 10 starts from. The polarity $+$ says we are feeding the contraction with the value $\mathtt{r}(\mathtt{e})$, coherently with the direction of its premise. We have to think that the top cell of a stack stores $\mathtt{r}(\mathtt{e})$. The contraction node replaces $\mathtt{r}(\mathtt{e})$ by $\mathtt{e}$ in the top. The axiom node behaves as an identity operator on the top. The first $\multimap_{\mathcal{L}}$ we meet pushes $\mathtt{a}$ on the top, so that such a symbol can be popped out by the second $\multimap_{\mathcal{R}}$ we meet. We keep going with these corresponding push-pop sequences until we reach the conclusion. By the way, we notice that the leftmost box is interpreted as an identity w.r.t. the stack content. This is because we are in a stratified setting.

Figure 11 recalls the set of transition steps that formally realize the stack machine that builds the CS-paths and that we can use to work out the details about how constructing the maximal CS-path just described.

The point of determining $R_{\Pi}(u)$. CS allows to define a weight $\mathrm{W}(\Pi)$, for every Π in a subsystem of MS. The weight has two relevant properties.

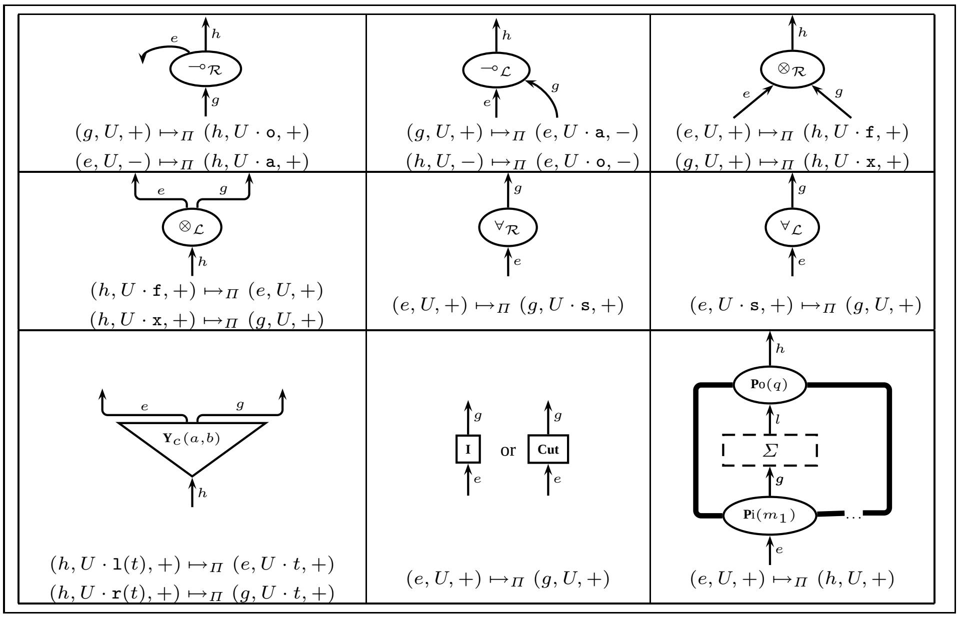


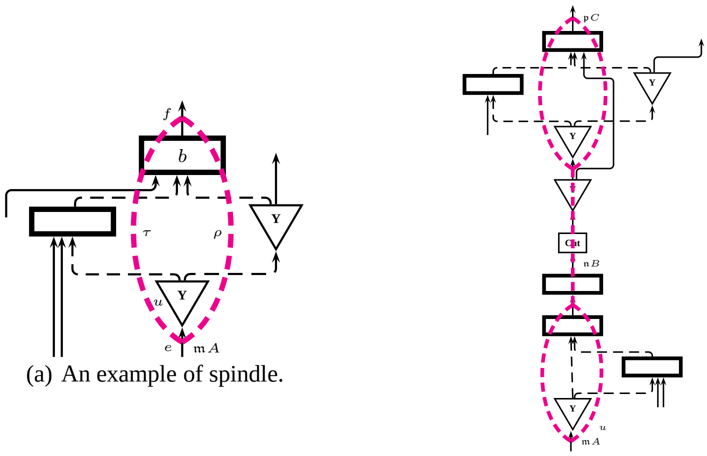
Fig. 11. Rewriting relation among contexts. If $(e, U, b) \mapsto_\Pi (e', U', b')$ then also $(e', U', -b') \mapsto_\Pi (e, U, -b)$, where $-b$ is the polarity opposed to b.

