A generalization of a Conservativity Theorem for Classical versus Intuitionistic Arithmetic

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Abstract

A basic result in Intuitionism is Π_2^0 -Conservativity. Take any proof p in Classical Arithmetic of some Π_2^0 -statement (some arithmetical statement $\forall x.\exists y.P(x, y)$, with P decidable). Then we may effectively turn p in some intuitionistic proof of the same statement. In a previous paper [1], we generalized this result: any classical proof p of an arithmetical statement $\forall x.\exists y.P(x, y)$, with P of degree k , may be effectively turned into some proof of the same statement, using Excluded Middle only over degree k formulas. When $k = 0$, we get the original conservativity result as particular case. This result was a by-product of a semantical construction. J. Avigad, of Carnegie Mellon University, found a short, direct syntactical derivation of the same result, using H. Friedman's A -translation. His proof is included here with his permission.

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1 Plan of the paper.

In section 2 we quickly introduce Classical and Intuitionistic Arithmetic, and we state Generalized conservativity. In section 3 we introduce Friedman's A -translation, and we explain how it was used to prove Π_2^0 -conservativity. In section 4 we use Friedman's A -translation to prove Generalized Conservativity.

Generalized Conservativity could look uninteresting, from an intuitionistic viewpoint. Indeed, it produces proofs using Excluded Middle, even if only over degree k formulas. We think of it instead as a

step toward extracting some constructive content from any classical proof, not just from proofs of Π_2^0 -statements. We defend our claim in section 5.

2 Stating Generalized Conservativity

We quickly recall what Intuitionistic and Classical Arithmetic are.

A Language L for Arithmetic Let L be the first order language including equality and function symbols for 0 (zero), s (successor) and any primitive recursive map. Any other primitive recursive predicate, like $x \leq y$, is introduced through its characteristic maps. We use the connectives $\perp, \wedge, \vee, \rightarrow, \forall, \exists$. We consider $\neg A$ as abbreviating $A \rightarrow \perp$. We call degree of a formula A , and we denote it by $\deg(A)$, the maximum number of nested quantifiers in A . Degree 0 formulas are propositional formulas. In the intended interpretation, they are all decidable (in fact, primitive recursive).

Axioms and Rules for Arithmetic. Call atomic rule any rule $A_1, \dots, A_n \vdash A$, with $A_1, \dots, A_n, A$ atomic, and closed under first order substitution. Rules for equality (reflexivity, symmetry, transitivity, compatibility with functions) are atomic. We call induction schema the axiom schema $A[x := 0] \rightarrow \forall x.(A \rightarrow A[x := s(x)]) \rightarrow \forall x.A$, for all formulas of L . We call Excluded Middle of degree k , or $\mathbf{EM}_k$ for short, the axiom schema $A \vee \neg A$ for all formulas of degree k . We call $\mathbf{EM}$ the axiom schema $A \vee \neg A$ for all formulas of L .

Minimal, Intuitionistic and Classical Logic for L . Rules of First Order Intuitionistic Logic are Rules for Equality, Introduction and Elimination rules for all connectives (rules for $\neg$ are obtained as particular case of rules for $\rightarrow$). Minimal Logic is obtained by removing Elimination of $\perp$. Without this rule, we cannot prove $\perp \rightarrow A$ for any A , and $\perp$ becomes a propositional constant we know nothing of. Classical Logic is obtained by adding $\mathbf{EM}$ to Intuitionistic Logic.

Minimal, Intuitionistic and Classical Arithmetic. They are obtained from, respectively, Minimal, Intuitionistic and Classical Logic, by adding all equational axioms for primitive recursive maps, Induction, and the axiom $\neg(0 = S(t))$.¹ We also call Intuitionistic Arithmetic "Heyting Arithmetic", and Classical Arithmetic "Peano Arith-

¹The axioms $s(t) = s(u) \rightarrow t = u$ and $0 = s(t) \rightarrow u = v$ for successor (with t, u any terms of L) are derivable from equational axioms for predecessor and conditional: $\mathbf{pred}(S(t)) = t$ and $\mathbf{if}(0, u, v) = u, \mathbf{if}(S(t), u, v) = v$.

