

## An Interactive Realizability Semantics for non-constructive proofs

Summer School  
"Réalizabilité à Chambéry"  
14-17 June, 2011

*Toward a model for Classical Logic  
through parallel computations  
and non-monotonic learning*

Stefano Berardi <http://www.di.unito.it/~stefano>  
C.S. Dept. Torino University



### Abstract

- Brouwer-Heyting-Kolmogorov-Realizability Semantics, from now on **BHK-Realizability**, takes a constructive mathematical proof of the existence of an individual with a given property and **automatically extracts a certified functional program** computing the individual. *Extracted programs are readable and may be improved.*
- A proof using Classical Logic (say, by contradiction) may still be interpreted as a program, but in a larger language, including extra features like **continuation or A-translation**. *Extracted program are often unreadable and hard to improve.*
- Our goal is to define a Interactive Realizability Semantics of Classical Logic, **I-Realizability** for short, based over the idea of **Learning** in the limit, in the sense of Gold [Go], which **interprets classical proofs as parallel, non-deterministic programs**, more readable and easier to improve.

### Acknowledgements

- I thank **P. Hyvernat** for inviting me to give a talk to **Chambéry Summer School on Realization**, and **C. Raffalli** and **T. Hirschowitz** for organizing the event.
- I thank **F. Aschieri**, **G. Birolo** and **U. de' Liguoro** for checking an earlier version of this talk, and for suggesting several improvements.

If there are mistakes left in these slides they are of the author ...

2

### § 1. Introduction: comparing the Functional paradigm and the Learning paradigm

*We outline:*

- *the principles of BHK Realizability, interpreting proofs without Excluded Middle as "constructions" in a typed functional language;*
- *the principle of Interactive Realizability, interpreting Excluded Middle as a learning operator.*

*This section only compare the respective ideas, and includes no formal definition. For an introduction to BHK realizability we refer to [Lo], Part 1.*

4

## Realizability Semantics and functional programs

- In BHK Realizability Semantics, all proofs of B with an hypothesis A are interpreted by recursive maps  $f:A \rightarrow B$ , called **realizers**. They take an individual  $a:A$  ("a of type A") and return an individual  $f(a):B$  ("f(a) of type B"). Proof axioms are interpreted by primitive maps, proof rules are interpreted by compositions of such maps.
- Properties of realizers  $f:A \rightarrow B$  are described by specifications of the form  $\forall x \in P. f(x) \in Q$ , with  $P \subseteq A$ ,  $Q \subseteq B$  properties of individuals of type A, B.
- Realizability Semantics defines realizers in a **functional language**, a typed lambda calculus extended with primitive for data types and recursion, called **system T**.

5

## Interactive Realizability Semantics and Monotonic Learning

- In this talk we introduce a more general notion of realizability, **Interactive Realizability**, which interprets classical proofs using **programs learning in the limit** in the sense of Gold [Go], and **monotonic** learning.
- We assume having a countable set  $\text{Atom}$  of **atoms of information**, a set  $S$  of consistent sets of such atoms, and a global **knowledge state**  $s \in S$ , common to all realizers.
- Proofs of B with assumption A are interpreted by recursive maps  $f = \langle f_1, f_2 \rangle: A \times S \rightarrow B \times P_{\text{fin}}(\text{Atom})$  we call **"interactive realizers"**. An interactive realizer takes a global state  $s \in S$ , some  $a:A$ , and returns some  $f_1(a,s):B$  and some finite set of atoms  $f_2(a,s) = X \subseteq \text{Atom}$ , **to be added to  $s \in S$** . We interpret adding X to s as a form of **"monotonic learning"**.

7

## Realizability Semantics and Excluded Middle

- Excluded Middle for a predicate  $A(x)$  over natural numbers is the axiom:  $\text{EM}_A = \forall x. (A(x) \vee \neg A(x))$
- **EM** is the schema  $\{EM_A \mid A(x) \text{ arithmetical formula}\}$
- A realizer  $r$  of  $\text{EM}_A$  in BHK Realizability Semantic is a map taking some  $m \in \mathbb{N}$  and returning a triple  $\langle b, r_1, r_2 \rangle$ , such that  $b$  is Boolean, and if  $b = \text{True}$  then  $r_1$  is a realizer of  $A(x)$ , if  $b = \text{False}$  then  $r_2$  is a realizer of  $\neg A(x)$ .
- Thus, if there is a realizer of  $\text{EM}_A$  in BHK Realizability Semantic, then the existence of a realizer for  $A(x)$  is a **decidable predicate**. This requirement forbids the existence of a BHK realizer of  $\text{EM}_A$  for most arithmetical predicates  $A(x)$ : **EM is false** in BHK Realizability

6

## The interaction between a realizer and the knowledge state

- Whenever  $f_2(a,s) = X \neq \emptyset$  (there is **something to learn**), we recompute  $f_2(a,s') = X' \subseteq \text{Atom}$  in the new state  $s'$  obtained adding X to s. We define in this way some increasing chain  $s \subseteq s' \subseteq s'' \subseteq \dots$  of states, and we assume that  $f_2(a,s^{(n)}) = \emptyset$  for some n (that eventually the realizer f has **nothing left to learn**).
- Properties of maps  $f = \langle f_1, f_2 \rangle: A \times S \rightarrow B \times P_{\text{fin}}(\text{Atom})$  are described by specifications of the form  $\forall x \in P. (X = \emptyset) \Rightarrow f(x) \in Q$ , with  $P \subseteq A$ ,  $Q \subseteq B$ . Whenever  $X = \emptyset$ , that is, "f has nothing left to learn", f behaves **like a construction of BHK-Realizability**, otherwise f extends the knowledge state s by adding X to s.

8

## Realizability Semantics and 1-Excluded Middle

$EM_1$  (1-Excluded Middle) =  $\{\forall x.(\exists y. P(x,y) \vee \forall y. P^\perp(x,y)) \mid$   
 $P(x,y)$  decidable,  $P^\perp$  complement of  $P\}$

- $EM_1$  is an axiom schema stronger than constructive Arithmetic, but weaker than EM [Ak]. There is **no realizer of  $EM_1$**  in BHK Realizability Semantics.
- There is **a realizer of  $EM_1$  in the I-Realizability**.  $EM_1$  is interpreted as a learning program, a construction of a more general kind than those considered in BHK Realizability.
- In order to interpret full EM as a learning program, we have to consider **non-monotonic learning** (not included in this talk), in which sometimes atoms **are removed from the knowledge state**. In monotonic learning we may only add atoms.

9

## Goedel's system $\mathcal{T}$ a simply typed $\lambda$ -calculus

- Goedel's system  $\mathcal{T}$  is a **simply typed lambda calculus**, having as atomic types the Data Types:  
 $Unit = \{\text{unit}\}$ ,  $Bool = \{\text{True}, \text{False}\}$ ,  $N = \{0, 1, 2, 3, \dots\}$ ,  
 $L = \{\text{nil}, \text{cons}(n, \text{nil}), \text{cons}(n, \text{cons}(m, \text{nil})), \dots\}$  ( $n, m \in N$ )
- Types of  $\mathcal{T}$  are closed under product types  $T \times U$  and arrow types  $T \rightarrow U$ . If  $u : U$  in  $\mathcal{T}$ , then  $\lambda x^T. u : T \rightarrow U$  in  $\mathcal{T}$ .
- Constants of  $\mathcal{T}$  are **if, unit, True, False, 0, Succ**, and primitive recursion **rec<sub>N</sub>, rec<sub>L</sub>** over integers and lists, n-ple  $\langle \cdot, \dots, \cdot \rangle$  and the  $i$ -th projection  $\pi_i$ , with the suitable typing (see [Bo] for more details).
- BHK Realizability Semantics takes an arithmetical proof and turns it into a program written in system  $\mathcal{T}$ .

11

## § 2. BHK Realizability Semantics

- We introduce Goedel's system  $\mathcal{T}$  and a version BHK-(Brouwer-Heyting-Kolmogorov) Realizability in which realizers are terms of  $\mathcal{T}$ .
- In the next section, we compare BHK Realizability with the I-Realizability Semantics.

