

# Some Complexity and Expressiveness results on Multimodal and Stratified Proof nets

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**Abstract.** We introduce a multimodal stratified framework  $MS$  that generalizes an idea hidden in the definitions of Light Linear/Affine logical systems: “More modalities means more expressiveness”.  $MS$  is a set of building-rule schemes that depend on parameters. We interpret the values of the parameters as modalities. Fixing the parameters yields deductive systems as instances of  $MS$ , that we call *subsystems*. Every subsystem generates stratified proof nets whose normalization preserves *stratification*, a structural property of nodes and edges, like in Light Linear/Affine logical systems. A first result is a sufficient condition for determining when a subsystem is strongly polynomial time sound. A second one shows that the ability to choose which modalities are used and how can be rewarding. We give a family of subsystems as complex as Multiplicative Linear Logic — they are linear time and space sound — that can represent Church numerals and some common combinators on them.

**Keywords:** Implicit Computational Complexity, Structural Proof-theory, Linear Logic, Polynomial Time Computations.

## 1 Introduction

This work relates to Implicit Computational Complexity (ICC), an area of Theoretical Computer Science that explores machine-independent characterizations of complexity classes.

**Motivations.** We are interested in polynomial time computations and in their characterizations by means of restrictions of Linear Logic (LL) [1]. Specifically, we focus on Light Affine Logic (LAL) [2], a simplification of Light Linear Logic (LLL) [3]. LAL is: (i) strongly polynomial time sound, and (ii) polynomial time complete, under the Curry-Howard (CH) correspondence. (i) means that every derivation  $\Pi$  of LAL normalizes in a time bounded by a polynomial in the size  $|\Pi|$  of  $\Pi$ , under any normalization strategy. (ii) says that every polynomial time Turing machine can be represented as a derivation of LAL.

*Stratification* is the key feature to obtain the bound in (i). Stratification is a property of nodes and edges, invariant under cut elimination, and follows from a careful interplay of bang (!) and paragraph (§) boxes. !-boxes are those we

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\* Both the authors have been supported by MIUR PRIN CONCERTO — protocol number 2007BHXCFFH

know since the introduction of LL, while §-boxes are specific to LAL. Operationally, both kinds of boxes can merge. In particular, !-boxes can merge into §-boxes. As usual, !-boxes are the sub-proof nets that can be duplicated. The key role of merging ! and §-boxes is twofold. On one side, merging realizes the stratification: the number of boxes enclosing a node/edge cannot change under cut elimination. On the other side, it gives expressiveness to the logical system. Without §-boxes Church numerals, used as iterators, could not exist in LAL.

**Contributions.** This work should be viewed as a first step towards making the slogan: “More modalities, more expressiveness” effective. We introduce the framework MS, a set of *proof nets building-rule schemes* that compose *nodes* to obtain proof nets. The edges labelling these formulæ, besides the usual LL connectives  $\rightarrow, \otimes, \forall$ , may contain logical operators to be interpreted as modalities. The choice of which modalities using is, in principle, arbitrary. So, we can have many more than the two ones of LAL inside our formulæ. This is why we dub MS as being “multimodal”.

Part of the nodes of MS are *standard*, corresponding to axiom, cut, linear implication, tensor and universal quantification of LL proof nets. Instead, the *modal* nodes serve to contract or weaken modal formulæ and to build boxes relatively to all the modalities we might be interested to deal with, in some fixed set of multimodal proof nets. Indeed, modal nodes and building-rule schemes of MS depend on parameters to be instantiated with modalities. For example, the contraction node of MS is  $Y_q(n, m)$ , with  $q, n, m$  modalities. If we let  $q, n, m = 1$ , its instance  $Y_1(1, 1)$  represents the contraction node of LAL, under the assumption that !<sup>1</sup> is ! inside MS. Analogously, we can represent the rule that builds “bang” boxes of LAL inside MS. Indeed, MS has a *Promotion building-rule scheme*  $P_q(m_1, \dots, m_k)$ , with  $q, m_1, \dots, m_k$  arbitrary modalities.  $P_q(m_1, \dots, m_k)$  can be applied to every proof net  $\Pi : A_1, \dots, A_k \vdash B$  of MS, with assumptions of type  $A_1, \dots, A_k$  and conclusion of type  $B$ . Its application yields a new proof net  $\Pi' : !^{m_1}A_1, \dots, !^{m_k}A_k \vdash !^qB$ . If we set  $k \leq 1$ , and  $q, m_k = 1$ , then both  $P_1(1), P_1()$  represent the !-box building-rules of LAL in MS. In particular, since we have  $P_q(m_1, \dots, m_k)$ , which puts a modality in front of all  $A_1, \dots, A_k, B$  of any  $\Pi$ , and since we do not have nodes corresponding to dereliction and digging, the sets of proof nets, generated by instantiating the schemes of MS, is stratified. This is why, besides “multimodal”, we dub MS also as “stratified”.

*Our goal is to propose MS as a generator of strong polynomial time sound systems, obtained as subsystems of MS by instances of node and building-rule schemes. The subsystems would generate stratified proof nets, typed with formulæ whose modalities control the normalization complexity.*

It should not be surprising that not every subsystem  $\mathcal{P}$  of MS can be polynomial time sound. We shall prove that both the use of a finite number of modalities and a bounded number of *spindles* in the proof nets of  $\mathcal{P}$ , assures  $\mathcal{P}$  is strongly polynomial time sound. “Spindle” is a technical notion pinpointing the proof net structure that, if iteratively composed an unbounded number of times, yields unsound polynomial time normalizations. The proof of strong polynomial time soundness exploits the Context Semantics in [4].

We conclude by showing how MS may be potentially useful to discover systems with interesting complexity bounds, just “playing” with modalities. We show how to instantiate the schemes of MS to obtain a family  $\{\mathcal{P}_{\text{LTS}}^M \mid M \in \mathbb{N}\}$  of systems such that every  $\mathcal{P}_{\text{LTS}}^M$  is *linear time-space sound* subsystem: every proof net  $\Pi \in \mathcal{P}_{\text{LTS}}^M$  both normalizes in linear time and in linear space. So  $\mathcal{P}_{\text{LTS}}^M$  belongs to the same complexity class as Multiplicative Linear Logic (MLL), but it extends MLL, since some Church numerals and some common combinators on them exist as proof nets of  $\mathcal{P}_{\text{LTS}}^M$ , for every  $M$ .

**Related works.** Our multimodal framework for polynomial time computations may recall ramified systems [5, 6]. The formal relation is still officially unclear, even though the results claimed in [7] support the idea that modalities will result in a formal, logically founded, refinement of ramification.

[8] introduces a *by-levels* analysis of elementary and polynomial time computations with a motivation orthogonal to ours: levels, in the spirit of 2-Sequents [9, 10], rule out the stratified structure of boxes. The effect seems analogous to the one we obtain with MS, since the resulting logical systems generalize the original ones. The relation among levels in [8] and our multimodalities has to be clarified.

**Acknowledgments.** We thank Ugo Dal Lago and Marco Gaboardi for the useful discussions leading to this work and the three referees for their high quality comments and suggestions.

**Paper outline.** We introduce MS in Section 2, the sufficient geometrical condition to determine which are the polynomial time subsystems of MS in 3, and the linear time-space subsystem of MS, strictly extending MLL, in 4. Conclusions, and further work, are in Section 5.

## 2 The framework MS

We proceed by: (i) giving the logical formulæ that label the edges of the proof nets, (ii) introducing the *nodes* and the inductive process to build the proof nets, (iii) defining some static measures on the proof nets, (iv) fixing the main normalization steps on the proof nets, and (v) setting the dynamic measures to assess the normalization cost.