metic". We denote Minimal, Intuitionistic and Classical Arithmetic respectively by $\mathbf{MA}$, $\mathbf{HA}$, $\mathbf{PA}$. For any theory T , we denote by $T \vdash A$ the statement: "the formula A is derivable in T ". By induction over A we may check that $\mathbf{MA} \vdash 0 = s(t) \rightarrow A$ for all A . In Intuitionistic Logic, $\mathbf{EM}_k$ proves $\mathbf{EM}_h$ for any $h \leq k$ (just add vacuous quantifiers in front of A in $A \vee \neg A$). $\mathbf{HA}$ proves $x = y \vee x \neq y$ (Excluded Middle for equality) using Induction twice. Then it proves (all formulas in) $\mathbf{EM}_0$ out of it. $\mathbf{HA}$ cannot prove $\mathbf{EM}_1$ instead. *A fortiori*, $\mathbf{HA}$ proves $\mathbf{EM}_k$ for no $k > 0$.

Replacing $\perp$ by any formula. If F, A are any formulas, call $F[\perp := A]$ the result of replacing $\perp$ with A everywhere in F (and possibly renaming bound variables, in order to avoid capture of free variables). In any proof of $\mathbf{MA}$ we may replace $\perp$ by any A , obtaining still a correct proof. The reason is that the only assumption over $\perp$ in $\mathbf{MA}$ is $0 = s(t) \rightarrow \perp$, and this axiom becomes $0 = s(t) \rightarrow A$, which is provable in $\mathbf{MA}$. Thus, if $\mathbf{MA} \vdash F$, then $\mathbf{MA} \vdash F[\perp := A]$ for all formulas A .

We intend to prove the following result:

Theorem 2.1 (*Generalized Conservativity*) *If $\deg(P) = k$, and $\mathbf{PA} \vdash \forall x. \exists y. P$, then $\mathbf{HA} + \mathbf{EM}_k \vdash \forall x. \exists y. P$.*

Π_2^0 -Conservativity is the statement: if $\deg(P) = 0$, and $\mathbf{PA} \vdash \forall x. \exists y. P$, then $\mathbf{HA} \vdash \forall x. \exists y. P$. It is obtained as particular case, by setting $k = 0$, then remarking that $\mathbf{HA}$ proves $\mathbf{EM}_0$. A quick way of proving Π_2^0 -Conservativity result (for Classical versus Intuitionistic Arithmetic) was introduced by H. Friedman. Avigad shown that Generalized Conservativity (of which we had a semantical proof) may be proved using Friedman A -translation.

3 Friedman's A -translation

Call Gödel's $\neg\neg$ -translation of F (see [3]) the formula $F^{\neg\neg}$, obtained replacing each sub-formula $t = u$ of F by $\neg\neg t = u$, and each sub-formula $B \vee C$ or $\exists x. B$ of F , respectively, by $\neg(\neg B \wedge \neg C)$ or $\neg\forall x. \neg B$.² For instance, $\perp^{\neg\neg} = \perp$. Gödel's translations of $\perp \rightarrow F$ and $F \vee \neg F$ are derivable within Minimal Logic. Gödel proved the following:

Theorem 3.1 *If $\mathbf{PA} \vdash F$, then $\mathbf{MA} \vdash F^{\neg\neg}$.*

²Actually, we replace $\vee, \exists$ by $\neg \wedge \neg, \neg\forall\neg$, while Gödel replaced them with $\neg\neg\vee, \neg\neg\exists$. However, two choices are intuitionistically equivalent.

The proof is by induction over the proof in $\mathbf{PA}$ (see [3]). Gödel's $\neg\neg$ -translation is a faithful interpretation of $\mathbf{PA}$ in $\mathbf{MA}$. Indeed, if $\deg(F) = k$, then in $\mathbf{HA} + \mathbf{EM}_k$ we may prove $F \leftrightarrow F^{\neg\neg}$. Thus, if $\mathbf{MA} \vdash F^{\neg\neg}$, then $\mathbf{HA} + \mathbf{EM}_k \vdash F$, hence $\mathbf{PA} \vdash F$. Summing up, $\mathbf{PA} \vdash F$ iff $\mathbf{MA} \vdash F^{\neg\neg}$ iff $\mathbf{HA} + \mathbf{EM}_k \vdash F$.³

Fix any formula A . Call Friedman A -translation of F (see [2]) the formula $F^{\neg\neg}[\perp := A]$.⁴ For instance: $\perp^A = \perp^{\neg\neg}[\perp := A] = \perp[\perp := A] = A$, while $(t = u)^A = (t = u)^{\neg\neg}[\perp := A] = \neg\neg(t = u)[\perp := A] =$ (by replacing $\neg$ with its definition) $((t = u \rightarrow A) \rightarrow A)$.