10

### A Realizability Interpretation of Formulas.

- Let  $A$  be any arithmetical formula. We define the type  $|A|$  of the realizers of  $A$  by induction over  $A$ . Let  $T = \{\text{Unit}, \text{Bool}, N, L\}$ .
- $|P(t_1, \dots, t_m)| = \text{Unit}$
- $|A_1 \wedge A_2| = |A_1| \times |A_2|$
- $|A_1 \vee A_2| = \text{Bool} \times |A_1| \times |A_2|$
- $|A_1 \rightarrow A_2| = |A_1| \rightarrow |A_2|$
- $|\forall x \in T. A| = T \rightarrow |A|$
- $|\exists x \in T. A| = T \times |A|$

12

## The extraction method implemented in Coq: BHK's Realizability

BHK's realizability is a way of associating to each closed arithmetical formula  $A$  all possible programs  $t:|A|$  of  $\mathcal{T}$  which **realize** what the formula **says**. We write  $t|-A$  for "t realizes A", and we call t a BHK **realizer** of A.

**Definition (Realizers).** Let t be a term of Goedel's system  $\mathcal{T}$ .

1.  $t|-P(t_1, \dots, t_n)$  iff  $P(t_1, \dots, t_n) = \text{True}$
2.  $t|-A \wedge B$  iff  $\pi_0 t |- A$  and  $\pi_1 t |- B$
3.  $t|-A \rightarrow B$  iff for all u, if  $u |- A$ , then  $t(u) |- B$
4.  $t|-\forall x A$  iff for all  $n \in \mathbb{N}$ ,  $t(n) |- A[n/x]$
5.  $t|-A \vee B$  iff  $\pi_0 t = \text{True}$ ,  $\pi_1 t |- A$ , or  $\pi_0 t = \text{False}$ ,  $\pi_2 t |- B$
6.  $t|-\exists x A$  iff  $\pi_0 t = n$  and  $\pi_1 t |- A[n/x]$

13

## BHK Realizers interprets proofs without EM as constructions

- There is a general procedure taking an arithmetical constructive proof of  $A$  (i.e., a proof without EM), and producing a BHK realizer of  $A$ , a program whose ideas mirrors the ideas of the proof. See Appendix for a sketch, [Re], § 1.2 for more details, and [Bo] for a full account.
- If the proof uses Peano Induction, then we decided to express the BHK realizer belongs to the system  $\mathcal{T}$ .
- If the proof also uses **induction over well-founded** decidable relations, we express the BHK realizer in the system  $\mathcal{T}$  + **fixed point operator**.

15

## Informative clauses in BHK's Realizability

- **Clauses 1** of BHK Realizability says that a proof of an atomic formula  $P(t_1, \dots, t_n)$  carries no information but the fact that  $P(t_1, \dots, t_n)$  is true, and corresponds to a trivial program.
- **Clauses 2-4** of BHK Realizability ( $A \wedge B$ ,  $A \rightarrow B$ ,  $\forall x A$ ) move the information from a realizer to another one, but they produce no new information.
- The clause **3** for  $t|-A \rightarrow B$  has the typical form  $\forall u \in \{a |- A\}$ .  $t(u) \in \{b |- B\}$  used in functional languages.
- **Clause 5** produces some new information: **True**  $\in$  **Bool** whenever the left-hand-side of  $A \vee B$  is realizable, **False**  $\in$  **Bool** whenever the right-hand-side of  $A \vee B$  is realizable.
- **Clause 6** produces some new information: some  $n \in \mathbb{N}$  such that  $A[n/x]$  is realizable.

14

## BHK Realizers allow to compute the witness of an existential statement

- A constructive proof of  $\forall x \exists y. P(x, y)$ , with  $P$  any formula, is interpreted by some  $r |- \forall x \exists y. P(x, y)$ , which takes some value  $a$  for  $x$  and return some value  $b$  for  $y$  such that  $P(a, b)$ .
- Such a  $b$  is called a "witness" of  $\exists y. P(a, y)$ .
- For instance, if  $L$  is the type of lists over  $\mathbb{N}$ , a proof of  $\forall l \in L. \exists m \in L. (\text{Perm}(l, m) \wedge \text{Sorted}(m))$  is interpreted by a realizer which is a sorting algorithm ([Re], § 2.1).
- The particular sorting algorithm we obtain depends on the idea of the proof: there are proofs corresponding to **InsertSort**, **MergeSort**, ...

16

## BHK Realizers does not interpret $EM_1$

- We cannot interpret in BHK Realizability Semantics an arithmetical proof including EM.
- The reason is that the boolean and the natural number in Clauses 5, 6 are computed by recursive maps from the parameters of the formula.
- This forbids realizers of some instance  $\forall x.(\exists y.P(x,y) \vee \forall y.P^\perp(x,y))$  of  $EM_1$ , for some decidable  $P(x,y)$ .
- Indeed, any realizer of  $EM_1$  should provide a map taking some  $n \in \mathbb{N}$  and returning True if  $\exists y.P(n,y)$  is realizable, False if  $\forall y.P^\perp(n,y)$  is realizable. By Turing's proof of undecidability of the Halting problem, there is no such a map for some **decidable**  $P(x,y)$ .

17

## The set **Atom** of atoms of information

- Assume  $D_1, \dots, D_n, D$  are data types in  $\{\text{Unit}, \text{Bool}, \mathbb{N}, L\}$ .
- Let  $\underline{d} = d_1, \dots, d_n$ . An **atom** is any sequence  $\langle P, \underline{d}, d \rangle$ , with  $P: D_1, \dots, D_n, D \rightarrow \text{Bool}$  any closed term of  $\mathcal{T}$ , and  $d_1 \in D_1, \dots, d_n \in D_n, d \in D$ , such that  $P(\underline{d}, d) = \text{True}$  in  $\mathcal{T}$ .
- An atom  $\langle P, \underline{d}, d \rangle$  includes the information:  $\exists x \in D. P(\underline{d}, x) = \text{True}$  is true, and provides an example of some  $d \in D$  such that  $P(\underline{d}, d) = \text{True}$ . Such a  $d \in D$  is called a **witness** of  $\exists x \in D. P(\underline{d}, x) = \text{True}$ .
- We denote with **Atom** the set of all atoms. A set  $s$  of atoms is **consistent** if it includes at most one witness for any existential statement  $\exists x \in D. P(\underline{d}, x) = \text{True}$ . A set  $s$  of atoms is **complete** if it includes exactly one witness for any existential statement  $\exists x \in D. P(\underline{d}, x) = \text{True}$ . Any complete set is infinite and is not recursive.

19

## § 3. Interactive Realizability Semantics

- We introduce Goedel's system  $\mathcal{T}$  extended with knowledge states, then Interactive Realizability Semantics, *I-Realizability* for short.
- We compare *I-Realizability* with *BHK-Realizability Semantics*.

18

## The set **S** of knowledge states

- **S** is the set of finite consistent sets of atoms.
- **S<sub>∞</sub>** is the set of (possibly infinite) consistent sets of atoms.
- **P<sub>fin</sub>(Atom)** is the set of (possibly inconsistent) finite sets of atoms.
- Any  $s = \{\langle P_1, \underline{d}_1, d_1 \rangle, \dots, \langle P_k, \underline{d}_k, d_k \rangle\} \in S$  includes the information that finitely many  $\exists x \in D. P(\underline{d}, x) = \text{True}$  are true, and exactly one witness for each of them.
- If  $s \in S$  includes no witness for  $\exists x \in D. P(\underline{d}, x) = \text{True}$ , we say that  $s$  "guesses"  $\forall x \in D. P(\underline{d}, x) = \text{False}$ .
- This "guess" may be used during the computation of a realizer, but often turns out to be false during the same computation.

20

## Merging sets of atoms

- The **merging** of a consistent  $s \in \mathcal{S}$  and of  $X \in P_{\text{fin}}(\text{Atom})$  is some  $s' \subseteq s \cup X$  obtained by selecting one atom  $\langle P, \underline{d}, d \rangle \in X$  for each  $\exists x \in D. P(\underline{d}, x) = \text{True}$  having no witness in  $s$  (such that  $\langle P, \underline{d}, e \rangle \notin s$  for all  $e \in D$ ), and adding it to  $s$ .
- An example of merging.** Let  $X = \{\langle P, \underline{d}, d \rangle, \langle P, \underline{d}, d' \rangle\}$ .
  - If  $\langle P, \underline{d}, e \rangle \in s$  for all  $e \in D$ , then the two possible merging of  $s, X$  are  $s' = s \cup \{\langle P, \underline{d}, d \rangle\}$  and  $s' = s \cup \{\langle P, \underline{d}, d' \rangle\}$ . We select and add to  $s$  one witness for  $\exists x \in D. P(\underline{d}, x) = \text{True}$ .
  - If  $\langle P, \underline{d}, e \rangle \in s$  for some  $e \in D$ , then the only possible merging of  $s, X$  is  $s' = s$ . We do not add a witness for  $\exists x \in D. P(\underline{d}, x) = \text{True}$  to  $s$ , because we already have one.