The first one is that the *r.steps* strictly decrease $\mathrm{W}(\Pi)$, for any Π. Namely, the weight bounds both the normalization time of Π and the size of the reducts of Π under a standard normalization strategy which is proved to be the worst one. So, the weight is a bound for any strategy.

The second property is that $\mathrm{W}(\Pi)$, for any Π, is dominated by $\sum_{b \in B_\Pi} R_\Pi(u)$, up to a polynomial. More formally, it exists a polynomial $p(x, y)$ such that, for every Π, $W(\Pi) \leq p\left(\sum_{b \in B_\Pi} R_\Pi(u), |\Pi|\right)$.

Spindles. They are specific configurations inside the proof nets of a given $\mathcal{P}$. If the structural properties of $\mathcal{P}$ limit the number of spindles we can concatenate in every of its proof nets, then we can deduce a polynomial bound on $\sum_{b \in B_\Pi} R_\Pi(u)$, giving us a sufficient condition for PTIME-soundness. Figure 12(a) shows an example of *spindle* between u and b. A spindle contains two, and only two, distinct CS-paths to go from one node of a proof net to another. Two CS-paths can duplicate structure, a potentially harmful behavior when the control over the normalization complexity is a concern.

Definition 1 (Spindles). *Let* $\mathcal{P} \subseteq \mathsf{MS}$*,* $\Pi \in \mathbf{PN}(\mathcal{P})$*;* $e \in E_\Pi$ *an edge labelled* $\mathfrak{m}A$ *entering a contraction* u*;* $f \in E_\Pi$ *an edge labelled* $\mathfrak{n}B$ *outgoing a* $\mathbf{Po}$ *node* b*;* $\partial(e) = \partial(f)$*. A* spindle $\mathfrak{m}A : \Sigma : \mathfrak{n}B$ *between* e *and* f*, or* u *and* b*, is a pair of* CS*-paths s.t.:* τ *is from* e *to* f*, through the left conclusion of* u*;* ρ *is from* e *to* f*, through the right conclusion of* u*;* τ *and* ρ *are the only* CS*-paths connecting* e *with* f*. The* principal premise *of* Σ *is* e*, the* principal conclusion *is* f*, and* u *the* principal contraction*. An*



(a) An example of spindle.

(b) A chain of two spindles.

Fig. 12. A spindle and a chain of two spindles.

edge that is premise (resp. conclusion) of a node of Σ, but that is not part of Σ, is said non-principal *premise (resp. conclusion). We shorten $\mathfrak{m}A:\Sigma:\mathfrak{n}B$ with $\mathfrak{m}:\Sigma:\mathfrak{n}$.*

Chains of spindles. Let the proof net Π contain both the spindles $\mathfrak{m}_i A_i:\Sigma_i:\mathfrak{n}_i B_i$, with $1 \leq i \leq r$, and, for every $1 \leq i < r$, a CS-path χ_i from the principal conclusion of Σ_i to the principal premise of Σ_{i+1}. Then, $\mathfrak{m}_1 A_1:\Theta:\mathfrak{n}_r B_r$, where $\Theta = (\bigcup_{i=1}^{r} \Sigma_i) \cup (\bigcup_{i=1}^{r-1} \chi_i)$, is a *chain of r spindles*, and $|\Theta| = r$ its size. Intuitively, a subsystem of MS that can build arbitrarily long chains of spindles cannot be PTIME-sound, for we cannot bound the amount of duplicated structure when normalizing. Arbitrarily long chains exist as soon as the rules of a subsystem allow to build $\mathfrak{m}A:\Theta':\mathfrak{m}A$, which can compose with itself an arbitrary number of times. The *dangerous chains* of spindles are the "prelude" to the construction of Θ' here above. A *dangerous chain* is $\mathfrak{m}A:(\Theta\cup\chi):\mathfrak{m}C$ where $\mathfrak{m}A:\Theta:\mathfrak{n}B$ is a chain, and χ a CS-path from the principal conclusion, with type $\mathfrak{n}B$, of the last spindle in Θ, and an edge whose type has modality $\mathfrak{m}$.
Finally, the sufficient condition for PTIME-soundness that extends the one in [8]:

Proposition 1 (PTIME-soundness: Sufficient Condition). *Let $\mathbb{X}$ be finite, and $\mathcal{P} \subseteq$ MS. If $\mathcal{P}$ cannot build dangerous spindles, then $\mathcal{P} \in$ PMS.*