We may characterize Friedman A -translation as follows. Use $\wedge^A, \rightarrow^A, \forall^A$ just as an alternative denotation for $\wedge, \rightarrow, \forall$. Define the A -negation of F , in symbols $\neg^A F$, as $F \rightarrow A$. Define $B \vee^A C$ as $\neg^A(\neg^A B \wedge^A \neg^A C)$. Define $\exists^A x.B$ as $\neg^A \forall^A x. \neg^A B$. Just by definition unfolding, we may check that F^A is obtained by replacing every atom a in F by a^A , and every connective c in F by c^A .

We may derive the following Theorem:

Theorem 3.2 (*Friedman A -Translation*) *If $\mathbf{PA} \vdash F$, then $\mathbf{MA} \vdash F^A$ for any A .*

Proof. If $\mathbf{PA} \vdash F$, then $\mathbf{MA} \vdash F^{\neg\neg}$, and $\mathbf{MA} \vdash F^{\neg\neg}[\perp := A]$ by replacing $\perp$ with A . Thus, $\mathbf{MA} \vdash F^A$.

We may derive Π_2^0 -Conservativity using A -translation. First of all, we remark that we have only to prove Σ_1^0 -Conservativity, which is the statement: if $\deg(P) = 0$, and $\mathbf{PA} \vdash \exists y.P$, then $\mathbf{HA} \vdash \exists y.P$. The reason is: for any theory T with Introduction and Elimination of $\forall$, we have $T \vdash A$ iff $T \vdash \forall x.A$. Second, we remark that every degree 0 formula P , with free variables $\underline{x} = x_1, \dots, x_n$ is equivalent in $\mathbf{HA}$ to some equation $f(\underline{x}) = 0$ (just take the primitive recursive characteristic map f of P). Therefore we may assume that P is an equational statement.

The next step is a curious Lemma, which C. Murphy proposed to call "Friedman Topmost Trick".⁵ Fix any $A = \exists x.P$, with P

³Faithfulness of Gödel's translation might seem to be enough to prove Generalized Conservativity. It is not. If $F = \forall x.\exists y.P$, and $\deg(P) = k$, then $\deg(F) = k + 2$. Faithfulness implies: if $\mathbf{PA} \vdash F$, then $\mathbf{HA} + \mathbf{EM}_{k+2} \vdash F$. Generalized Conservativity requires, instead: if $\mathbf{PA} \vdash F$, then $\mathbf{HA} + \mathbf{EM}_k \vdash F$.

⁴Friedman A -translation was originally intended from Intuitionistic to Minimal Logic. To be accurate, therefore, we should call F^A "the composition of A -translation with Gödel's translation, applied to F ". We prefer to call F^A directly "the A -translation of F ", for short.

⁵The proof of this Lemma plays a key role in Continuation Theory.

equational. The Lemma says: if we A -translate the negation $\forall x.\neg P$ of A itself, we obtain a statement true in Minimal Logic. If we A -translate A itself, we get a statement equivalent to A in Minimal Logic.

Lemma 3.3 (*Friedman Topmost Trick*) *Fix any $A = \exists x.P$, with P equational.*

- $(\forall x.\neg P)^A$ is derivable in Minimal Logic.
- A^A is equivalent to A in Minimal Logic.

Proof.

- We have $(\forall x.\neg P)^A = \forall^A x.\neg^A P^A = \forall x.(P^A \rightarrow A) = (P \text{ is equational})$:

$$(1) \quad \forall x.(((P \rightarrow A) \rightarrow A) \rightarrow A)$$

Since $A = \exists x.P$, by $\exists$ -Introduction we obtain $P \rightarrow A$. We conclude that (1) above is equivalent to $\forall x.(A \rightarrow A)$, hence it is derivable.