Merging corresponds to an **(apparent) conflict** in a parallel computation, when two processes try to write **two solutions of the same goal** over **the same memory**. It does not matter which process wins: **the goal is fulfilled**.<sub>21</sub>

## An extension $T_s$ of Goedel's system $T$ with knowledge states

We add to Goedel's system  $T$  and to the language of arithmetic the following constants.

- Atomic types:**  $\mathcal{S}$  denoting the set of finite consistent sets of atoms, and  $P_{\text{fin}}(\text{Atom})$ , denoting the set of finite sets of atoms.
- One **constant**  $s$  for each  $s \in \mathcal{S}$ , and one **constant**  $X$  for each  $X \in P_{\text{fin}}(\text{Atom})$
- The **union** map  $U: P_{\text{fin}}(\text{Atom}), P_{\text{fin}}(\text{Atom}) \rightarrow P_{\text{fin}}(\text{Atom})$

For any  $P: D_1, \dots, D_n, D \rightarrow \text{Bool}$  closed term of  $T$  we add:

- the **Skolem map**:  $\phi_P: S, D_1, \dots, D_n \rightarrow D$ ,
- the **oracle**  $\chi_P: S, D_1, \dots, D_n \rightarrow \text{Bool}$
- the **update map**:  $\text{Add}_P: S, D_1, \dots, D_n, D \rightarrow P_{\text{fin}}(\text{Atom})$  <sub>23</sub>

## Monotonic Learning and knowledge states

- A program with **monotonic learning** has a state  $s \in S_{\text{fin}}$ , and uses all information and assumptions from  $s$ .
- Whenever the program finds some example  $P(\underline{d}, d) = \text{True}$  which **falsifies** an assumption  $\forall x \in D. P(\underline{d}, x) = \text{False}$  of  $s$ , **it merges the one-element set  $\{\langle P, \underline{d}, d \rangle\}$  with  $s$**  and restarts **all subcomputations which used this wrong assumption**.
- This idea of monotonic learning is better expressed using processes executed **in parallel** and **non-deterministically**. However, in order to compare I-Realizability with BHK-Realizability, we express monotonic learning programs in some **extension of Goedel's system  $T$** . The relation between learning programs and functional programs may be made formal in term of **Monads** [Be]. <sub>22</sub>

## Reduction rules for $T_s$

$T_s$  is defined by adding to  $T$  the algebraic reductions corresponding to the following equations:

- $U(X, Y) = X \cup Y$
  - $\text{Add}_P(s, \underline{d}, d) = \{\langle P, \underline{d}, d \rangle\} \in P_{\text{fin}}(\text{Atom})$  if  $P(\underline{d}, d) = \text{True}$  and  $\langle P, \underline{d}, d \rangle \notin s$  for all  $d \in D$ ,  $= \emptyset \in P_{\text{fin}}(\text{Atom})$  otherwise.
  - $\phi_P(s, \underline{d}) = \underline{d} \in D$  if  $\langle P, \underline{d}, d \rangle \in s$ ,  $=$  some dummy value  $\underline{d}_0 \in D$  otherwise.
  - $\chi_P(s, \underline{d}) = \text{True}$  if  $\langle P, \underline{d}, d \rangle \in s$  for some  $d \in D$ ,  $= \text{False}$  otherwise.
- $U$  is union map.  $\text{Add}_P$  returns at most **one fresh atom** (i.e., not in  $s$ ) to be added to the knowledge state  $s$ .
  - $\phi_P$  is a Skolem map providing a witness for  $\exists x \in D. P(\underline{d}, x) = \text{True}$  if any exists **in  $s$** ,  $\chi_P$  is an oracle deciding whether  $\exists x \in D. P(\underline{d}, x) = \text{True}$  is true **using  $s$** . The maps  $\phi_P, \chi_P$  are **relativized to some  $s \in \mathcal{S}$** . <sub>24</sub>

### Some examples for $\chi_P$ and $\phi_P$

- Let  $s = \{\langle P, 0, 13 \rangle, \langle P, 13, 205 \rangle\}$ . Assume  $P: N, N \rightarrow \text{Bool}$  is a binary closed term of  $T_S$ .
- We have  $\chi_P(s, 0) = \text{True}$  and  $\phi_P(s, 0) = 13$ , because  $\langle P, 0, 13 \rangle \in s$ .
- We have  $\chi_P(s, 13) = \text{True}$  and  $\phi_P(s, 13) = 205$ , because  $\langle P, 13, 205 \rangle \in s$ .
- We have  $\chi_P(s, 205) = \text{False}$  and  $\phi_P(s, 205) = \text{some dummy value in } N$ , because  $\langle P, 205, m \rangle \notin s$  for all  $m \in N$ .
- Even if  $\chi_P(s, 205) = \text{False}$ , we might have  $\exists x \in N. P(205, x) = \text{True}$  because, say,  $P(205, 133) = \text{True}$  but  $s$  "does not know it". By this we mean:  $\langle P, 205, m \rangle \notin s$  for all  $m \in N$ .

25

### Terms and formulas having a "hole" of type S

- We call each  $s \in S$  a (finite) "**knowledge state**".
- We call S-terms and S-formulas all terms  $t[\cdot]$  and formulas  $A[\cdot]$  having a free variable  $(\cdot): S$  as unique subterm of type S.
- We denote by  $A[s]$  the result of replacing  $(\cdot)$  with some  $s: S$ . The constant  $s$  is the only subterm of type S in  $t[s]$ ,  $A[s]$ , and it represents the **current knowledge state** of the realizer  $t[\cdot]$  and of the formula  $A[\cdot]$ .

### The Skolem maps and the oracles of $T_S$ may be wrong

- The Skolem maps  $\phi_P(s, \underline{d})$  and the oracle  $\chi_P(s, \underline{d})$  of  $T_S$  have an extra argument  $s$ , they use the information and the guesses from  $s$ , and they are **computable**, while ordinary Skolem maps are not.
- The price to pay is that  $\phi_P(s, \underline{d})$  may fail to produce a witness for  $\exists x \in D. P(\underline{d}, x) = \text{True}$  even if some exists, in the case no such witness is available in  $s$ .
- For the same reason, we may have  $\chi_P(s, \underline{d}) = \text{False}$  even if  $\exists x \in D. P(\underline{d}, x) = \text{True}$  is true, if no witness for such statement is available in  $s$ .
- The outputs of  $\phi_P, \chi_P$  rely on the guesses made by  $s$ , which may turn out **to be wrong**. However, through a learning mechanism, a realizer of a of simply existential formula in  $T_S$  will eventually return a **correct witnesses** for the formula. 26

### Interactive Realizability (w.r.t. a knowledge state s)

- For each arithmetical formula we define a type  $\|A\|$  for the interactive realizers of  $A$ . The definition is the same as in BHK, but in the case of an atomic formula, in which we choose:  $\|P(t_1, \dots, t_m)\| = P_{\text{fin}}(\text{Atom})$ . Interactive realizers of atomic formulas are (possibly inconsistent) **sets of atoms**, while BHK realizers of atomic formulas are dummy constants.
- For any S-term  $t$ , S-formula  $A$  such that  $t: \|A\|$ , we define a realizability notion  $t \Vdash_s A$ , to be read: "**t realizes A w.r.t. to a knowledge state  $s \in S$** ". We call it "**Interactive Realizability**".
- The goal of a realizer  $t$  w.r.t. the knowledge state  $s \in S$  is to **interact** with the global knowledge state  $s$ , extending it in order to make the formula  $A$  true.

## Interactive Realizability (w.r.t. a knowledge state s)

The definition of Interactive Realizability is by induction over the S-formula **A**. Differences with BHK's Realizability are marked **red**.