**Formulae of MS.** Let  $M \in \mathbb{N}$  and  $\mathcal{V}$  be a countable set of propositional variables, ranged over by  $x, y, w, \dots$ . For every fixed  $M$ ,  $\mathcal{F}^M$  is the set of the *formulae* generated by

$$F ::= L \mid E \quad L ::= x \mid F \otimes F \mid F \multimap F \mid \forall x.F \quad E ::= !^n F \quad (n \in \{1, 2, \dots, M\})$$

using  $F$  as start symbol.  $\mathcal{F}$  is  $\bigcup_{M \in \mathbb{N}} \mathcal{F}^M$ .  $E$  is the start symbol of *modal* formulæ;  $L$  the one of *linear* or *non-modal* formulæ.  $A, B, C, \dots$  will range over formulæ belonging to  $\mathcal{F}$ , and  $\Gamma, \Delta, \Phi, \Psi$  over multisets of formulæ.  $A \left[ \frac{B}{y} \right]$  will denote substitution of  $B$  for  $y$  in  $A$ .

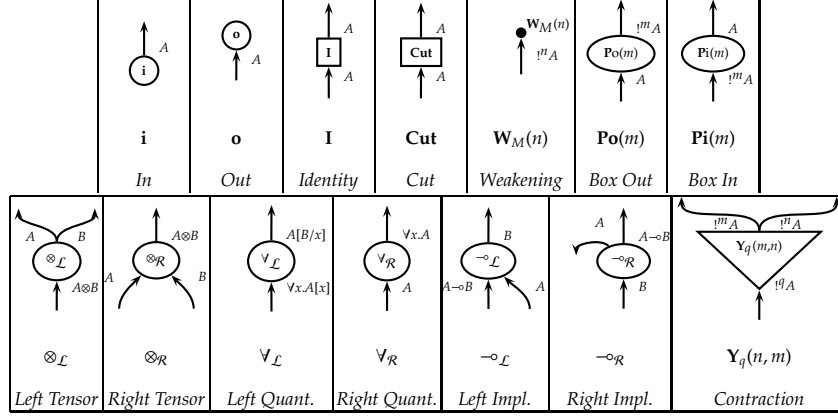


Fig. 1. The nodes, with  $M, m, n, q \in \mathbb{N}$ .

*The nodes for the proof nets of MS.* They are in Figure 1 together with their abbreviated and long names, respectively.  $u, v, w, \dots$  will range over (occurrences) of nodes.

*The proof nets of MS.* The nodes  $i, I$ , and  $o$ , connected as in the leftmost graph of Figure 2 form a proof net. Moreover, if the two rightmost graphs in Figure 2 are proof nets, then all the graphs in Figure 3 are proof nets as well, built using the associated *building-rule scheme*. Namely, we build the proof nets as we were using a sequent calculus rule schemes. For example, “ $Y_q(n, m)$  building-rule scheme with  $m, n, q \in \mathbb{N}$ ” says that we build a new proof net starting from the leftmost generic proof net in Figure 2, eliminating two of its  $i$  nodes, plugging the two outgoing edges of a  $Y_q(n, m)$  node into the two dangling edges of  $I$ , labeled  $!^n A$  and  $!^m A$ .

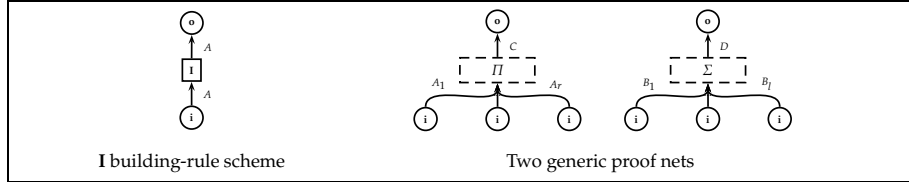


Fig. 2. The basic proof net and two generic proof nets.

Another example is “ $P_q(m_1, \dots, m_r)$  building-rule scheme with  $q, m_1, \dots, m_r, r \in \mathbb{N}$ ”. It *simultaneously* introduces exactly a single  $Po(m)$  node and a, possibly empty, sequence of  $Pi(m)$  nodes, to form a (modal) box, starting from the leftmost generic proof net in Figure 2. The notation  $P_q(m_1, \dots, m_r)$  summarizes the parameters of the box it introduces:  $q$  is the (index of the) modality of the edge

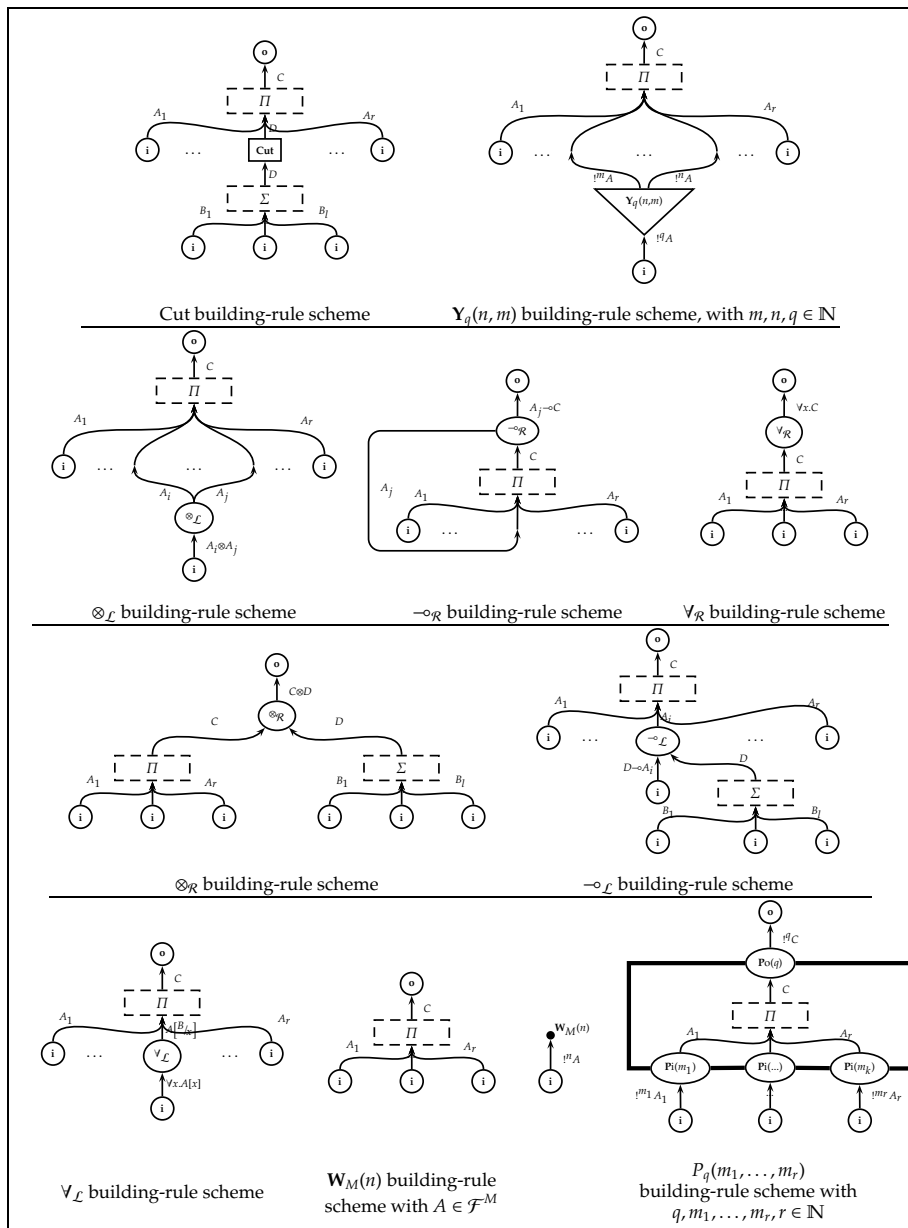


Fig. 3. The building-rule schemes that inductively define the proof nets.

outgoing  $\mathbf{Po}(m)$ , and every  $m_i$ , with  $0 \leq i \leq r$ , is the (index of the) modality associated to the edge incoming the  $i$ th instance of  $\mathbf{Pi}(m)$ , counting from the left. We shall tend to identify the unique instance of  $\mathbf{Po}(m)$ , used by  $P_q(m_1, \dots, m_r)$ ,

with the box it introduces. Namely, we shall call *box* that instance of  $\mathbf{Po}(m)$ . Finally, we want to remark that the  $\mathbf{W}_M(n)$  building-rule scheme only operates on a modal formula  $!^n A$ . This choice simplifies the presentation of MS. However, we could relax the building scheme-rule of  $\mathbf{W}_M(n)$  to any  $A$ , without affecting our results, concerning the complexity. Now, let  $\Pi$  be a proof net.  $V_\Pi$  is the set of its nodes.  $E_\Pi$  is the set of its edges.  $B_\Pi$  is the set of its  $\mathbf{Po}(m)$  nodes. Moreover,  $P_\Pi : B_\Pi \rightarrow \mathbb{N}$  counts the premises of a box.