- We have $A^A = \exists^A x.P^A = \neg^A \forall^A x.\neg^A P^A$. By unfolding the first $\neg^A$ we obtain $A^A = ((\forall^A x.\neg^A P^A) \rightarrow A) = ((\forall x.\neg P)^A \rightarrow A)$. By the previous point, $(\forall x.\neg P)^A$ is derivable in minimal logic. We conclude that A^A is equivalent to A in minimal logic.

Π_2^0 -Conservativity is an immediate consequence of Friedman Topmost Trick. Assume we may prove $A = \exists x.P$ in $\mathbf{PA}$, with P equational. Then in $\mathbf{MA}$ we may prove A^A , hence A . We conclude $\mathbf{HA} \vdash A$, Q. E. D..

4 Proving Generalizing Conservativity

Our goal is now to extend the Conservativity result to $A = \exists x.P(x, y)$, with $\deg(P) = k$, and to $\mathbf{PA}, \mathbf{HA} + \mathbf{EM}_k$. Therefore we have to extend Friedman Topmost Trick as follows: in $\mathbf{HA} + \mathbf{EM}_k$, we may prove both $(\forall x.\neg P)^A$, and $A^A \leftrightarrow A$.

Fix any formula A , any $k \in \mathbb{N}$, any connective c of L . The first step is to prove, in $\mathbf{HA} + \mathbf{EM}_k$, that if c has all arguments of degree $\leq k$, then the map $B \mapsto B \vee A$ commutes with c, c^A .

Lemma 4.1 (*Commutativity Lemma*) *Fix any formula A , any $k \in \mathbb{N}$. The following equivalences are provable in $\mathbf{HA} + \mathbf{EM}_k$, for any $B, C, \forall x.D, \exists x.E$ of degree $\leq k$:*

- $(\wedge) \quad (B \wedge C) \vee A \leftrightarrow (B \vee A) \wedge^A (C \vee A)$
- $(\rightarrow) \quad (B \rightarrow C) \vee A \leftrightarrow (B \vee A) \rightarrow^A (C \vee A)$
- $(\forall) \quad (\forall x.D) \vee A \leftrightarrow \forall^A x.(D \vee A)$
- $(\vee) \quad (B \vee C) \vee A \leftrightarrow (B \vee A) \vee^A (C \vee A)$
- $(\exists) \quad (\exists x.E) \vee A \leftrightarrow \exists^A x.(E \vee A)$

Proof

- $(\wedge)$. By distributivity of $\wedge^A = \wedge$ with $\vee$.
- $(\rightarrow)$. $\Rightarrow$. Assume $(B \rightarrow C) \vee A$, in order to prove $(B \vee A) \rightarrow (C \vee A)$. We assume $B \vee A$ and we have to prove $C \vee A$. If $B \rightarrow C$, then $C \vee A$ because we assumed $B \vee A$. If A , then $C \vee A$. $\Leftarrow$. Assume $(B \vee A) \rightarrow (C \vee A)$, in order to prove $(B \rightarrow C) \vee A$. We use $B \vee \neg B$, and $C \vee \neg C$. If either $\neg B$ or C , we deduce $B \rightarrow C$, and we conclude $(B \rightarrow C) \vee A$. If B and $\neg C$, we deduce $B \vee A$, hence $C \vee A$ by assumption. By $\neg C$ we obtain A , and we conclude $(B \rightarrow C) \vee A$.
- $(\forall)$. $\Rightarrow$. Assume $(\forall x.D) \vee A$, in order to prove $\forall x.(D \vee A)$. x is a bound variable of $\forall x.D$, and we may assume that x is not free in A . If $\forall x.D$, then $\forall x.(D \vee A)$. If A , then $D \vee A$, and $\forall x.(D \vee A)$ because x is not free in the assumption A . $\Leftarrow$. Assume $\forall x.(D \vee A)$, in order to prove $(\forall x.D) \vee A$. We use $(\exists x.\neg D) \vee (\neg \exists x.\neg D)$ and $D \vee \neg D$. Out of them we may intuitionistically derive $(\exists x.\neg D) \vee (\forall x.D)$. If $\exists x.\neg D$, from $\forall x.(D \vee A)$ we obtain A , then $(\forall x.D) \vee A$. If $\forall x.D$, then $(\forall x.D) \vee A$.
- $(\vee)$. $\Rightarrow$. Assume $B \vee C \vee A$, in order to prove $(B \vee A) \vee^A (C \vee A)$. By definition of $\vee^A$, we assume $\neg^A(B \vee A)$, $\neg^A(C \vee A)$, and we have to prove A . The new assumptions are, by definition, $(B \vee A) \rightarrow A$ and $(C \vee A) \rightarrow A$. Our thesis, A , follows by case reasoning over $B \vee C \vee A$ and Modus Ponens. $\Leftarrow$. Assume $(B \vee A) \vee^A (C \vee A)$, that is, $\neg^A(\neg^A(B \vee A) \wedge \neg^A(C \vee A))$, in order to prove $B \vee C \vee A$. We use $B \vee \neg B$ and $C \vee \neg C$. If B or C , then $(B \vee C) \vee A$. If both $\neg B$, $\neg C$, we deduce $(B \vee A) \rightarrow A$ and $(C \vee A) \rightarrow A$. By definition, we obtain $\neg^A(B \vee A)$, $\neg^A(C \vee A)$ and $\neg^A(B \vee A) \wedge \neg^A(C \vee A)$. Our assumption is the A -negation of this latter: we conclude A .
- $(\exists)$. $\Rightarrow$. Assume $(\exists x.E) \vee A$, in order to prove $\exists^A x.(E \vee A)$. By definition of $\exists^A$, we assume $\forall x.\neg^A(E \vee A)$ and we have to prove