1.  $t \Vdash_s P(t_1, \dots, t_k)$  iff  **$t[s]-s = \emptyset$  implies  $P(t_1, \dots, t_k)[s] = \text{True}$**
  2.  $t \Vdash_s A \wedge B$  iff  $\pi_0 t \Vdash_s A$  and  $\pi_1 t \Vdash_s B$
  3.  $t \Vdash_s A \rightarrow B$  iff for all  $u \Vdash_s A$  we have  $tu \Vdash_s B$
  4.  $t \Vdash_s \forall x A$  iff for all  $n \in \mathbb{N}$  we have  $tn \Vdash_s A[n/x]$
  5.  $t \Vdash_s A \vee B$  iff either  $\pi_0 t[s] = \text{True}$  and  $\pi_1 t \Vdash_s A$ , or  $\pi_0 t[s] = \text{False}$  and  $\pi_2 t \Vdash_s B$
  6.  $t \Vdash_s \exists x A$  iff  $\pi_0 t[s] = n$  and  $\pi_1 t \Vdash_s A[n/x]$
- $t \Vdash A$  iff  $\forall s \in S. t \Vdash_s A$

29

## Atomic formulas in Interactive Realizability

- The clause 1 for  $t \Vdash_s P(t_1, \dots, t_k)$  has the form  $\forall s \in S. t[s]-s = \emptyset$  **implies**  $P(t_1, \dots, t_k)[s] = \text{True}$ . It is the only clause different from BHK Realizability.
- Clause 1 defines the following loop, which we call **the learning loop**: the realizer  $t$  merges some  $\emptyset \subset X \subseteq t[s]-s$  with  $s$ , forming  $s'$ , then some  $\emptyset \subset X' \subseteq t[s']-s'$  with  $s'$ , forming  $s''$ , and so forth, producing some increasing chain  $s \subset s' \subset s'' \subset \dots$  of states. If and when we have  $\emptyset = t[s^{(n)}]-s^{(n)}$  (no fresh atoms are added to  $s^{(n)}$ ) we reached some state  $s^{(n)}$  in which, according to clause 1, we have  $P(t_1, \dots, t_k)[s^{(n)}] = \text{True}$ .
- We may prove that if a realizer is **extracted from a proof**, then it adds **only fresh atoms** (i.e.,  $t[s] \cap s = \emptyset$  for all  $s \in S$ ). In this case, **the learning loop ends when**  $t[s^{(n)}] = \emptyset$

30

## The Fixed Point Theorem

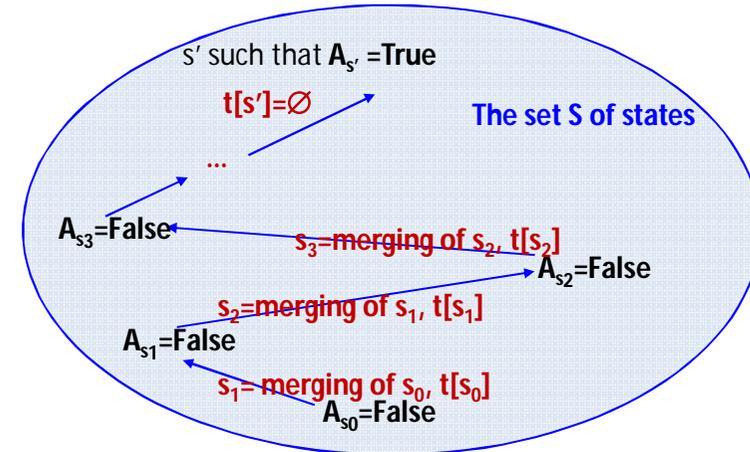
- We may prove (using the fact that the map  $s:S \rightarrow t[s]: P_{\text{fin}}(\text{Atom})$  is continuous w.r.t. the Scott topology over  $S$ ) the following Fixed Point result, which guarantees termination of the learning loop.

**Fixed Point Theorem.** Assume  $t[\cdot]: P_{\text{fin}}(\text{Atom})$  is any S-term. Then any sequences  $s \subset s' \subset s'' \subset \dots$  of states defined by  $s^{(i)} \subset s^{(i+1)} \subseteq$  some merging of  $s^{(i)}, t[s^{(i)}]$  for all  $i$ , **terminates in**  $\emptyset = t[s^{(n)}]-s^{(n)}$ , **for some  $n$ .**

In the next slide we represent one possible learning loop associated to a realizer  $t[\cdot]$  validating an atomic formula **A**. In this particular loop we add the **maximum possible of atoms** at each step. By Fixed Point Theorem, though, we are **not forced** to add the maximum of atoms at each step.

31

## A possible learning loop for a realizer $t[\cdot]$ of an atomic formula **A**



### Informative clauses in Interactive Realizability

- **Clauses 2-4** of Interactive Realizability ( $A \wedge B$ ,  $A \rightarrow B$ ,  $\forall xA$ ) move some information from a realizer to another one, but they produce no new information.
- **Clause 5** produces some new information: if  $\pi_0 t[s] = \text{True} \in \text{Bool}$  then the left-hand-side of  $A \vee B$  is realizable, if  $\pi_0 t[s] = \text{False} \in \text{Bool}$  then the right-hand-side of  $A \vee B$  is realizable.
- The value  $\pi_0 t[s]$  (and the side of  $A \vee B$  which we realize) may change as the knowledge state increase. By a continuity argument, in any increasing chain  $s \subset s' \subset s'' \subset \dots$  of states,  $\pi_0 t[s]$  is **eventually stationary** either to **True** or to **False**. In general,  $\pi_0 t[s]$  is not stationary to the same value **True (or False)** on all sequences. 33

### Interactive Realizers interprets proofs with EM1 as learning programs

- We may interpret proofs using  $EM_1$  into Interactive Realizers of system  $T_S$  (of  $T_S$  + **fixed point operators** if the proof uses well-founded induction). The procedure is **almost the same** interpreting arithmetical constructive proof of A into BHK realizer of A. There are two differences:
  1. We change BHK interpretation of **"atomic rules"**, that is, of all rules having atomic premises and conclusion. For instance: reflexivity, symmetry and transitivity of equality.
  2. We produce **an interactive realizer of  $EM_1$** .
- We explain these changes in the next slides. We refer to the Appendix for a sketch of the interpretation of proofs into interactive realizers, and to [As],[As3] for a full account. 35

### Informative clauses in Interactive Realizability

- **Clause 6** produces some new information: some witness  $\pi_0 t[s] = n \in \mathbb{N}$  of  $\exists xA$  (some  $n \in \mathbb{N}$  such that  $A[n/x]$  is realizable).
- The witness  $\pi_0 t[s] \in \mathbb{N}$  of  $\exists xA$  may change as the knowledge state increases. By a continuity argument, in any increasing chain  $s \subset s' \subset s'' \subset \dots$  of states,  $\pi_0 t[s]$  is **eventually stationary** to some value  $n_0$  (not to the same  $n_0$  on all sequences, though).
- The term  $\pi_0 t[s]$  has a multi-value limit, one for each increasing chain  $s \subset s' \subset s'' \subset \dots$  of states.
- These different limit values arise in different computations, therefore are not in contradiction each other. 34

### The interactive realizer associated to an atomic rule

$$\begin{array}{c} \dots & & \dots \\ r_1[s] \parallel \vdash P_1(t_1) & \dots & r_n[s] \parallel \vdash P_n(t_n) \\ \hline r_1[s] \cup \dots \cup r_n[s] \parallel \vdash P(t) \end{array}$$

- **Why is it correct to take the union of all realizers?** In order to reach a state in which  $P(t)$  is true it is enough to reach a state in which  $P_1(t), \dots, P_n(t)$  are true, i.e., a state  $s$  in which  $r_1[s] = \dots = r_n[s] = \emptyset \in P_{fin}(Atom)$ .
- If we define  $r[s] = r_1[s] \cup \dots \cup r_n[s]$ , when  $r[s] = \emptyset$  we have  $r_1[s] = \dots = r_n[s] = \emptyset \in P_{fin}(Atom)$ , therefore  $P_1(t), \dots, P_n(t)$  are true, hence  $P(t)$  is true. Thus,  $r[s] \parallel \vdash P(t)$ . 36

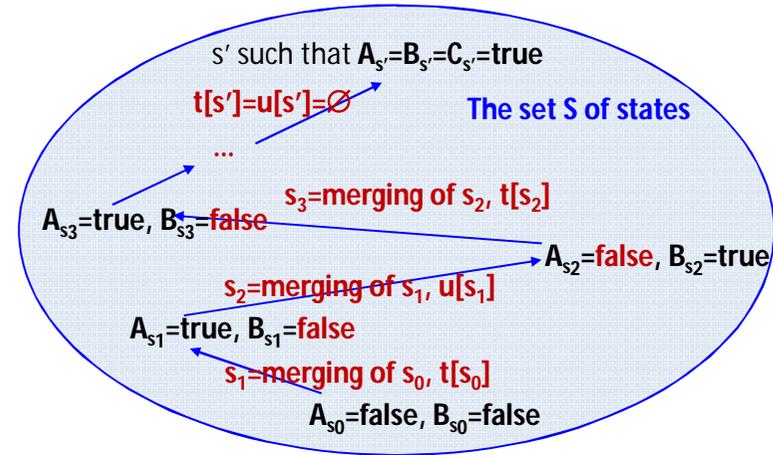
### How does a realizer work for an atomic rule?