Proof (Sketch). Both the absence of dangerous spindles in any proof net Π of $\mathcal{P}$ and the finiteness of $\mathbb{X}$ lead to a constant bound L on the length of the chains of spindles in Π. L only depends on $\mathcal{P}$. The bound L implies the existence of a polynomial that only depends on $\mathcal{P}$ and on the depth of Π, which bounds $\sum_{b\in B_\Pi} R_\Pi(u)$. For what observed above, this implies the PTIME-soundness of $\mathcal{P}$. □

3.2 Necessary condition for PTIME-soundness

Here we present a necessary condition for PTIME-soundness of subsystems in MS.

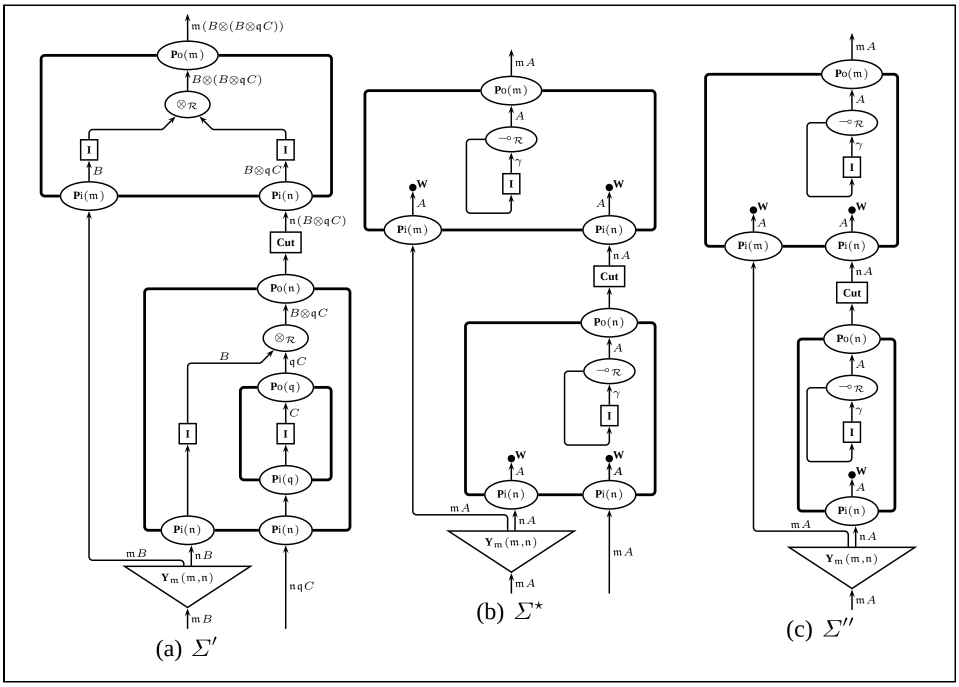


Fig. 13. Examples relative the proof of Lemma 1.

Proposition 2 **(**PTIME**-soundness Necessary Condition).** *Let* $\mathcal{P} \subseteq$ MS*. If* $\mathcal{P} \in$ PMS *then* $\mathcal{P}$ *cannot build dangerous spindles.*

The idea to prove Proposition 2 is that if $\mathfrak{m}A : \Sigma \cup \chi : \mathfrak{m}B$ exists in $\mathcal{P}$, then $\Sigma \cup \chi$ can be constructively transformed into $\mathfrak{m}C : (\Sigma' \cup \chi') : \mathfrak{m}C$ that freely composes with itself, leading to an exponential blow-up. Now, the formal steps to prove Proposition 2 follow.

Fact 1. *Let* $\mathcal{P} \subseteq$ MS*.*

1. *For every* $l \geq 0$*,* $\mathcal{P}$ *contains* $\Pi_l \triangleright A, \ldots, A \vdash A$*, with* $A = \gamma \multimap \gamma$*,* l *occurrences of* A *as assumptions, and* $\partial(\Pi_l) = 1$*.*
2. *If* $\mathcal{P}$ *proves* $A, \ldots, A \vdash A$*, with* l *occurrences of* A*, then it proves also* $\mathfrak{m}_1 A, \ldots, \mathfrak{m}_k A \vdash \mathfrak{q}A$ *for* $k < l$*, for every* $P_{\mathfrak{q}}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k) \in \mathcal{P}$*.*

For example, Π_l in Fact 1 can just contain the nodes $(\multimap_{\mathcal{R}})$ and $\bullet\mathbf{W}$.