A . The new assumption unfolds to $\forall x.((E \vee A) \rightarrow A)$. We reason by cases over $(\exists x.E) \vee A$. x is a bound variable of $\exists x.E$, and we may assume that x is not free in A . In the case $\exists x.E$ holds, we first deduce $\exists x.(E \vee A)$. From it and $\forall x.((E \vee A) \rightarrow A)$ we conclude A , by $\exists$ -Elimination and x not free in A . In the case A holds, we conclude A again. $\Leftarrow$. Assume $\exists^A x.(E \vee A)$, that is, $(\forall x.(E \vee A \rightarrow A)) \rightarrow A$, in order to prove $(\exists x.E) \vee A$. We use $(\exists x.E) \vee \neg(\exists x.E)$. If $(\exists x.E)$, we conclude $(\exists x.E) \vee A$. If $\neg\exists x.E$, then we intuitionistically deduce $\forall x.\neg E$ and $\forall x.((E \vee A) \rightarrow A)$ from it. By our assumption and Modus Ponens we deduce A . We conclude $(\exists x.E) \vee A$.

The next step is to prove, in $\mathbf{HA} + \mathbf{EM}_k$ again, that for any formula A , and for any formula F of degree $\leq k$, F^A is equivalent to $F \vee A$, no matter what A is.

Lemma 4.2 (*Avigad Lemma*) *If $\deg(F) \leq k$, then $F^A \leftrightarrow F \vee A$ in $\mathbf{HA} + \mathbf{EM}_k$.*

Proof. By induction over F .

Assume $F = \perp$. Then the thesis is $A \leftrightarrow \perp \vee A$, derivable in Intuitionistic Logic.

Assume F is any equation. Then $F \vee \neg F$ holds in $\mathbf{HA}$, and the thesis is $((F \rightarrow A) \rightarrow A) \leftrightarrow F \vee A$. $\Rightarrow$. Assume $(F \rightarrow A) \rightarrow A$. If F , then $F \vee A$. If $\neg F$, then $F \rightarrow A$. From our assumption we deduce A , then $F \vee A$. $\Leftarrow$. Assume $F \vee A$. If F , then $F \rightarrow A$ implies A , hence $(F \rightarrow A) \rightarrow A$. If A , then $(F \rightarrow A) \rightarrow A$ again.

Assume $F = (B \wedge C), (B \rightarrow C), \forall x.D, (B \vee C), \exists x.E$. Then $F^A = (B^A \wedge^A C^A), (B^A \rightarrow^A C^A), \forall^A x.D^A, (B^A \vee^A C^A), \exists^A x.E^A$. By induction hypothesis over B, C, D, E , the formula F^A is equivalent in $\mathbf{HA} + \mathbf{EM}_k$ to, respectively, $((B \vee A) \wedge^A (C \vee A)), ((B \vee A) \rightarrow^A (C \vee A)), \forall^A x.(D \vee A), ((B \vee A) \vee^A (C \vee A)), \exists^A x.(E \vee A)$. By applying the previous Lemma, F^A is equivalent in $\mathbf{HA} + \mathbf{EM}_k$ to $(B \wedge C) \vee A, (B \rightarrow C) \vee A, (\forall x.D) \vee A, (B \vee C), (\exists x.E) \vee A$. In all cases, what we got is $F \vee A$.