**Static measures.** Let  $\Pi$  be a proof net.  $\partial(u)$  denotes the *depth*, or *level*, for every  $u \in V_\Pi \cup E_\Pi$ . Specifically,  $\partial(u)$  is the greatest number of (nested) boxes that contain  $u$ .  $\partial(\Pi)$  denotes the *depth of  $\Pi$* , the greatest  $\partial(u)$ , for every  $u \in V_\Pi \cup E_\Pi$ .

$V_\Pi, E_\Pi$ , and  $B_\Pi$  can be made relative to a given level  $0 \leq d \leq \partial(\Pi)$ . Namely,  $V_\Pi^d$  is the set of nodes at level  $d$ . Analogous definitions hold for  $E_\Pi^d, B_\Pi^d$ .

$s(\Pi)$  denotes the *size of  $\Pi$* . It counts the number of the nodes in  $\Pi$ .  $s_d(\Pi)$  denotes the *size of  $\Pi$  at level  $d$* . It counts the number of nodes that occur at level  $d$ . Finally,  $b_d(\Pi)$  counts the number of boxes, or, equivalently, of occurrences of  $\mathbf{Po}(m)$  nodes, at level  $d$ .

For example, for the net, say  $\Pi$ , in Figure 9(b)  $\partial(\Pi) = 1$ ,  $s_0(\Pi) = 15$  since the occurrences of  $\mathbf{Pi}(m)$  and  $\mathbf{Po}(m)$  are not inside the box they delimit.  $s(\Pi) = 21$  and  $b_0(\Pi) = 1$ .

**Subsystems of MS.** We call *subsystems of MS* all the sets of *instances* of the building rules in Figure 2 and 3.  $\mathcal{P} \subseteq \mathbf{MS}$  denotes that  $\mathcal{P}$  is a *subsystem of MS*. Two examples are  $\mathcal{P} = \{\otimes_{\mathcal{L}}, \mathbf{I}, \mathbf{Y}_3(1, 2), P_1(1, 1)\}$  and  $\mathcal{P}' = \{\otimes_{\mathcal{L}}, \mathbf{I}, \mathbf{Y}_1(1, 2), P_1(1), P_1(2, 3)\}$ . We also use regular expressions to identify sets of scheme instances.  $P_q(a^?, m_1, \dots, m_r)$  denotes  $\{P_q(m_1, \dots, m_r), P_q(a, m_1, \dots, m_r)\}$ , while  $P_q(a^*, m_1, \dots, m_r)$  denotes the infinite set  $\{P_q(m_1, \dots, m_r), P_q(a, m_1, \dots, m_r), P_q(a, a, m_1, \dots, m_r), \dots\}$ . Finally, the *number of modalities of any  $\mathcal{P} \subseteq \mathbf{MS}$*  is the number of modalities occurring in the instances of building rule-schemes that form  $\mathcal{P}$ .

**Normalization.** The *normalization steps (norm. step)* on the proof nets of MS are identified by:  $[\otimes_{\mathcal{R}}/\otimes_{\mathcal{L}}]$ ,  $[-\circ_{\mathcal{R}} / -\circ_{\mathcal{L}}]$ ,  $[\forall_{\mathcal{R}}/\forall_{\mathcal{L}}]$ ,  $[P_q(m_1, \dots, m_r)/\mathbf{W}_M(q)]$ ,  $[P_n(n_1, \dots, n_l)/P_q(n, m_1, \dots, m_r)]$ ,  $[P_q(m_1, \dots, m_r)/\mathbf{Y}_q(n, m)]$ ,  $[\mathbf{I}/-]$ ,  $[-/\mathbf{I}]$ . The first three norm. steps are the standard ones, relative to the logical operators that occur in the name. The fourth one corresponds to a standard box erasure by a weakening node. The fifth norm. step merges two boxes in the obvious way. Figure 4 shows the details of the remaining two norm. steps. Also, the *linear norm. steps* are  $[\otimes_{\mathcal{R}}/\otimes_{\mathcal{L}}]$ ,  $[-\circ_{\mathcal{R}} / -\circ_{\mathcal{L}}]$ ,  $[\forall_{\mathcal{R}}/\forall_{\mathcal{L}}]$ ,  $[\mathbf{I}/-]$ ,  $[-/\mathbf{I}]$ . The *polynomial ones* are  $[P_n(n_1, \dots, n_l)/P_q(n, m_1, \dots, m_r)]$ ,  $[P_q(m_1, \dots, m_r)/\mathbf{Y}_q(n, m)]$ ,  $[\mathbf{I}/-]$ ,  $[-/\mathbf{I}]$ . The *garbage steps* are  $[P_q(m_1, \dots, m_r)/\mathbf{W}_M(q)]$ ,  $[\mathbf{I}/-]$ ,  $[-/\mathbf{I}]$ . In particular, the underscore of  $[\mathbf{I}/-]$  and  $[-/\mathbf{I}]$  stands for any logical operator or building scheme-rule.  $\rightarrow$  will be the contextual closure of  $\triangleright$ , so that  $\Pi \rightarrow \Sigma$  whenever  $\Pi$  rewrites to  $\Sigma$ .  $\rightarrow^*$  is the reflexive and transitive closure of  $\rightarrow$ ,  $\Pi \rightarrow^n \Sigma$  denotes *exactly  $n$  steps of  $\Pi \rightarrow^* \Sigma$* ,  $\Pi \rightarrow_d \Sigma$  is a norm. step at depth  $d$ , and, finally,  $\Pi \rightarrow_d^n \Sigma$  denotes *exactly  $n$  steps of  $\Pi \rightarrow^* \Sigma$  at depth  $d$* . A proof net  $\Pi$  is *normal* when none of the above norm. steps rewrites it.

Notice that  $[P_q(m_1, \dots, m_r)/\mathbf{Y}_q(n, m)]$  implies that, in general,  $\rightarrow$  is not deterministic. Nevertheless, every reduction always terminates. It is enough to

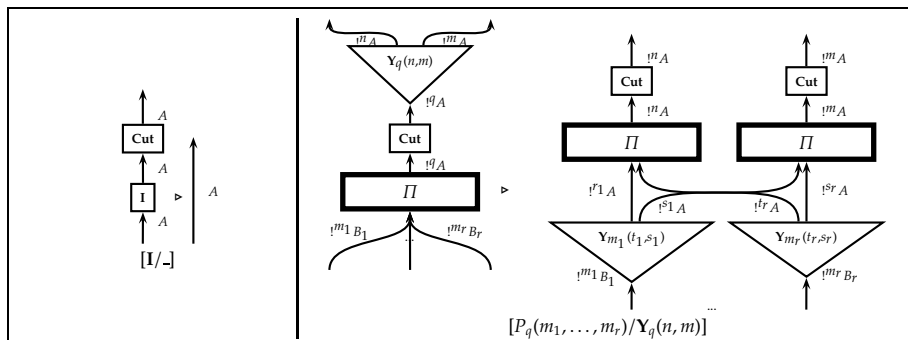


Fig. 4. The norm. steps  $[I/-]$  and  $[P_q(m_1, \dots, m_r)/Y_q(n, m)]$ .

consider the *forgetful map* from proof nets of MS to proof nets of ELL that transforms every modality  $!^n$  into  $!$ . Under that mapping, every norm. step in MS becomes a cut elimination step in ELL.