$t[s]: A$        $u[s]: B$

-----  
 $t[s] \text{ U } u[s]: C$

- The realizer  $r[s] = t[s] \text{ U } u[s]$  searches for some  $s$  such that  $r[s] = \emptyset$ , that is,  $t[s] = u[s] = \emptyset$ , in order to validate the atomic formulas  $A$  and  $B$  at the same time, and  $C$  as a consequence.
- The search for  $s$  such that  $t[s] = u[s] = \emptyset$  terminates in finitely many steps by the Fixed Point Theorem. However, this search may be more complex than just searching for some  $s$  such that  $t[s] = \emptyset$ . For instance, if we look first for a state in which  $t[s] = \emptyset$  and  $A$  is true it might be that  $u[s] \neq \emptyset$  and  $B$  false, and conversely (see next slide for an example). 37

### A possible learning loop for a realizer $r[.] = t[.] \text{ U } u[.]$ of the conclusion of an atomic rule



### The interactive realizer of $EM_1$

- A realizer  $E_p \mid \vdash \forall x. (\exists y. P(x,y) \vee \forall y. P^\perp(x,y))$  of an instance of  $EM_1$  may be defined by
 
$$E_p[s](x) = \langle \chi_p(s,x), \langle \phi_p(s,x), \emptyset \rangle, \lambda y. \text{Add}_p(s,x,y) \rangle$$
- Given any value  $n$  for  $x$ , the realizer  $E_p[s](n)$  returns the truth value  $\chi_p(s,n)$ , that is, the assumption made by  $s$  about the truth value of  $\exists y. P(n,y)$ , and a realizer either of  $\exists y. P(n,y)$ , or of  $\forall y. P^\perp(n,y)$ , according to the truth value of  $\chi_p(s,n)$ .
- Assume  $\chi_p(s,n) = \text{True}$ .** Then  $s$  has some atom  $\langle P, n, m \rangle$  proving  $\exists y. P(n,y)$ . In this case  $\langle \phi_p(s,n), \emptyset \rangle$  is a realizer of  $\exists y. P(n,y)$ : indeed,  $m = \phi_p(s,n)$  is a witness of  $\exists y. P(n,y)$ , and  $\emptyset : P_{\text{fin}}(\text{Atom})$  is a realizer of  $P(n,m)$ , because  $P(n,m)$  is true.

### The learning loop for $EM_1$

- Assume  $\chi_p(s,n) = \text{False}$ .** Then  $s$  has no atom  $\langle P, n, m \rangle$  proving  $\exists y. P(n,y)$ , and  $\lambda y. \text{Add}_p(s,n,y)$  is a realizer of  $\forall y. P^\perp(n,y)$ , that is, for any  $m$ ,  $\text{Add}_p(s,n,m)$  realizes  $P^\perp(n,m)$ . Indeed:
    - If  $m$  is a witness of  $\exists y. P(n,y)$ , then the realizer  $E_p$  learns that  $\exists y. P(n,y)$  is true.  $\text{Add}_p(s,n,m)$  returns the singleton  $\{\langle P, n, m \rangle\} \in P_{\text{fin}}(\text{Atom})$ , to be merged to the state  $s$ .
    - If  $m$  is no witness, then  $P(n,m)$  is false, and therefore  $P^\perp(n,m)$  is trivially realizable by  $\text{Add}_p(s,n,m) = \emptyset \in P_{\text{fin}}(\text{Atom})$ .
- Remark that the behaviour of each instance  $E_p(n)$  of the realizer  $E_p$  is quite simple.  $E_p[.](n)$  may add at most one atom  $\langle P, n, m \rangle$  to the knowledge state  $s$ .

## Program extraction in Interactive Realizability

- Any proof of  $\forall x \exists y. P(x, y)$ , with P **atomic**, and using  $EM_1$ , is interpreted by some  $r \Vdash \forall x \exists y. P(x, y)$ , which takes some value  $a$  for  $x$ , and returns some value  $b[s] = \pi_0 r(a)[s]$  for  $y$  and some state-extending operator  $t[s] = \pi_1 r(a)[s]: P_{fin}(Atom)$ .
- After a finite loop of state-extending operations, we may reach some  $s$  such that  $t[s] = \emptyset$ : for such an  $s$ , the value of  $b[s]$  is a **witness** of  $\exists y. P(a, y)$ .
- Interactive Realizability provides a model of the fragment  $EM_1$  of classical logic in which **all connectives, including  $\forall, \exists$ , are interpreted as in BHK Realizability**. This is not the case with all other constructive interpretations of classical logic.
- The model is a conservative extension of BHK Realizability model for formulas  $\exists y. P(a, y)$  with P atomic. 41

## What is the use of a witness “in the limit”?

- A realizer  $r \Vdash \forall x \exists y. P(x, y)$  for a non-atomic P has an interest in computations, even if it provides a witness  $b[s] = \pi_0 r(a)[s]$  only in the limit of an increasing chain  $s \subset s' \subset s'' \subset \dots$  of states, and even if this limit is not computable in general.
- Indeed, assume that we use  $\forall x \exists y. P(x, y)$  as a Lemma to prove a goal  $\exists z. Q(z)$ , with Q atomic. Then, by a continuity argument, we may prove that we only need to know the value of  $b[s]$  over some finite state  $s \in S$  in order to compute a witness  $c$  for  $\exists z. Q(z)$ . We do not have to compute the limit of  $b[s]$ , we **only have to know some “approximation” of  $b[s]$**  in some finite state in order to fulfill our goal  $\exists z. Q(z)$ , and such  $b[s]$  is **not a true witness** of  $\exists y. P(a[s], y)$  in general. 43

## The interpretation of $\exists y. P(x, y)$ for a non-atomic P

- Any proof of  $\forall x \exists y. P(x, y)$ , with P **not atomic**, and using  $EM_1$ , is interpreted by some  $r \Vdash \forall x \exists y. P(x, y)$ , which takes some value  $a$  for  $x$ , and returns some value  $b[s] = \pi_0 r(a)[s]$  for  $y$  and some realizer  $t[s] = \pi_1 r(a)[s]$  of  $P(a, b)$ .
- By a continuity argument, we may prove that  $b[s]$  stabilizes to some limit value  $v$  on all increasing chains  $s \subset s' \subset s'' \subset \dots$  of states (not to the same value  $v$  on all sequences, though).
- We may prove that  $v$  is a **witness** of  $\exists y. P(a, y)$  only if the knowledge state  $s = \bigcup_{n \in \mathbb{N}} s^{(n)} \in S_\infty$  limit of the chain is **complete**. The limit  $v$  over the chain is **not computable** from the input  $a$  when P is not atomic. 42

## Summing up

- Interactive Realizability w.r.t. a knowledge state interprets a classical proof of an existential statement  $\exists y. P(a, y)$  with P atomic as a realizer finding a witness, and using a knowledge state  $s \in S$  increasing with time.
- Whenever the proof, by Excluded Middle, uses the truth value of a formula  $\exists y. P(n, y)$  which is not known in  $s$ , the realizer makes a **guess  $\forall y. P^\perp(n, y)$**  about this truth value.
- If and when the realizer finds a witness  $m$  for the opposite statement  $\exists y. P(n, y)$ , it **merges  $\{P, n, m\}$  with the current knowledge state  $s$** . Then it removes all subcomputations built over the guess  $\forall y. P^\perp(n, y)$ .
- **Program extracted from classical proofs are associated to many state-extending operators  $C, C', C'', \dots$**

44

## Comparing Realizability and Games models for Classical Logic

- Interactive Realizability is a Realizability model of  $EM_1$  and monotonic learning. Interactive Realizability **originates from Game-Theoretical model of  $EM_1$  and monotonic learning**, which uses the idea of 1-backtracking [As2].
- **“Backtracking”** in Game Theory is the possibility for a player of coming back finitely many times to a previous position of the play and changing his/her move from it. Adding backtracking to Game Theory allows us to model full Classical Logic [Coq].
- **“1-Backtracking”** is a restricted form of backtracking, when coming back to a previous move is an **irreversible choice**. 1-backtracking models the fragment  $EM_1$  of EM [Be1], [Be2], [Be3].