Lemma 1. *Let* $\mathcal{P} \subseteq$ MS *be a subsystem that can build a dangerous spindle* $\Sigma \cup \chi$*. In* $\mathcal{P}$ *there is another dangerous spindle* $\Sigma'' \cup \chi''$*, obtained constructively from* $\Sigma \cup \chi$*, such that: (a)* $\Sigma'' \cup \chi''$ *contains only Contractions, Box-out and Cut nodes at level 0; (b) Every edge* e *at level* 0 *has label* $\mathfrak{q}A$*, for some fixed* A*; (c)* $\partial(\Sigma'' \cup \chi'') = 1$*; (d) The only premise of* $\Sigma'' \cup \chi''$ *is the principal one.*

Proof. Let $\Pi \in \mathbf{PN}(\mathcal{P})$ be the proof net containing $\Sigma \cup \chi$.

We reduce all the linear cuts in Π at level 0. We get to $\Pi' \in \mathbf{PN}(\mathcal{P})$ with $\Sigma' \cup \chi'$, the reduct of $\Sigma \cup \chi$, in it. An example of a possible $\Sigma' \cup \chi'$ is in Figure 13(a), where χ' is empty. $\Sigma' \cup \chi'$ is still a dangerous spindle, as well as the residuals of the three CS-paths ρ, τ, χ of $\Sigma \cup \chi$ are still CS-paths in Π'. The three CS-paths cannot contain linear nodes. Indeed, we just observe that, e.g., τ must begin in a contraction, with label $\mathfrak{n}A$, and must stop in a box, with label $\mathfrak{m}B$. So it cannot cross neither any right-node, otherwise it would add a non-modal symbol to $\mathfrak{n}A$, and so the last formula could not be $\mathfrak{m}B$, nor any left-node, otherwise it would remove a non-modal symbol from $\mathfrak{n}A$.
We transform $\Sigma' \cup \chi'$, just generated, into another graph $\Sigma^\star \cup \chi^\star$ by replacing every proof net enclosed in a box $P_{\mathfrak{q}}(\mathfrak{m}_1, \ldots, \mathfrak{m}_k)$ of $\Sigma' \cup \chi'$ by $\Pi_k \rhd A, \ldots, A \vdash A$ as in point 1 of Fact 1. Consequently, every $\mathfrak{q}B$ at level 0 in Σ' becomes $\mathfrak{q}A$ in $\Sigma^\star$, for some $\mathfrak{q}$. This replacement implies $\partial(\Sigma^\star \cup \chi^\star) = 1$ (Figure 13(b)).
$\Sigma^\star \cup \chi^\star$ is a graph satisfying the requirements (a)-(b)-(c) in the statement we want to prove. However, in general, it does not satisfy (d). Moreover, $\Sigma^\star \cup \chi^\star$ is not necessarily a spindle because it may not be contained in a proof net of $\mathcal{P}$. We can modify $\Sigma^\star \cup \chi^\star$ to get a $\Sigma'' \cup \chi''$ that satisfies the point (d). If e is a non-principal premise of $\Sigma^\star \cup \chi^\star$, entering some box b of $\Sigma^\star \cup \chi^\star$, then we remove the edge e and we reduce the number of premises of b, according to point 2 of Fact 1. Now, we can plug every non-principal conclusion of $\Sigma'' \cup \chi''$, exiting upward from some contraction u of $\Sigma'' \cup \chi''$, with a weakening node: we get a proof net Π'' containing $\Sigma'' \cup \chi''$, thus showing that $\Sigma'' \cup \chi''$ is a dangerous spindle. □

Proof (of Proposition 2). By contraposition, we show that: "If $\mathcal{P}$ can build a dangerous spindle $\Sigma \cup \chi$, then it is not PTIME-sound." Let us assume $\mathfrak{m} : (\Sigma \cup \chi) : \mathfrak{m}$ has premises $\Gamma, \mathfrak{m}B$ and conclusions $\Delta, \mathfrak{m}C$, remembering that Σ may not be a proof net, so admitting more non-principal conclusions Δ. By Lemma 1 we build a dangerous $\Sigma'' \cup \chi''$ of depth 1 with premise $\mathfrak{m}A$ and conclusions $\mathfrak{m}A, \tilde{\Delta}$. Then $\Sigma'' \cup \chi''$ becomes a proof net $\Pi'' \rhd \mathfrak{m}A \vdash \mathfrak{m}A$, using weakening. Now we concatenate as many copies of Π'' as we want, with a closed box b whose conclusion has type $\mathfrak{m}A$. We get a family $\langle \Theta_n \rhd \vdash \mathfrak{m}A \mid n \in \mathbb{N} \rangle$ of proof nets. Every Θ_n has depth 1 and size $O(n)$. The canonical strategy replicates b, until the level 0 is normal. This takes linear time. Though, the final size at level 1 has grown exponentially, implying that Θ_n reduces in time $O(2^n)$. So, $\mathcal{P} \notin$ PMS. □