We observe that the instances of the norm. steps  $[P_n(n_1, \dots, n_l)/P_q(n, m_1, \dots, m_r)]$ , and  $[P_q(m_1, \dots, m_r)/Y_q(n, m)]$ , induced by fixing some  $\mathcal{P}$  in MS, may not lead to a proof net of  $\mathcal{P}$ , starting from a proof net of  $\mathcal{P}$ . This is why the set of norm. steps we can consider as valid for any given  $\mathcal{P}$  in MS, are those ones that map a proof net of  $\mathcal{P}$  to a proof net of  $\mathcal{P}$ .

Focusing on any given  $\Pi \rightarrow \Sigma$  we can see that every node  $x$  of  $\Sigma$  different from a cut can be mapped back to its (unique) *source* node  $y$ :  $x$  is the *residual* of  $y$ . Conversely, not every node has a residual. Finally, we remark that both the structure of proof nets and the behavior of the norm. steps make MS *stratified*: the residual  $x$  of any node  $y$  at level  $d$  different from a cut in a given proof net  $\Pi$  keeps being at level  $d$ .

**Dynamic measures.** Let  $\Pi$  be a proof net.  $[\Pi]$  is the *reduction time* of  $\Pi$ . It is defined as  $\max_{\Pi \rightarrow^n \Sigma} n$ .  $\|\Pi\|$  is the *used space* of  $\Pi$ . It is defined as  $\max_{\Pi \rightarrow \Sigma} s(\Sigma)$ . If  $0 \leq i, d \leq \partial(\Pi)$ , then  $[\Pi]^d = \max_{\Pi \rightarrow_d^n \Sigma} n$ ,  $\|\Pi\|^d = \max_{\Pi \rightarrow_d \Sigma} s(\Sigma)$  and  $\|\Pi\|_i^d = \max_{\Pi \rightarrow_d^i \Sigma} s_i(\Sigma)$ .

### 3 Polynomial time and MS

We proceed by: (i) defining what a *polynomial time sound* subsystem of MS is, (ii) recalling *Context Semantics* [4], (iii) defining the notion of *spindle*, and (iv) using (ii) and (iii) to assess the normalization cost.

**Strong polynomial time sound subsystems.** Let  $\mathcal{P} \subseteq \text{MS}$ .  $\mathcal{P}$  is *strongly polynomial step* whenever for every  $d \in \mathbb{N}$  there is a polynomial  $p_d(n)$  such that, for every  $\Pi \in \text{PN}(\mathcal{P})$ , with  $\partial(\Pi) \leq d$ ,  $[\Pi] \leq p_d(s(\Pi))$ .

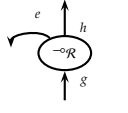
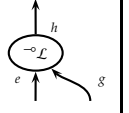
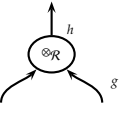
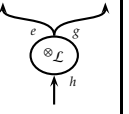
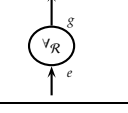
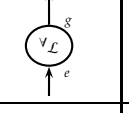
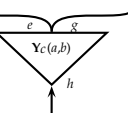
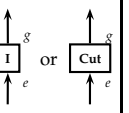
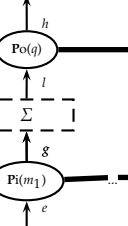
$\mathcal{P}$  is *strongly polynomial size* whenever for every  $d \in \mathbb{N}$  there is a polynomial  $p_d(n)$  such that, for every  $\Pi \in \text{PN}(\mathcal{P})$ , with  $\partial(\Pi) \leq d$ ,  $\|\Pi\| \leq p_d(s(\Pi))$ .

$\mathcal{P}$  is *strongly polynomial time sound* (or *ptime*) if it is both polynomial step and size. PMS denotes the class of the ptime subsystems of MS. This distinction is

meaningful since, for example, the multiplicative and exponential fragment of Elementary Affine Logic is a subsystem of MS. Our goal is to give a sufficient condition to say when any  $\mathcal{P} \subseteq \text{MS}$  belongs to PMS.

**Context Semantics.** We recall and simplify Context Semantics in [4], developed for quantitative analysis of Linear Logic. The simplification works because we rule out both *digging* and *dereliction*, and we perform the reductions *level-by-level*. This is the reason why we need just one stack instead of the traditional two stacks. *Multimodality* is harmless because we manage all the modalities as they were equal.

*Exponential signatures* are generated by  $t ::= e \mid r(t) \mid 1(t)$ .  $\mathcal{E}$  is their set. A *stack element* belongs to  $S = \{\mathbf{a}, \mathbf{o}, \mathbf{f}, \mathbf{s}, \mathbf{x}\} \cup \mathcal{E}$ . A *stack*  $s$  is a finite non empty sequence of stack elements, i.e.  $s \in S^+$ . A *polarity* is an element of  $\mathcal{B} = \{+, -\}$ . If  $c$  is a polarity, we denote  $c \downarrow$  the other one. If  $\Pi$  is a proof net, a *context* of  $\Pi$  is an element of  $C_\Pi = E_\Pi \times S^+ \times \mathcal{B}$ . If  $C, C' \in C_\Pi$  we write  $C \mapsto_\Pi C'$  if  $C'$  can be obtained from  $C$  through the *rewriting relation* in Figure 5.

	$(g, U, +) \mapsto_\Pi (h, U \cdot \mathbf{o}, +)$ $(e, U, -) \mapsto_\Pi (h, U \cdot \mathbf{a}, +)$ $(h, U \cdot \mathbf{o}, -) \mapsto_\Pi (g, U, -)$ $(h, U \cdot \mathbf{a}, -) \mapsto_\Pi (e, U, +)$		$(g, U, +) \mapsto_\Pi (e, U \cdot \mathbf{a}, -)$ $(h, U, -) \mapsto_\Pi (e, U \cdot \mathbf{o}, -)$ $(e, U \cdot \mathbf{a}, +) \mapsto_\Pi (g, U, -)$ $(e, U \cdot \mathbf{o}, +) \mapsto_\Pi (h, U, +)$
	$(e, U, +) \mapsto_\Pi (h, U \cdot \mathbf{f}, +)$ $(g, U, +) \mapsto_\Pi (h, U \cdot \mathbf{x}, +)$ $(h, U \cdot \mathbf{f}, -) \mapsto_\Pi (e, U, -)$ $(h, U \cdot \mathbf{x}, -) \mapsto_\Pi (g, U, -)$		$(h, U \cdot \mathbf{f}, +) \mapsto_\Pi (e, U, +)$ $(h, U \cdot \mathbf{x}, +) \mapsto_\Pi (g, U, +)$ $(e, U, -) \mapsto_\Pi (h, U \cdot \mathbf{f}, -)$ $(g, U, -) \mapsto_\Pi (h, U \cdot \mathbf{x}, -)$
	$(e, U, +) \mapsto_\Pi (g, U \cdot \mathbf{s}, +)$ $(g, U \cdot \mathbf{s}, -) \mapsto_\Pi (e, U, -)$		$(e, U \cdot \mathbf{s}, +) \mapsto_\Pi (g, U, +)$ $(g, U, -) \mapsto_\Pi (e, U \cdot \mathbf{s}, -)$
	$(h, U \cdot 1(t), +) \mapsto_\Pi (e, U \cdot t, +)$ $(h, U \cdot r(t), +) \mapsto_\Pi (g, U \cdot t, +)$ $(e, U \cdot t, -) \mapsto_\Pi (h, U \cdot 1(t), -)$ $(g, U \cdot t, -) \mapsto_\Pi (h, U \cdot r(t), -)$		$(e, U, +) \mapsto_\Pi (g, U, +)$ $(g, U, -) \mapsto_\Pi (e, U, -)$
	$(e, U, +) \mapsto_\Pi (h, U, +)$ $(h, U, -) \mapsto_\Pi (e, U, -)$		

**Fig. 5.** Rewriting Relation among contexts. Notice that if  $(e, U, b) \mapsto_\Pi (e', U', b')$  then also  $(e', U', b' \downarrow) \mapsto_\Pi (e, U, b \downarrow)$ .