45

## Bibliography

- [Ak] Y. Akama, S. Berardi, S. Hayashi, U. Kohlenbach, *An Arithmetical Hierarchy of the Law of Excluded Middle and Related Principles*. LICS 2004, pages 192-201.
- [As] F. Aschieri and S. Berardi. *A Realization Semantics for  $EM_1$ -Arithmetic*. TLCA, 2009.
- [As2] F. Aschieri. *Learning Based Realizability for HA +  $EM_1$  and 1-Backtracking Games: Soundness and Completeness*. Submitted to APAL.
- [As3] F. Aschieri. *Learning, Realizability and Games in Classical Arithmetic*. Ph. D. thesis, Torino, 2011.

47

## “Retracting” and Classical Logic: a mathematical study

- The common ground between Interactive Realizability and Game Theory with backtracking is the possibility of **“retracting a previous choice”**: retracting a guess in Interactive Realizability, retracting a previous move in Game Theory with backtracking.
- The notion of **retracting may be studied as a mathematical notion**, without any reference to Realizability, nor to Game Theory.
- It turns out that retracting is a suitable notion for defining a **constructive model of Predicative Classical Arithmetic** [Be4] and of **non-monotonic learning**.

46

## Bibliography

- [Be] S. Berardi and U. de’ Liguoro. *Interactive realizers. A new approach to program extraction from non constructive proofs*. To appear on TOCL 2011:  
<http://tocl.acm.org/accepted/451deLiguoro.pdf>  
*Interactive Realizers and Monads* (Preliminary and simpler version of the journal paper, 2010):  
<http://www.di.unito.it/~deligu/papers/InterRealMonads.pdf>
- [Be1] S. Berardi, T. Coquand, and S. Hayashi. *Games with 1-backtracking*. APAL, 2010.
- [Be2] S. Berardi and M. Tatsuta. *Positive Arithmetic Without Exchange Is a Subclassical Logic*. In Zhong Shao, editor, APLAS, volume 4807 of Lecture Notes in Computer Science, pages 271-285. Springer, 2007.

48

## Bibliography

- [Be3] S. Berardi and Y. Yamagata. *A Sequent Calculus for Limit Computable Mathematics*. APAL, 153(1-3):111-126, 2008.
- [Be4] S Berardi, U. de' Liguoro: *Toward the interpretation of non-constructive reasoning as non-monotonic learning*. Inf. Comput. 207(1): 63-81 (2009)
- [Bo] L. Boerio *"Optimizing Programs Extracted from Proofs"*. Ph. D. Thesis, C. S. Dept. Turin University, 1997: <http://www.di.unito.it/~stefano/BoerioPhD-OptimizingProgramsExtractedFromProofs.ps>
- [Coq] T. Coquand. *A Semantics of Evidence for Classical Arithmetic*. JSL, 60(1):325-337, 1995.

49

### § 4. Learning and Parallel computations: an example

- We introduce some classical proofs of simple existential statements, and we use Interactive Realizability in order to extract a non-trivial program mirroring the ideas from the proof.
- We stress that we if allow non-determinism and parallelism in our interpretation, we may extract different and subtler programs from the same proofs.

51

## Bibliography

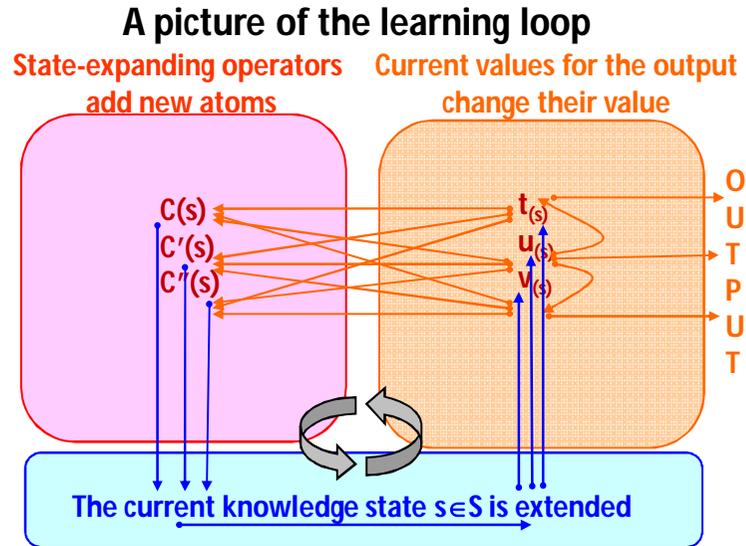
- [Go] E. M. Gold, *Limiting Recursion*, Journal of Symbolic Logic 30, p. 28-48 (1965)
- [Lo] J. Longley, *Realizability: a short course*, Summer School "Réalizabilité à Chambéry", 2011. Part 1.
- [Re] *Realizability: Extracting Programs from proofs*, Summer School of Bertinoro, Italy, 2007, § 1.2 and § 2.1:  
<http://www.di.unito.it/~stefano/Berardi-ProgramExtraction-SummerSchool-Bertinoro2007.ppt>

50

### Learning and Parallel computations

- Evaluating the **learning loop** associated to an interactive realizers require the study of a **parallel computation**.
- The reason is that such a realizer may be the **union** of state-extending operators  $C, C', C'', \dots$ , which may **return at the same time** different and possibly alternative witnesses to be added to our knowledge base.
- We obtain a different result if we add **one witness** at the time, sequentially, or **many witnesses in parallel**.
- Two witnesses of the same statement may be **in (apparent) conflict** with each other, and may require a non-deterministic choice. However, we may prove: if we start from a logically correct proof we obtain a **correct, terminating** and **deadlock-free** parallel computation.

52



## The Minimum Principle

- Assume  $f: \mathbb{N} \rightarrow \mathbb{N}$  is any map. A **minimum point** of  $f$  is any  $x \in \mathbb{N}$  such that  $f(x) \leq f(y)$ . The minimum principle is the statement  $\text{Min} = \exists x \forall y. f(x) \leq f(y)$ , that is, " **$f$  has a minimum point**".
- A BHK realizer of  $\text{Min}$  should define some computable functional  $F[f]$ , taking a parameter  $f: \mathbb{N} \rightarrow \mathbb{N}$ , and returning some minimum point  $n = F[f]$  of  $f$ .
- By a continuity argument, we may show that a computable functional  $F$  should produce a minimum point  $n$  **out of finitely many values of  $f$** . This is impossible for some  $f$ . Thus, there is no such an  $F$ , and no BHK realizer of  $\text{Min}$ .
- We describe now a classical proof of  $\text{Min}$ , then the interactive realizer of  $\text{Min}$  extracted from it.

54

## A classical proof of $\text{Min}$

- Assume  $f: \mathbb{N} \rightarrow \mathbb{N}$  is any map. We prove  $\text{Min} = \exists x \forall y. f(x) \leq f(y)$  using  $\text{EM}_1$ , by induction over the well-founded relation  $P(x, y) \equiv (f(x) > f(y))$ .
- **A Proof of  $\text{Min}$  by  $\text{EM}_1$  and induction over  $P$ .** We assume that if  $f(y) < f(x)$  for some  $y$ , then  $\text{Min}$  holds, and we have to prove  $\text{Min}$ . We use  $\text{EM}_1$  on  $P$  and  $x$ :  $\exists y. f(x) > f(y) \vee \forall y. f(x) \leq f(y)$ , and case reasoning. **Left-hand-side.** If  $\exists y. f(x) > f(y)$ , we pick some  $y$  such that  $f(x) > f(y)$ , we apply the induction hypothesis on  $y$ , and we deduce  $\text{Min}$ . **Right-hand-side.** If  $\forall y. f(x) \leq f(y)$ , then  $x$  is a minimum point of  $f$ . **Q.E.D..**
- If we express this proof in Natural Deduction, then we apply the translation sketched in the Appendix, we obtain an interactive realizer  $r[s] = \langle \mu_0[s], \lambda y \in \mathbb{N}. C(y)[s] \rangle$ , with  $\mu_0: \mathbb{N}$ , and  $C \mid \mid \neg \forall y. f(\mu_0) \leq f(y)$  a realizer.