We may now derive a generalization of Friedman Topmost Trick, and generalized Conservativity out of it.

Lemma 4.3 (*Generalized Friedman Topmost Trick*) *Fix any $A = \exists x.P$, with $\deg(P) = k \in \mathbb{N}$.*

- $(\forall x. \neg P)^A$ is derivable in $\mathbf{HA} + \mathbf{EM}_k$.
- A^A is equivalent to A in $\mathbf{HA} + \mathbf{EM}_k$.

Proof.

- We have $(\forall x. \neg P)^A = \forall^A x. \neg^A P^A = \forall x. (P^A \rightarrow A)$. By Avigad's Lemma, this latter is equivalent to $\forall x. (P \vee A \rightarrow A)$ in $\mathbf{HA} + \mathbf{EM}_k$. Since $A = \exists x. P$, by $\exists$ -Introduction we deduce $P \rightarrow A$ is true. We deduce $P \vee A \rightarrow A$ out of it, then $\forall x. (P \vee A \rightarrow A)$ by $\forall$ -introduction. We conclude $(\forall x. \neg P)^A$ in $\mathbf{HA} + \mathbf{EM}_k$.
- We have $A^A = \exists^A x. P^A = \neg^A \forall^A x. \neg^A P^A$. By unfolding the first $\neg^A$ we obtain $A^A = ((\forall^A x. \neg^A P^A) \rightarrow A) = ((\forall x. \neg P)^A \rightarrow A)$. By the previous point, $(\forall x. \neg P)^A$ is derivable in $\mathbf{HA} + \mathbf{EM}_k$. We conclude that A^A is equivalent to A in $\mathbf{HA} + \mathbf{EM}_k$.

Theorem 4.4 (*Generalized Conservativity*) *If $\deg(P) = k$, and $\mathbf{PA} \vdash \forall x. \exists y. P$, then $\mathbf{HA} + \mathbf{EM}_k \vdash \forall x. \exists y. P$.*

Proof. Assume we may prove $A = \exists x. P$ in $\mathbf{PA}$, with $\deg(P) = k$. Then in $\mathbf{MA}$ we may prove A^A . By $\mathbf{HA} + \mathbf{EM}_k \vdash A^A \leftrightarrow A$, we conclude $\mathbf{HA} + \mathbf{EM}_k \vdash A$, Q.E.D. .

5 Discussing the result

We claim that using Generalized Conservativity we may find some constructive content in any Classical Arithmetical proof, not just in proofs of Π_2^0 -statements.

For instance, fix any recursive map $f : N \rightarrow N$, any $a \in N$, and consider the following Minimum Principle M : " f has a minimum point x ", in symbols $M = \exists x. \forall y. f(x) \leq f(y)$. M is a Σ_2^0 statement, hence Σ_1^0 -conservativity does not apply to it. Indeed, M is derivable in $\mathbf{PA}$, but not in $\mathbf{HA}$.⁶ Generalized Conservativity says that we

⁶ M is provable in $\mathbf{PA}$. Assume, for contradiction, that M is false (we are using $\mathbf{EM}_2$ here). Then, using Classical Logic again, $\forall x. \exists y. f(x) > f(y)$. We will prove, by total induction over $v \in N$, that for all $c \in N$ we have $f(c) > v$. By induction hypothesis, we assume therefore that $f(c) > v'$ for all $c \in N$, all $v' < v$, in order to prove $f(c) > v$. Then $f(c) \geq v$ for all $c \in N$. By $\forall x. \exists y. f(x) > f(y)$, we deduce that for some $c' \in N$ we have $f(c) > f(c') \geq v$. We conclude $f(c) > v$. A contradiction now follows by selecting a free variable $a \in N$, then choosing $c = a, v = f(a)$, and deducing $f(a) > f(a)$. Q. E. D..