**Definition 2 (Number of Paths).** Let  $\Pi \in \mathbf{PN}(\mathbf{MS})$ , and  $b \in B_\Pi$ .  $R_\Pi(b)$  is the number of maximal paths from  $b$  to some  $v \in V_\Pi$ . It is called number of paths of  $b$ .

**Lemma 1 (Number of Copies).** Let  $\Pi \rightarrow_d^* \Sigma$ ,  $b \in B_\Pi^d$  and  $\Theta$  the proof net inside  $b$  at depth  $d + 1$ . Then, in  $\Sigma$ , there are at most  $R_\Pi(b)$  equal residuals (“copies”) of  $\Theta$ .

So, sometimes we will call *number of copies of  $\Pi$*  the number  $R_\Pi(b)$ . The proof generalizes the work of [4].

**Canonical Reductions.** A reduction sequence  $\Pi = \Pi_0 \rightarrow^* \Pi'_0 \rightarrow^* \Pi_1 \rightarrow^* \Pi'_1 \rightarrow^* \dots \rightarrow^* \Pi'_\partial \rightarrow^* \Pi_{\partial+1} \rightarrow^* \Sigma$  is *canonical* if  $\partial = \partial(\Pi)$ ,  $\Pi_i \rightarrow_i^* \Pi'_i$ , reducing all the linear norm. steps at level  $i$ ,  $\Pi'_i \rightarrow_i^* \Pi_{i+1}$ , reducing all the polynomial norm. steps at level  $i$ , and  $\Pi_{\partial+1} \rightarrow^* \Sigma$ , reducing all the garbage steps. The coming lemmas will imply Proposition 1. Firstly, a generalization of the polynomial soundness in [2], provable by induction on  $l$ , is:

**Lemma 2 (Canonical Reductions Are The Worst Ones).** Let  $\Pi$  be a proof net. For every reduction  $\sigma : \Pi = \Pi_0 \rightarrow \Pi_1 \rightarrow \dots \rightarrow \Pi_l = \Sigma$  there exists another reduction  $\tau : \Pi = \Pi'_0 \rightarrow \Pi'_1 \rightarrow \dots \rightarrow \Pi'_l = \Sigma$  such that (i)  $\tau$  is canonical, (ii)  $l' \geq l$ , and (iii)  $\max\{s(\Pi'_i) \mid i \leq l'\} \geq \max\{s(\Pi_j) \mid j \leq l\}$ . Moreover, if  $\sigma$  is a reduction performed in some subsystem  $\mathcal{P}$ , then also  $\tau$  is performed in  $\mathcal{P}$ .

**Lemma 3 (A Condition for Polynomiality).** Let  $\mathcal{P} \subseteq \mathbf{MS}$ . Let us assume that for every  $\partial \in \mathbb{N}$  there exist two polynomials  $p_\partial(x), q_\partial(x)$  such that, for every  $\Pi \in \mathbf{PN}(\mathcal{P})$  with  $\partial(\Pi) \leq \partial$  and for every  $d \leq \partial(\Pi)$ ,  $[\Pi]^d \leq p_\partial(s(\Pi))$  and  $\|\Pi\|^d \leq q_\partial(s(\Pi))$ . Then  $\mathcal{P} \in \mathbf{PMS}$ .

*Proof.* The result holds for canonical reductions because iterating  $\partial$  times a polynomial still gives a polynomial. Then, Lemma 2 implies the thesis.  $\square$

**Weight for the proof nets.**  $T_d(\Pi)$  is  $\sum_{u \in V_\Pi^d} T_d(\Pi, u)$ , where<sup>3</sup>:

$$T_d(\Pi, u) \stackrel{\text{def}}{=} \begin{cases} 1 & u \in \{\mathbf{I}, \mathbf{Cut}, \mathbf{W}_M(n), \mathbf{Pi}(m), \mathbf{i}, \mathbf{o}\} \\ 3 & u \in \{\otimes_{\mathcal{L}}, \otimes_{\mathcal{R}}, \forall_{\mathcal{L}}, \forall_{\mathcal{R}}, \neg_{\mathcal{L}}, \neg_{\mathcal{R}}, \mathbf{Y}_q(m, n)\} \\ 2 \cdot (P_\Pi(u) + 1) \cdot R_\Pi(u)^2 & u \in \{\mathbf{Po}(m)\}. \end{cases}$$

**Lemma 4 (The Weight bounds Time and Space).** Let  $\Pi$  be a proof net. (i)  $[\Pi]^d, \|\Pi\|_d^d \leq T_d(\Pi)$ , and (ii)  $\|\Pi\|_i^d \leq s_i(\Pi) \cdot T_d(\Pi)$ . So, we also have  $\|\Pi\|^d \leq s(\Pi) \cdot T_d(\Pi)$ .

*Proof.* (i) Let us consider a reduction  $\Pi \rightarrow_d^n \Sigma$ . A case analysis on cuts shows that  $T_d(\cdot)$  strictly decreases during reduction. So  $n \leq T_d(\Pi)$ . Since every node has weight at least 1,  $s_d(\Pi) \leq T_d(\Pi)$ . So  $\|\Pi\|_d^d \leq T_d(\Pi)$ .

<sup>3</sup> We adapt the *modified weight* in [4]. We notice that we might have chosen a simpler definition of weight, for example with  $T_d(\Pi, u) = 1$ , for every node, but  $\mathbf{Po}(m)$ . However, we plan to generalize  $\mathbf{MS}$  with *unconstrained wakening*. So, we adopt a more *flexible* definition than the one strictly required.

- (ii) Let us consider a reduction  $\Pi \rightarrow_d^* \Sigma$  and fix a level  $i > d$ . For every  $u$  node at level  $i$ , inside a box  $b \in B_\Pi^d$ , at most  $R_\Pi(b)$  copies of  $u$  will appear in  $\Sigma$  (Lemma 1). Let  $b_1, \dots, b_k$  be all the boxes at level  $d$ , and  $S_1, \dots, S_k$  their sizes at level  $i$ . Then,  $\|\Pi\|_i^d \leq S_1 \cdot R_\Pi(b_1) + \dots + S_k \cdot R_\Pi(b_k) \leq s_i(\Pi) \cdot T_d(\Pi)$ , since  $R_\Pi(b_i) \leq T_d(\Pi, b_i)$ .  $\square$

**Proposition 1 (Polynomially bounded  $R_\Pi$  implies Ptime).** *Let  $\mathcal{P} \subseteq \text{MS}$ . Let us assume there exists a polynomial  $r(x)$  such that, for every proof net  $\Pi$  of  $\mathcal{P}$  and for every box  $b \in B_\Pi$ ,  $R_\Pi(b) \leq r(s(\Pi))$ . Then,  $\mathcal{P} \in \text{PMS}$ .*

*Proof.* Algebraic manipulations show that  $T_d(\Pi) \leq 3s_d(\Pi) + 2s_d(\Pi) \cdot \sum_{b \in B_\Pi^d} R_\Pi(b) \leq 3s(\Pi) + 2s^2(\Pi) \cdot r(s(\Pi))$ . Lemma 4 and Lemma 3 imply the statement.  $\square$

The sufficient condition for ptime we prove relies on the technical notion of *spindle*, a particular configuration of nodes. Too many spindles composed in a proof net of some  $\mathcal{P} \subseteq \text{MS}$  may lead to exponential blow up.