Remark 1. The proof of Proposition 2 highlights a peculiar aspect of PTIME-sound subsystems. PTIME-soundness combines two more primitive properties we can call *polynomial step soundness* (pstep) and *polynomial size soundness* (psize). A subsystem Π is pstep (resp. psize) iff there is a polynomial $p(x)$ such that, for every $\Pi \in \mathcal{P}$, $\Pi \to^k \Sigma$ implies that k (resp. $|\Sigma|$) is bounded by $p(|\Pi|)$. So, the proof of Proposition 2 shows also that "*if* $\mathcal{P} \subseteq$ MS *is psize, then it is pstep as well, namely* PTIME-*sound*".

4 PTIME-maximal subsystems

This section supplies the local criterion that distinguishes PTIME-sound and PTIME-*un*sound subsystems of MS, just looking at their rules.

Definition 2 **(PTIME-maximal subsystems).** *Let $\mathcal{P} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$ be a subsystem of* MS*. We say that $\mathcal{P}$ is* PTIME-maximal *if, for every $\mathcal{P}' = (\mathbb{X}, \mathcal{B}'_{\mathbb{X}}, \mathcal{R}'_{\mathbb{X}})$, $\mathcal{B}_{\mathbb{X}} \subset \mathcal{B}'_{\mathbb{X}}$ implies $\mathcal{P}' \notin$* PMS*.*

Let $\mathcal{P} \subseteq$ MS. We write $\mathfrak{m} \preceq \mathfrak{n}$ whenever there exists a proof net $\Pi \in \mathbf{PN}(\mathcal{P})$ containing a CS-path whose edges are all labeled by modal formulæ, the first one exiting upwards from a $\mathbf{Po}$ node with modality $\mathfrak{m}$, and the last one with modality $\mathfrak{n}$. $\preceq$ is clearly transitive.

Fact 2. *Let $\mathcal{P}$ be a* PTIME*-maximal subsystem.*

1. *If $\mathcal{P}$ has $P_{\mathfrak{q}}(\overrightarrow{\mathfrak{m}}, \mathfrak{n}, \mathfrak{n})$ it also has $P_{\mathfrak{q}}(\overrightarrow{\mathfrak{m}}, \mathfrak{n}^*)$, where $\mathfrak{n}^*$ stands for any unlimited and finite sequence of assumptions with a $\mathfrak{n}$-modal formula.*
2. *If $\mathcal{P}$ has $P_{\mathfrak{q}}(\mathfrak{m}, \mathfrak{n})$ and $P_{\mathfrak{q}}(\mathfrak{n}, \mathfrak{r})$ it also has $P_{\mathfrak{q}}(\mathfrak{m}, \mathfrak{n}, \mathfrak{r})$.*
3. *$\forall \mathfrak{q} \in \mathbb{X}$, $\mathcal{P}$ has at least $P_{\mathfrak{q}}(\mathfrak{q}^?)$, where $\mathfrak{n}^?$ stands for at most one assumption with a $\mathfrak{n}$-modal formula. This is like requiring a reflexive $\preceq$.*

The justification to the first point develops as follows. $\mathcal{P}$ cannot have $P_{\mathfrak{q}}(\overrightarrow{\mathfrak{m}}, \mathfrak{n}^*)$ if such a rule generates a spindle. In that case, the same spindle exists thanks to $P_{\mathfrak{q}}(\overrightarrow{\mathfrak{m}}, \mathfrak{n}, \mathfrak{n})$ against the PTIME-maximality of $\mathcal{P}$. An analogous argument can be used to justify the second point. The third one is obvious because $P_{\mathfrak{q}}(\mathfrak{q}^?)$ has at most one assumption.