M is not provable in $\mathbf{HA}$. Roughly speaking, the reason is that cannot find x , in general, just by computing finitely many values of f .

may turn any proof p of M into some proof q of the same M using only EM_1 . EM_1 is intuitionistically equivalent to the axiom schema $\exists z.Q(\underline{z}, z) \vee \forall z.\neg Q(\underline{z}, z)$, for any equation Q . EM_1 may be imagined as an oracle $\mathcal{O}$. $\mathcal{O}$ takes a list of values for $\underline{z}$, and returns some pair $\langle b, c \rangle$. If $b = 1$, then $Q(\underline{z}, c)$ is true, if $b = 2$, then $Q(\underline{z}, z)$ is false for all z . $\mathcal{O}$ is a Δ_2^0 -map (often a Δ_2^0 -complete map). Using $\mathcal{O}$ as oracle, q finds, for any a , some x such that $\forall y.f(x) \leq f(y)$. Thus, q defines some Δ_2^0 -map $\phi(a)$ ($a \in N$ is the free variable we selected in the end of the proof of p). We claim that ϕ is a constructive content for the original proof p . For instance, in the case of the proof p we included in the footnote, ϕ is recursively defined as follows. Let $\mathcal{O}$ be an oracle for $(\exists y.f(a) > f(y)) \vee (\forall y.f(a) \leq f(y))$, and $\mathcal{O}(a) = \langle b, c \rangle$. If $b = 1$ (and $f(a) > f(c)$), then $\phi(a)$ is $\phi(c)$. If $b = 2$ (and $f(a) \leq f(y)$ for all y), then $\phi(a)$ is a . $\phi(a)$ is always a minimum point for f .

In fact, the definition of ϕ reflects the ideas in p . ϕ uses the oracle $\mathcal{O}$ to produce some decreasing chain $f(a) > f(a') > f(a'') \dots$, terminating because N is well-founded. When the chain stops, it stops in some c such that $f(c) \leq f(y)$ for all y . ϕ is well-defined in Classical Arithmetic. The obvious objection against a constructive use of ϕ is that we cannot compute $y = \phi(x)$. But we may think of $\phi(x)$ as a formal notation, which we are able to turn into a real integer whenever we used it to solve a concrete problem. Suppose for instance we have an inequality $f(t) \leq f(t + 27)$ (This example is due to T. Coquand). We may solve it using any minimum point $y = \phi(0)$ of f . We define in this way a proof of a simple existential statement $\exists t.f(t) \leq f(t + 27)$ in Classical Arithmetic. Using Σ_1^0 -conservativity, we may turn it into some constructive proof of the same result, effectively defining some $t \in N$ such that $f(t) \leq f(t + 27)$. After a careful work over the classical proof (not to be included here) we conclude that the definition of t is the following variant of the definition of ϕ . We look for the first point in which the succession $f(0) > f(27) > f(27+27) > f(27+27+27) > \dots$ stops decreasing (for the first $27i$ such that $f(27i) \leq f(27i+27)$). Definition of t retains ideas from the definition of $\phi(0)$ (but the oracle $\mathcal{O}$ of ϕ is now gone).

In general, we claim: by Σ_1^0 -conservativity, we may use Δ_k^0 -maps in constructive reasoning, to describe ideas in classical proofs of Σ_k^0 -statements. This is because we may eventually get rid of them when proving Σ_1^0 -statements. We also have a more elaborate argument. In a previous paper [1] we defined an intuitionistic model $\mathcal{N}$ of Δ_k^0 -maps.

Thus, it is sound to use Δ_k^0 -maps in intuitionistic reasoning.

The role of Generalized Conservativity is to provide a bound to the complexity of non-recursive maps we may have to use. Suppose we have a classical arithmetical proof p of a Σ_k^0 -statement. By Generalized Conservativity we may describe the ideas in p using only Δ_k^0 -maps. The bound k is independent from the degree of Excluded Middle we use in p . It only depends on the degree of the conclusion of p . In the previous example, a proof p of a Σ_2^0 -statement using $\mathbf{EM}_2$ may be turned into a proof q using only $\mathbf{EM}_1$. The constructive content of q is then some Δ_2^0 -map ϕ . Whenever we use A as a Lemma in a proof of some Σ_1^0 -statement $\exists x.P(x)$, then by Σ_1^0 -conservativity we may use some suitable recursive variant ϕ' of ϕ to compute x .

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