**Definition 3 (Spindles).** *Let  $\Pi \in \text{PN}(\mathcal{P})$ ,  $u \in V_\Pi$  be a contraction,  $b$  be a Box Out node, and  $d$  another node (possibly  $b = d$ ), with  $\partial(u) = \partial(b) = \partial(d)$ . Let  $\tau, \rho, \chi$  be three paths such that: (i) both  $\tau$  and  $\rho$  connect  $u$  to  $b$ , (ii)  $\tau$  starts from one of the conclusions of  $u$ , and  $\rho$  from the other one, (iii) no other paths exist connecting  $u$  to  $b$ , (iv)  $\chi$  (possibly empty) is between  $b$  and  $d$ . The subgraph  $\Sigma$  that contains all and only the nodes of  $\tau, \rho, \chi$  is a **spindle** between  $u$  and  $d$ .<sup>4</sup>*

**Definition 4 (Chains of Spindles).** *Let  $u_1, \dots, u_r$  be contractions and  $d_1, \dots, d_r$  other nodes of  $\Pi \in \text{PN}(\mathcal{P})$ , all at the same depth. Let us assume that, for every  $i \leq r$ , both there is a spindle  $\Sigma_i$  between  $u_i$  and  $d_i$ , and  $d_i$  is connected through an edge to  $u_{i+1}$ . The subgraph  $\Sigma$  that contains all the nodes and edges of  $\Sigma_1, \dots, \Sigma_r$  is a **chain of spindles between  $u_1$  and  $d_r$**  of length  $|\Sigma| = r$ .  $\Sigma$  is **dangerous** if its initial and final edges are labeled by two formulae  $!^i A$  and  $!^i B$ .*

If  $\Sigma$  is a dangerous chain of  $r$  spindles with both initial and final edges labeled by  $!^i A$ , using the **Cut** building-rule scheme,  $\Sigma$  can compose with itself an arbitrary number  $L$  of times, yielding a chain  $\Theta_L$  of spindles with length  $r \cdot L$ .

**Lemma 5 (Bounding the Chains limits the Number of Paths).** *Let  $\mathcal{P} \subseteq \text{MS}$ . Let us assume that every chain of spindles inside a proof net  $\Pi \in \text{PN}(\mathcal{P})$  cannot be longer than  $L$ , for some fixed  $L \in \mathbb{N}$ . Then, there is a polynomial  $p(x)$  such that  $R_\Pi(b) \leq p(s(\Pi))$ , for every  $\Pi \in \text{PN}(\mathcal{P})$ , and  $b \in B_\Pi$ .*

*Proof.* Let  $b$  be a fixed box at depth  $d$  in  $\Pi$ .

<sup>4</sup> Point (iii) is equivalent to “ $\tau$  and  $\rho$  share only the node  $u$  and the box  $b$ , up to *linear reductions*.”

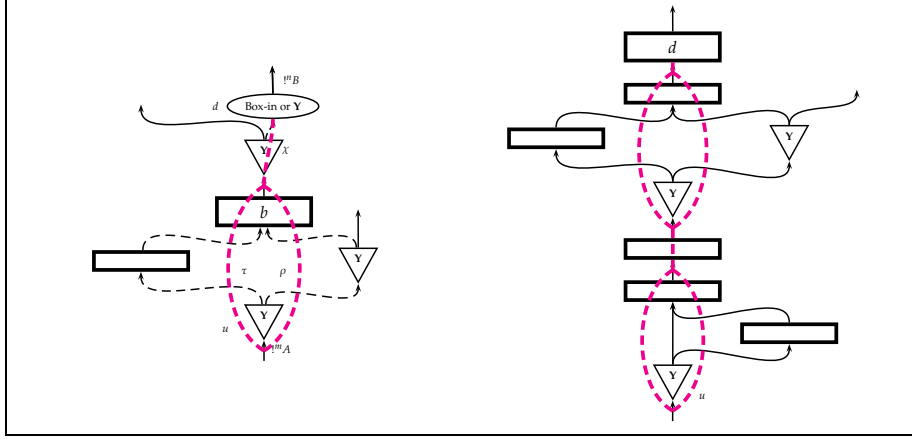


Fig. 7. An example: a spindle and a chain of spindles, both between  $u$  and  $d$ .

1. We want to classify the links  $u \in V_{\Pi}^d$  in classes that we call class-0, class-1, class-2,  $\dots$  according to how many consecutive spindles there are between  $b$  and  $u$ . Let us consider all the possible paths  $\tau$  between  $b$  and  $u$ . Let us call  $\Sigma_b^u$  the sub-graph that contains all the nodes of all the  $\tau$ 's. Let us consider all the possible chains of spindles  $\Phi_b^u, \Psi_b^u, \dots, \Omega_b^u$  whose nodes are among the nodes of  $\Sigma_b^u$ .  $u$  is class- $i$  if  $i = \max\{|\Phi_b^u|, |\Psi_b^u|, \dots, |\Omega_b^u|\}$ . Notice that there are at most  $L + 1$  classes  $(0, \dots, L)$  by hypothesis. Also, notice that some links may remain unclassified (e.g. the links that cannot be reached from  $u$ ).
2. Let  $m_i$  be the number of Contractions of class- $i$  (they are all at depth  $d$ ), and  $m = m_0 + \dots + m_L$ . We observe that  $m \leq s_d(\Pi)$ .
3. For  $0 \leq i \leq L$ , let  $\Pi_i$  be the sub-graph containing all and only the class- $j$  nodes, for every  $j \leq i$ .
4. By definition, a path  $\tau$  starting from  $b$  is maximal relatively to  $\Pi_i$  if there exists a maximal path  $\tau'$  starting from  $b$  whose intersection with  $\Pi_i$  is exactly  $\tau$ .
5. We call  $R_i$ , for  $0 \leq i \leq L$ , the number of paths starting from  $b$  and maximal relatively to  $\Pi_i$ . The definitions imply  $R_{\Pi}(b) = R_L$ .

We prove that  $R_i \leq \prod_{j=0}^i (m_j + 1)$  by induction on  $i$ . Two paths separate when we traverse a contraction node, but never crossing  $\otimes_L$  or  $\otimes_R$  nodes, which, instead, force to go in one specific direction. In  $\Pi_0$  there are no spindles, meaning  $R_0 \leq m_0 + 1$ . By induction, let  $R_{i-1} \leq \prod_{j=0}^{i-1} (m_j + 1)$ . Then, the paths maximal relatively to  $\Pi_i$  are at most  $R_i \leq R_{i-1} \cdot (m_i + 1) \leq \prod_{j=0}^i (m_j + 1)$ . So,  $R_{\Pi}(b) = R_L \leq \prod_{j=0}^L (m_j + 1) \leq s(\Pi)^{L+1}$ .  $\square$

Some comments on the strategy we use to prove Lemma 5 are worth doing. Let  $b \in B_{\Pi}^d$  be the box and  $\Sigma$  be the graph we deal with in the proof just concluded. The goal of the proof is to find a bound on the number of copies of  $b$ , produced

by  $\Pi \rightarrow_d^* \Sigma$ . The intuition driving the proof strategy lies on the observation that the number of copies of  $b$  depends, essentially, on both a *vertical* and a *horizontal* component. The *vertical* component is the number of consecutive spindles that we can have moving upward in  $\Sigma$ , starting from  $b$ . Instead, if a spindle exists between  $b$  and some node  $u$ , then the *horizontal* component counts all the paths from  $b$  to  $u$  external to any spindle from  $b$  to  $u$ . Specifically, the proof of Lemma 5 shows that the reduction time is more affected by the vertical component than by the horizontal one. The reason is that the horizontal component can be bounded by  $s_d(\Pi)$ , at least restricting to some right  $u$ 's, as we do.