55

## An interactive realizer of $\text{Min}$

We define the interactive realizer  $r[s] = \langle \mu_0[s], \lambda y \in \mathbb{N}. C(y)[s] \rangle$ , with  $\mu_0: \mathbb{N}$ , and  $C \mid \mid \neg \forall y. f(\mu_0) \leq f(y)$ . Let  $P(x, y) \equiv (f(x) > f(y))$ .

1. The axiom  $\text{EM}_1$  on  $P$ ,  $x$  is translated by  $\chi_p(s, x)$ .
2. Case reasoning is translated by **if**( $\chi_p(s, x)$ , ..).
3. In the case  $\exists y. f(x) > f(y)$  (when  $\chi_p(s, x) = \text{True}$ ) we pick an  $y$  such that  $f(x) > f(y)$  with the Skolem map  $\phi_p(s, x)$ , then we translate ind. hyp. by a **recursive call**  $\mu(\phi_p(s, x))$ .
4. In the case  $\forall y. f(x) \leq f(y)$  (when  $\chi_p(s, x) = \text{False}$ ),  $x$  is a minimum point of  $f$ , and we return  $x$ .

The realizer  $C(m)$  of  $f(\mu_0) \leq f(m)$  is an instance of the *right-hand-side of  $\text{EM}_1$* , and is equal to update map  $\text{Add}_p(s, \mu_0, m)$ . In the case  $f(\mu_0) > f(m)$  (the guess  $\forall y. f(\mu_0) \leq f(y)$  is wrong)  $C(m)[s]$  adds the atom  $\langle P, n, m \rangle$  with  $n = \mu_0[s]$  to the knowledge state  $s$ .

56

## Defining a Realizer of the Minimum Principle

- Let  $P(x,y) \equiv (f(x) > f(y))$ . We define  $r \Vdash \text{-Min}$  by  $r[s] = \langle \mu_0[s], \lambda y \in \mathbb{N}. C(y)[s] \rangle$ , with  $\mu_0: \mathbb{N}$ , and  $C \mid \mid \text{-} \forall y. f(\mu_0) \leq f(y)$  a realizer.
- 1.  $\mu(x)[s] = \text{if}(\chi_p(s,x), \mu(\phi_p(s,x)), x) : \mathbb{N}$
- 2.  $\mu_0 = \mu(0) : \mathbb{N}$
- 3.  $C(y)[s] = \text{Add}_p(s, \mu_0, y) \mid \mid \text{-} f(\mu_0) \leq f(y)$
- Let  $s = \{ \langle P, 0, 13 \rangle, \langle P, 13, 205 \rangle \}$ . In §2 we checked that:  $\chi_p(s,0) = \text{True}$ ,  $\phi_p(s,0) = 13$ ,  $\chi_p(s,13) = \text{True}$ ,  $\phi_p(s,13) = 205$  and  $\chi_p(s,205) = \text{False}$ . Thus,  $\mu_0 = \mu(0) = (\text{by } \chi_p(s,0) = \text{True}) \mu(\phi_p(s,0)) = \mu(13) = (\text{by } \chi_p(s,13) = \text{True}) \mu(\phi_p(s,13)) = \mu(205) = (\text{since } \chi_p(s,205) = \text{False}) 205$ .  $s$  “guesses” that 205 is a minimum point of  $f$  because  $s$  includes no witness for  $\exists y. f(205) > f(y)$ .
- Let  $f(205) > f(133)$ . Then  $C(133)[s] = \{ \langle P, 205, 133 \rangle \}$ :  $C$  finds a counterexample 133 to the “guess” of  $s$  and adds it to  $s$ .  
57

## What is the use of a witness “in the limit”?

- The interactive realizer  $r \mid \mid \text{-} \exists x \forall y. f(x) \leq f(y)$  provides a witness for a non-atomic property  $\forall y. f(x) \leq f(y)$ . It has an interest for computations, even if it provides a witness  $\mu_0[s]$  only in the limit of an increasing chain  $s \subset s' \subset s'' \subset \dots$  of states, and even if this limit is not computable in general.
- In the rest of the talk, we use  $\exists x \forall y. f(x) \leq f(y)$  as a Lemma to prove goals of the form  $\exists z. Q(z)$ , with  $Q$  atomic. Then, by a continuity argument, we may prove that we only need to know the value of  $\mu_0[s]$  over some finite state  $s \in S$  in order to compute a witness  $c$  for  $\exists z. Q(z)$ . We will not have to compute the limit of  $\mu_0[s]$ , we will **only have to know some “approximation”**  $\mu_0[s]$  in some finite state  $s$ , and this is **not a true minimum** point of  $f$  in general.  
59

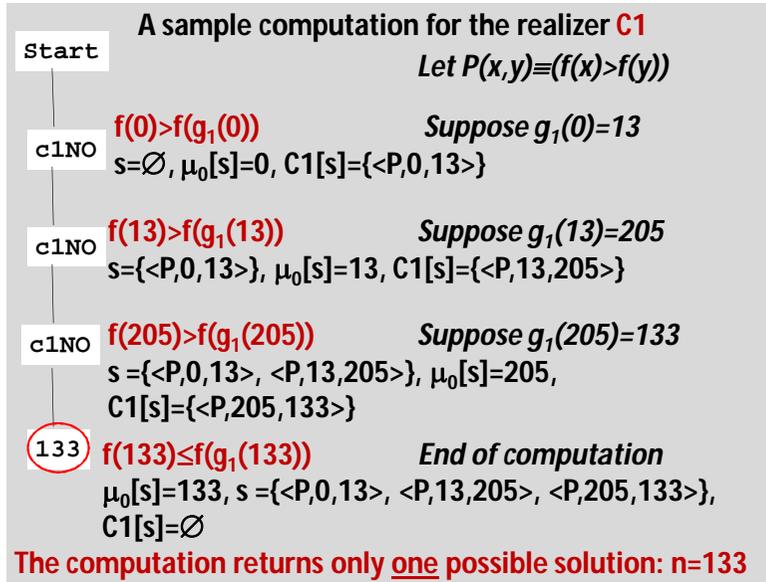
## Discussing the Realizer of the Minimum Principle

- The component  $\mu_0: \mathbb{N}$  returns some  $\mu_0[s]$ , which is a minimum point of  $f$  **w.r.t. the knowledge state  $s$** :  $s$  makes the “guess”  $\forall y. P^\perp(\mu_0[s], y)$ , that is,  $\forall y. f(\mu_0[s]) \leq f(y)$ , because it has no evidence of the opposite.
- However, the guess made by  $s$  may be wrong, in this case  $\mu_0[s]$  is no minimum point of  $f$ , and we have  $f(\mu_0[s]) > f(p)$  for some  $p$ .
- The realizer  $C(m)[s]: P_{\text{fin}}(\text{Atom})$  asks for some  $m \in \mathbb{N}$ . In the case we have  $f(\mu_0[s]) > f(m)$ , then  $C(m)[s]$  adds the atom  $\langle P, \mu_0[s], m \rangle$  to the knowledge state  $s$ : it “learns that  $\mu_0[s]$  is wrong”.