Lemma 2. *Let $\mathcal{P}$ be a* PTIME*-maximal subsystem. Then $\preceq$ is a* linear quasi-order*, i.e. $\preceq$ is transitive, reflexive and connected.*

Proof. $\preceq$ is always transitive. Here it is also reflexive, because of the presence of $P_{\mathfrak{q}}(\mathfrak{q})$. We have to prove that it is connected, that is: $\forall \mathfrak{m}, \mathfrak{n}\ (\mathfrak{m} \preceq \mathfrak{n} \vee \mathfrak{n} \preceq \mathfrak{m})$. Let us assume that $\mathcal{P}$ is PTIME-maximal and $\neg(\mathfrak{m} \preceq \mathfrak{n})$. We show that $\mathfrak{n} \preceq \mathfrak{m}$. $\neg(\mathfrak{m} \preceq \mathfrak{n})$ implies that adding $P_{\mathfrak{n}}(\mathfrak{m})$ to $\mathcal{P}$ gives a PTIME-*un*sound system $\mathcal{P}'$. So we can find a dangerous spindle $\mathfrak{q} : (\Sigma \cup \chi) : \mathfrak{q}$ in $\mathcal{P}'$ containing $P_{\mathfrak{n}}(\mathfrak{m})$. $\Sigma \cup \chi$ is not in $\mathcal{P}$. However, in $\mathcal{P}$, we can find a graph Σ' that, extended with a suitable number of $P_{\mathfrak{n}}(\mathfrak{m})$ boxes, gives Σ. In particular, we find in Σ' (and so in $\mathcal{P}$) two paths, one from modality $\mathfrak{q}$ to $\mathfrak{m}$ and the other one from $\mathfrak{n}$ to $\mathfrak{q}$, that can be composed to create a path from $\mathfrak{n}$ to $\mathfrak{m}$. So, $\mathfrak{n} \preceq \mathfrak{m}$. $\square$

Remark 2. We focus only on PTIME-maximal systems with a *linear order* $\preceq$. This because any $\mathcal{P}$ whose $\preceq$ is not as such is as expressive as another $\mathcal{P}'$ with less modalities and whose $\preceq$ is linear. For example, if $\mathcal{P}$ is PTIME-maximal and contains $P_{\mathfrak{n}}(\mathfrak{m})$ and $P_{\mathfrak{m}}(\mathfrak{n})$, then $\mathfrak{m}$ and $\mathfrak{n}$ cannot be distinguished, as $\preceq$ is transitive, so $\mathfrak{n}, \mathfrak{n}$ can be identified. We shall write $\prec$ for the strict order induced by $\preceq$.

Theorem 1 **(Structure of PTIME-maximal Subsystems).** *Let $\mathcal{P}$ be a subsystem of* MS *with a linear $\preceq$. $\mathcal{P}$ is* PTIME*-maximal iff $\mathcal{P}$ contains exactly the following rules:*

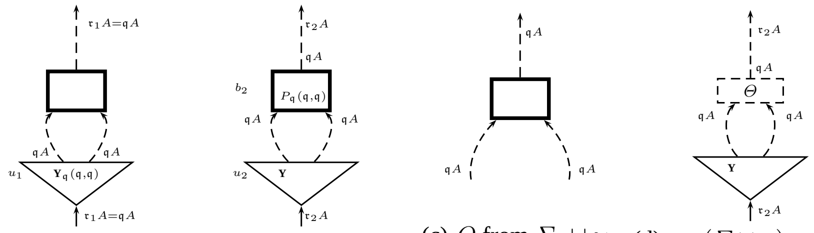
1. *all the rules $\mathbf{Y}_{\mathfrak{q}}(\mathfrak{m}, \mathfrak{n})$ for every $\mathfrak{q} \prec \mathfrak{m}, \mathfrak{n}$;*
2. *all the rules $\mathbf{Y}_{\mathfrak{q}}(\mathfrak{q}, \mathfrak{n})$ for every $\mathfrak{q} \prec \mathfrak{n}$;*
3. *all the rules $P_{\mathfrak{q}}(\mathfrak{q}^?, \mathfrak{m}_1^*, \dots, \mathfrak{m}_k^*)$ for every $\mathfrak{m}_1, \dots, \mathfrak{m}_k \prec \mathfrak{q}$;*
4. *only one among $\mathbf{Y}_{\mathfrak{q}}(\mathfrak{q}, \mathfrak{q})$ and $P_{\mathfrak{q}}(\mathfrak{q}^*, \mathfrak{m}_1^*, \dots, \mathfrak{m}_k^*)$, for every $\mathfrak{m}_1 \dots, \mathfrak{m}_k \prec \mathfrak{q}$.*

For example, LAL is not maximal. It can be extended to a PTIME-maximal system letting $! \prec \S$, and adding the missing rules.