Proposition 1 and Lemma 5 imply:

**Proposition 2 (Bounded Chains imply Ptime).** *Let  $\mathcal{P} \subseteq \text{MS}$ . Let us assume that every chain of spindles inside a proof net  $\Pi \in \text{PN}(\mathcal{P})$  cannot be longer than  $L$ , for some fixed  $L \in \mathbb{N}$ . Then  $\mathcal{P} \in \text{PMS}$ .*

**Proposition 3 (The absence of Dangerous Chains imply Ptime).** *If the number of modalities that occur in the instances of the building-rule schemes that define  $\mathcal{P} \subseteq \text{MS}$  is finite, and no dangerous chain of spindles can occur in any  $\Pi \in \text{PN}(\mathcal{P})$ , then  $\mathcal{P} \in \text{PMS}$ .*

*Proof (of Proposition 3).* Let  $M$  be the number of modalities of  $\mathcal{P}$ . Proposition 2 implies the thesis, if we prove  $\mathcal{P}$  only builds chains of spindles at most  $M$  long. By contradiction, let  $\mathcal{P}$  build a chain of  $r > M$  spindles, whose spindles begin with the contractions  $u_1, \dots, u_r$ . By the pigeons-hole principle, at least two contractions  $u_i$  and  $u_j$  share the same input label. If  $\Sigma$  is the chain of spindles between  $u_i$  and  $u_j$ ,  $\Sigma$  is dangerous, yielding a contradiction.  $\square$

## 4 Linear Time and Space subsystems in MS

Here we want to usefully exploit MS. We define a partial order  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$ , and a family  $\{\mathcal{P}_{\mathbb{LTS}}^M \mid M \in \mathbb{N}\}$  of subsystems of MS, such that, for every fixed  $M$ : (i)  $\mathcal{P}_{\mathbb{LTS}}^M$  has at most  $M$  modalities, (ii)  $\mathcal{P}_{\mathbb{LTS}}^M$  contains proof nets that normalize in linear time and space, and (iii) the first  $M$  Church numerals together with a successor, a sum and a function which, essentially, is a predecessor exist as proof nets of  $\mathcal{P}_{\mathbb{LTS}}^M$ . So, every  $\mathcal{P}_{\mathbb{LTS}}^M$  strictly extends the expressiveness of Multiplicative Linear Logic (MLL), whose proof nets, recall, have linear time and space complexity as well.

**The partial order  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$ .**  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$  is a partial order with  $\mathbb{LTS}$  as carrier and  $\leq_{\mathbb{LTS}}$  as order relation. The antireflexive restriction of  $\leq_{\mathbb{LTS}}$  is  $<_{\mathbb{LTS}}$ . We write  $x \not\leq_{\mathbb{LTS}} y$  whenever  $x, y$  cannot be compared under  $\leq_{\mathbb{LTS}}$ .  $x \uparrow^z y$  means both that  $x \not\leq_{\mathbb{LTS}} y$  and that  $x, y <_{\mathbb{LTS}} z$ .  $x \sqcap y$  denotes the *greatest lower bound* (glb) of  $x, y \in \mathbb{LTS}$  under  $\leq_{\mathbb{LTS}}$ .  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$  is such that, for every  $x, y \in \mathbb{LTS}$ : (i) there is  $\perp \in \mathbb{LTS}$  such that  $\perp \leq_{\mathbb{LTS}} x$ , and (ii) if  $x \uparrow^z y$ , for some  $z$ , then  $x \sqcap y = \perp$ .

$P_0(0^*)$	$P_{m_0}(m_1, \dots, m_k)$ whenever $\perp \neq x_{m_0} \leq_{\mathbb{LTS}} x_{m_1}, \dots, x_{m_k}$
$Y_k(0, j)$ whenever $x_j \leq_{\mathbb{LTS}} x_k$	$Y_k(i, j)$ whenever $x_i \uparrow^{x_k} x_j$
$W(i)$ for every $x_i \in \mathbb{LTS}$	

Fig. 8. The modal rules of  $\mathcal{P}_{\mathbb{LTS}}^M$ , with  $m_0, \dots, m_k \in \{1, \dots, M\}$ .

**The family**  $\{\mathcal{P}_{\mathbb{LTS}}^M \mid M \in \mathbb{N}\}$ . Let  $x_0 = \perp, x_1, \dots, x_M$  be distinct elements of  $\mathbb{LTS}$ .  $\mathcal{P}_{\mathbb{LTS}}^M$  is a subsystem of MS whose modal rules, in Figure 8, only use the modalities  $!^0 = \S, !^1, \dots, !^M$ . The structure of  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$  might look somewhat “overdimensioned” w.r.t. the system we finally obtain. However, intuitively, a system which is linear time and space sound, and which is not the multiplicative fragment of Linear Logic, necessarily requires a careful control over the ways we can compose structure, the goal being forbidding any situation where the size grows too much. Specifically,  $(\mathbb{LTS}, \leq_{\mathbb{LTS}})$  rules out chains of spindles of length  $> 1$ . So, every  $\mathcal{P}_{\mathbb{LTS}}^M$  is ptime thanks to Proposition 2. The more, we can prove:

**Proposition 4 (Linear Time and Space Soundness of  $\mathcal{P}_{\mathbb{LTS}}^M$ ).** *Let  $\mathcal{P}_{\mathbb{LTS}}^M$  be given. There exist  $k_1, k_2$  such that, for every  $\Pi \in \mathbf{PN}(\mathcal{P}_{\mathbb{LTS}}^M)$ ,  $\|\Pi\| \leq k_1 \cdot s(\Pi)$  and  $[\Pi] \leq k_2 \cdot s(\Pi)$ .*

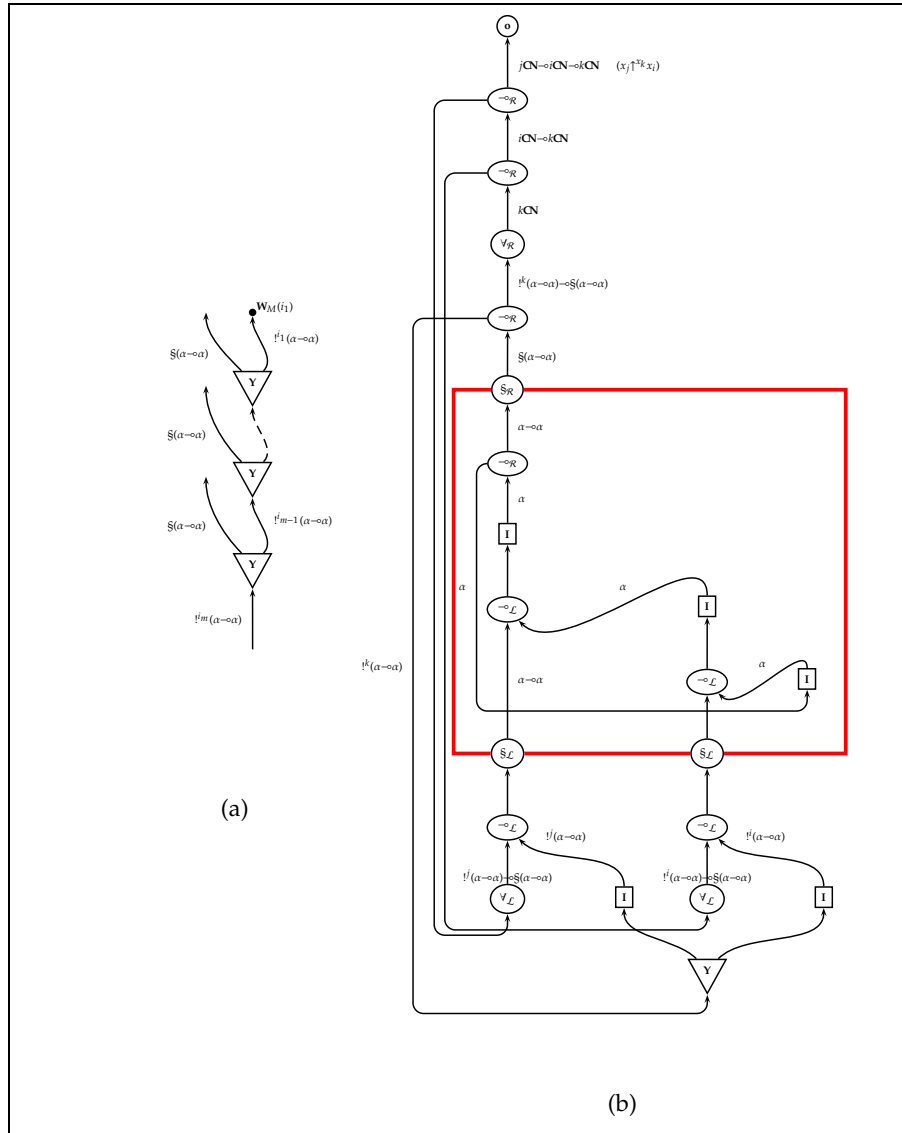
*Proof.* Let  $b$  be a box at depth  $d$  in  $\Pi$ . We recall that the paths outgoing  $b$  split only in Contraction nodes. Also, we observe that they can build only spindles entering a  $\S$ -box, and that paths, entering a  $\S$ -box, cannot split anymore, since  $\S$  never labels the premise of a contraction. So, the paths form a binary tree rooted at  $b$ . Finally, the modalities labelling edges along a path decrease. So, no more than  $M$  contractions may lie on a single path, and we get  $R_\Pi(b) \leq 2^M$  ( $\star$ ). Now, we recall that  $T_d(\Pi)$  strictly decreases during the normalizations at level  $d$ . So:

$$T_d(\Pi) \leq 3s(\Pi) + 2 \sum_{b \in \mathcal{B}_\Pi^d} (P_\Pi(b) + 1) \cdot R_\Pi(b)^2 \leq 3s(\Pi) + 2 \cdot 2^{2M} \sum_{b \in \mathcal{B}_\Pi^d} (P_\Pi(b) + 1) \leq s(\Pi) (3 + 2^{2M+1})$$

the first and third inequalities holding thanks to algebraic manipulations, and the second one thanks to ( $\star$ ). So,  $[\Pi]^d, \|\Pi\|_d^d$  are linear in  $s(\Pi)$  because of Lemma 4.(i). However, for  $i > d$ , the bound  $\|\Pi\|_i^d \leq s_i(\Pi) \cdot T_d(\Pi)$  implied by Lemma 4.(ii) is too high. To lower it, recall that if  $u$  is a link at level  $i > d$ , inside a box  $b$  at level  $d$ ,  $u$  can be copied at most  $R_\Pi(b) \leq 2^M$  times:  $\|\Pi\|_i^d \leq 2^M \cdot s_i(\Pi)$  is linear, as well as  $\|\Pi\|_i^d$ . The final statement follows from composing, level by level, the obtained bounds. We obtain the constants  $k_1, k_2$  that depend on  $M$  and  $\partial(\Pi)$ .  $\square$

Concerning Proposition 4,  $M$  limits the height of the chains we build, level by level, in the course of the normalization. An arbitrary  $M$  would yield a polynomial size bound, because the (level by level) size of the proof net would replace  $M$  as bound on the length of consecutive contractions.

**Church numerals.** The type of a Church numeral  $\bar{n}$  can be  $i\mathbf{CN} \equiv \forall \alpha. !^i(\alpha \multimap \alpha) \multimap \S(\alpha \multimap \alpha)$ , for every  $i \in \{0, \dots, M\}$ . The *spine* in Figure 9(a) is the usual key structure to represent a Church numeral  $\bar{m}$  as a proof net, like, for example, in LAL.



**Fig. 9.** (a) A spine of contractions inside a Church numeral. (b) The Sum between two Church numerals.

*The sums.* A type for the sum can be  $j\mathbf{CN} \rightarrow i\mathbf{CN} \rightarrow k\mathbf{CN}$ , whenever  $x_j \uparrow^{x_k} x_i$ . Figure 9(b) shows its proof net. It takes  $\bar{m}$  and  $\bar{n}$  to yield  $\bar{m} + \bar{n}$ . A further type for the proof net summing two Church numerals can be  $j\mathbf{CN} \rightarrow i\mathbf{CN} \rightarrow k\mathbf{CN}$ , for every  $j, i$  such that  $x_j \leq_{\text{LTS}} x_i$ , or  $x_i \leq_{\text{LTS}} x_j$ , and  $x_j, x_i <_{\text{LTS}} x_k$ . The final proof net would be analogous to the one in Figure 9(b).

**A “moral” Predecessor.** Every  $i \in \mathcal{P}_{\text{ITS}}^M$  contains a proof net with type  $i\text{CN} \multimap (\forall \alpha. !^i(\alpha \multimap \alpha) \multimap \S(\alpha \multimap ((\alpha \multimap \alpha) \otimes \alpha)))$ , taking  $\bar{n}$  to yield a proof net of type  $(\forall \alpha. !^i(\alpha \multimap \alpha) \multimap \S(\alpha \multimap ((\alpha \multimap \alpha) \otimes \alpha)))$ . The second element of the pair in the result would essentially correspond to  $\bar{n} - 1$ . We cannot get exactly  $\bar{n} - 1$  because we lack the unconstrained weakening. Like in LAL, weakening would allow to erase the first element. Recall that we have ruled out unconstrained weakening from PMS, in order to keep this introductory work as simple as possible.

## 5 Conclusions and further work

We introduce MS, that we want to use as a framework to find deductive systems that, under the proof-as-programs analogy, only develop strongly polynomial time computations, and that, thanks to their multimodal nature, can be reasonably expressive. Concerning strong polynomial time soundness, currently, we can supply a sufficient condition on subsystems of MS. Concerning expressiveness, we can show how to exploit multimodality to set a linear time and space sound subsystem of MS, which strictly extends MLL.

Active and future work can develop in many directions.

Certainly, it will be necessary to investigate the conditions under which the subsystems we can obtain from MS enjoy good computational properties, like confluence, for example.

However, we think that, currently, the most stimulating directions are the following ones.

We look for a ptime soundness criterion. In fact, we expect that the inverse of Proposition 3 holds as well.

Also, we want to exploit the expressiveness that seems implicitly supplied by the multimodality. “Playing” with the modalities, we are currently defining subsystems whose iteration schemes are more expressive than those we can represent in LAL. The least goal is to reformulate, at least in part, the results claimed in [7], or, symmetrically, to extend the results of [11].

We want to characterize *maximal* subsystems of MS. The intuition can be given by a simple example. The relevant fragment of LAL corresponds to a subsystem  $\mathcal{P}_{\text{LAL}}$  with  $K = \{\mathbf{Y}_1(1, 1), P_1(1^?), P_2(1^*, 2^*)\}$  as modal rules. We can extend  $K$  as  $K \cup \{\mathbf{Y}_1(1, 2), \mathbf{Y}_1(2, 2)\}$ , which, interestingly enough, is still strongly ptime thanks to Proposition 3. Namely,  $\mathcal{P}_{\text{LAL}}$  is not maximal w.r.t. ptime.

Finally, we want to relate the class  $\{\mathcal{P}_{\text{ITS}}^M \mid M \in \mathbb{N}\}$  with the polynomial time and non-size-increasing system in [12]. The conjecture is that we can embed (at least) an additive-pair free fragment of it, with only boolean lists, simultaneously giving a proof-theoretical meaning to the type  $\diamond$ .

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