Proof. "Only if" direction. We want to prove that every system $\mathcal{P} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, \mathcal{R}_{\mathbb{X}})$ containing the rules described in points *1*, *2*, *3*, 4 is PTIME-maximal. The order $\preceq$ prevents to build dangerous spindles thanks to its linearity. So, Proposition 1 implies $\mathcal{P}$ is PTIME-sound. Moreover, if $\mathbf{Y}_{\mathfrak{q}}(\mathfrak{q},\mathfrak{q})$ belongs to $\mathcal{P}$, adding $P_{\mathfrak{q}}(\mathfrak{q}^*, \mathfrak{m}_1^*, \ldots, \mathfrak{m}_k^*)$, against point *4*, would allow to construct a dangerous spindle. The same would be by starting with $P_{\mathfrak{q}}(\mathfrak{q}^*, \mathfrak{m}_1^*, \ldots, \mathfrak{m}_k^*)$ in $\mathcal{P}$ and, then, adding $\mathbf{Y}_{\mathfrak{q}}(\mathfrak{q},\mathfrak{q})$.
"If" direction. We assume $\mathcal{P}$ is PTIME-maximal and we want show that it must contain at least all the rules described in the statement. We prove it by contradiction first relatively to points *1*, *2*, and *3*, then to point 4. A contradiction relative to points *1*, *2*, and *3* requires to assume that $\mathcal{P}$ has not a rule R described in *1*, *2*, *3*. Let $\mathcal{B}'_{\mathbb{X}} = \mathcal{B}_{\mathbb{X}} \cup \{R\}$. The PTIME-maximality of $\mathcal{P}$ implies that $\mathcal{P}' = (\mathbb{X}, \mathcal{B}'_{\mathbb{X}}, \mathcal{R}'_{\mathbb{X}})$, for the suitable $\mathcal{R}'_{\mathbb{X}}$, is not PTIME-sound. Thanks to the contraposition of Proposition 1, $\mathcal{P}' \notin$ PMS implies that $\mathcal{P}'$ can build a dangerous spindle $\mathfrak{r} : (\Sigma \cup \chi) : \mathfrak{r}$. Now we show that this is ht source of various kinds of contradictions.
Let u be an instance of $R = \mathbf{Y}_{\mathfrak{q}}(\mathfrak{m},\mathfrak{n})$ inside $\mathfrak{r} : (\Sigma \cup \chi) : \mathfrak{r}$. For example, we can assume that one of the two distinct CS-paths in Σ crosses u from $\mathfrak{q}A$ to $\mathfrak{m}A$. By definition of $\preceq$, $\mathfrak{r} \preceq \mathfrak{q} \prec \mathfrak{m} \preceq \mathfrak{r}$, so contradicting the linearity of $\preceq$.
Let u be an instance of $R = \mathbf{Y}_{\mathfrak{q}}(\mathfrak{q},\mathfrak{n})$ inside $\mathfrak{r} : (\Sigma \cup \chi) : \mathfrak{r}$. The previous case excludes that one of the two distinct CS-paths in $\Sigma \cup \chi$ crosses u from $\mathfrak{q}A$ to $\mathfrak{n}A$. So, it must cross u from its premise, labeled $\mathfrak{q}A$ to its conclusion, labeled $\mathfrak{q}A$. This means that $\mathfrak{r} : (\Sigma' \cup \chi') : \mathfrak{r}$, obtained from $\Sigma \cup \chi$ just removing u, exists in $\mathcal{P}$, which could not be PTIME-sound.
The case $R = P_{\mathfrak{q}}(\mathfrak{q}, \mathfrak{m}_1, \ldots, \mathfrak{m}_k)$ combines the two previous ones.
Now we move to prove point *4*. Without loss of generality, we can prove the thesis for $P_{\mathfrak{q}}(\mathfrak{q},\mathfrak{q})$ instead of $P_{\mathfrak{q}}(\mathfrak{q}^*, \mathfrak{m}_1^*, \ldots, \mathfrak{m}_k^*)$. By contradiction, let $\mathcal{P}$ be PTIME-maximal, and let the thesis be false. So, there are $R_1 = \mathbf{Y}_{\mathfrak{q}}(\mathfrak{q},\mathfrak{q})$ and $R_2 = P_{\mathfrak{q}}(\mathfrak{q},\mathfrak{q})$ such that either $\mathcal{B}_{\mathbb{X}}$ has both of them, *or* $\mathcal{B}_{\mathbb{X}}$ has none of them. In the first case the two rules would build a spindle, making $\mathcal{P}$ not PTIME-sound. In the second case, neither R_1 nor R_2 belong to $\mathcal{P}$. This means that neither $\mathcal{P}_1 = (\mathbb{X}, \mathcal{B}_{\mathbb{X}} \cup \{R_1\}, \mathcal{R}^1_{\mathbb{X}})$, nor $\mathcal{P}_2 = (\mathbb{X}, \mathcal{B}_{\mathbb{X}} \cup \{R_2\}, \mathcal{R}^2_{\mathbb{X}})$, for suitable $\mathcal{R}^1_{\mathbb{X}}, \mathcal{R}^2_{\mathbb{X}}$, are PTIME-sound, because, recall, $\mathcal{P}$ is PTIME-maximal. So, there has to be $\mathfrak{r}_1 : (\Sigma_1 \cup \chi_1) : \mathfrak{r}_1$ in $\mathcal{P}_1$ that involves an instance u_1 of R_1 (Figure 14(a)), and $\mathfrak{r}_2 : (\Sigma_2 \cup \chi_2) : \mathfrak{r}_2$ in $\mathcal{P}_2$ involving the instance b_2 of R_2 (Figure 14(b)). Thanks to Lemma 1 we can assume that all the labels in $\Sigma_1 \cup \chi_1$ and $\Sigma_2 \cup \chi_2$ are of the form $\mathfrak{n}A$, for a fixed A and some $\mathfrak{n}$'s. Then, u_1 must be the *principal* contraction of Σ_1. If not, we could eliminate it as we did for u in point 2, proving that $\mathcal{P}$ is not PTIME-sound. This is why $\mathfrak{r}_1 = \mathfrak{q}$ in $\Sigma_1 \cup \chi_1$ of Figure 14(a). Now, let Θ be the graph $\Sigma_1 \cup \chi_1$ without u_1; we can build Θ in $\mathcal{P}$ (Figure 14(c)). Θ has two premises $\mathfrak{q}A$ and one conclusion $\mathfrak{q}A$, exactly as the box b_2 in Σ_2. So, we can replace b_2 in $\Sigma_2 \cup \chi_2$ with Θ getting a new spindle $\mathfrak{r}_2 : (\Sigma \cup \chi) : \mathfrak{r}_2$ in $\mathcal{P}$. But $(\Sigma \cup \chi) \in \mathbf{PN}(\mathcal{P})$ implies $\mathcal{P}$ is not PTIME-sound. □

5 Conclusions

We supply a criterion that discriminates PTIME-sound stratified systems, based on a quite general structure of proof nets, from the PTIME-*un*sound ones. Such systems



(a) $\mathfrak{r}_1 : (\Sigma_1 \cup \chi_1) : \mathfrak{r}_1$ in $\mathcal{P}_1$ (b) $\mathfrak{r}_2 : (\Sigma_2 \cup \chi_2) : \mathfrak{r}_2$ in $\mathcal{P}_2$ (c) Θ from $\Sigma_1 \cup \chi_1$ (d) $\mathfrak{r}_2 : (\Sigma \cup \chi) : \mathfrak{r}_2$ in $\mathcal{P}$

Fig. 14. The proof nets used in the proof of Theorem 1

are subsystems of the framework MS. Roughly, every subsystem is a rewriting system, based on proof nets. The proof nets are typable with modal formulæ whose modalities can be quite arbitrary. We justify the "arbitrariness" of deciding the set of modalities to use by our will to extend as much as we can the set of programs-as-stratified-proof nets. To this respect, the "largest" subsystems are the PTIME-maximal ones that we can recognize thanks to the form of their proof nets building rules. The "state transition" from a PTIME-maximal subsystem to a PTIME-*un*sound one corresponds to moving from a system that composes chains of spindles with bounded length to a system with unbounded long chains.
Finally, we can state that MS accomplishes the reason we devised it. The subsystem sLAL in Section 2 can replace (2) in Fig. 1 for a suitable extension of BC^{-}, inside SRN, in place of (1). To show how, however, really requires a whole work, whose technical details will be included in Vercelli's doctoral thesis, which shall also present some conditions able to assure if a subsystem enjoys cut-elimination.

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