58

## An Interactive Realizer for the corollary $\exists x. f(x) \leq f(g_1(x))$ of Min (by T. Coquand)

- Let  $P(x,y) \equiv (f(x) > f(y))$ . Assume  $f, g_1: \mathbb{N} \rightarrow \mathbb{N}$ . Consider the unequation  $(c_1) f(n) \leq f(g_1(n))$ . The existence of a solution of  $(c_1)$  is a corollary of Min, if we set  $n = \text{minimum point } \mu_0$  of  $f$ .
- Let  $C1 = C(g_1(\mu_0)): P_{\text{fin}}(\text{Atom})$ . Then  $\langle \mu_0, C1 \rangle$  is an interactive realizer of  $\exists x. f(x) \leq f(g_1(x))$  interpreting the classical proof of existence of a solution.  $C1$  is a state-extending operator, adding the atom  $\langle P, n, g_1(n) \rangle$  to  $s$  whenever  $n = \mu_0[s]$  and  $f(n) > f(g_1(n))$  (i.e.,  $c_1$  is false). In the new state  $s'$ ,  $\mu_0[s'] \neq n$ .
- In the next picture, we fix a random choice of  $f, g_1$ , then we draw **the only possible computation** finding a solution of the unequation  $(c_1)$  using the operator  $C1$ . Whenever  $c_1$  is false, we write **c1NO**. There is a unique state-extending operator, therefore the computation is **deterministic**.  
60



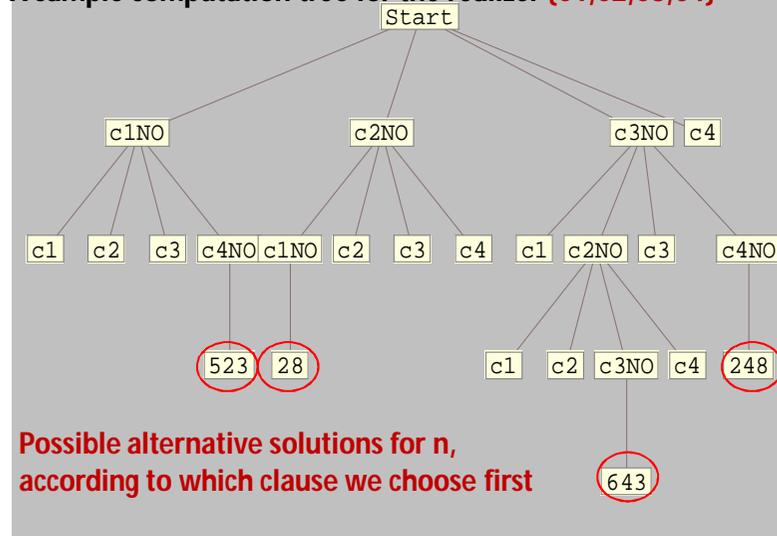
### A sample computation tree for a sequential non-deterministic Realizer

- Let  $P(x,y) \equiv (f(x) > f(y))$ . Assume  $f, g_1, g_2, g_3, g_4 : \mathbb{N} \rightarrow \mathbb{N}$ . Consider the 4-equations system:  
 $(c_i) f(n) \leq f(g_i(n)) \quad (i=1, \dots, 4)$
- The existence of a solution for the system  $(c_1) \wedge \dots \wedge (c_4)$  is a corollary of Min, if we set  $n = \text{minimum point } \mu_0 \text{ of } f$ .
- The interactive realizer interpreting the classical proof of existence of a solution is  $\langle \mu_0, C_1 U \dots U C_4 \rangle$ , with  $C_i = C(g_i(\mu_0))[s]$  for  $i=1, \dots, 4$ .  $C_i$  requires to add the atom  $\langle P, \mu_0[s], g_i(\mu_0[s]) \rangle$  to the current state  $s$ , whenever it is true (whenever  $f(\mu_0[s]) > f(g_i(\mu_0[s]))$  is true, i.e.,  $c_i$  is false).
- In the next picture, we draw the tree of all possible computations finding a solution of this system, using  $C_1, \dots, C_4$ . Whenever  $c_i$  is false, we write **ciNO**.

### A sample computation tree for a sequential non-deterministic Realizer

- Each  $C_i$  tries to make the subgoal  $(c_i)$  true: whenever the current value  $n = \mu_0[s]$  for the minimum of  $f$  is wrong,  $C_i$  adds the atom  $\langle P, n, g_i(n) \rangle$  to the knowledge state  $s$ . As a result,  $s$  increases to  $s'$ , and the current value  $n$  for the minimum is replaced by  $g_i(n) = \mu_0[s']$ .
- We have a tree of possible computation because the computation is non-deterministic. When more than one  $c_i$  is false we choose which  $C_i$  to apply, by choosing one node of the form **ciNO**: the tree forks. The computation is sequential: we can never apply in parallel  $C_i, C_j$ , because two atoms  $\langle P, n, g_i(n) \rangle, \langle P, n, g_j(n) \rangle$  define different witnesses for  $\exists y. f(n) > f(y)$ , hence are inconsistent each other.

### A sample computation tree for the realizer {C1,C2,C3,C4}



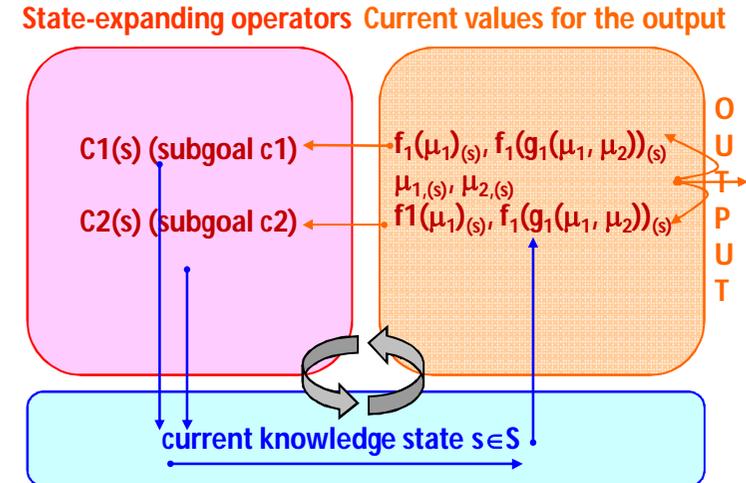
### A realizer corresponding to a parallel program

- **Thesis:**  $\forall f_1, f_2: \mathbb{N} \rightarrow \mathbb{N}. \forall g_1, g_2: \mathbb{N}, \mathbb{N} \rightarrow \mathbb{N}. \exists n, m \in \mathbb{N}$  s.t.
 
$$\begin{cases} (c1) & f_1(n) \leq f_1(g_1(n, m)) \\ (c2) & f_2(m) \leq f_2(g_2(n, m)) \end{cases}$$
  - **Proof (using Min of  $f_1$  and  $f_2$ ).**  $n$ =minimum point  $\mu_1$  of  $f_1$ ,  $m$ =minimum point  $\mu_2$  of  $f_2$ .
  - The realizer associated to the proof is  $\langle \mu_1, \mu_2, C1UC2 \rangle$  with  $C_i = C(g_i(\mu_1, \mu_2))$ . The current values of  $n, m$  rely on **guesses** made by the state  $s$ .
1. C1 tries and make **the subgoal (c1)** true: whenever the current value  $n$  for the minimum of  $f_1$  is wrong, C1 adds  $f_1(n) > f_1(g_1(n, m))$  to the knowledge state. As a result, the current value  $n$  for the minimum **is replaced by  $g_1(n, m)$ .**
  2. C2 tries and make **the subgoal (c2)** true, in the same way.

### A sample computation tree for a parallel non-deterministic realizer

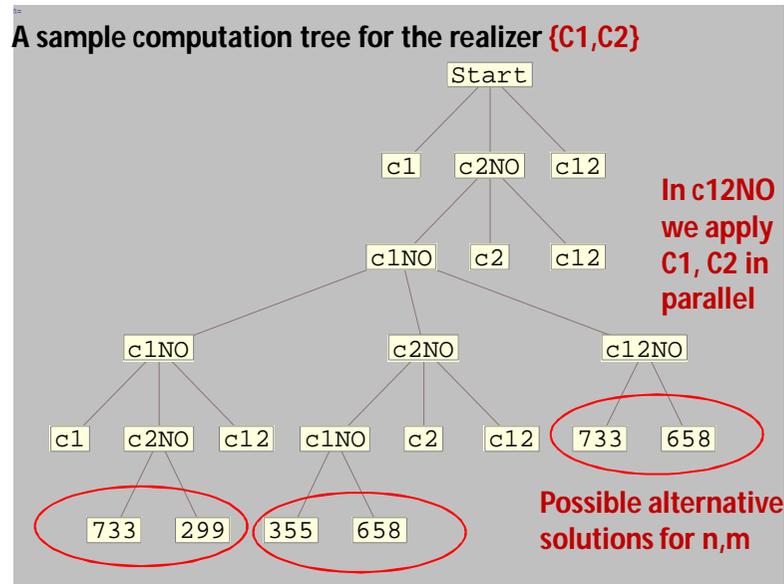
- In the next picture we assume to be fixed some random maps  $f_1, f_2, g_1, g_2$ , and we draw the computation tree for the union realizers **C1 U C2** in a sample case.
- A node labelled "**c1NO**" represent a situation in which the subgoal  $c1$  is false and we apply C1 to try and make  $c1$  true. The same for a node labelled "**c2NO**".
- A node labelled "**c12NO**" represent a situation in which both subgoals  $c1, c2$  are false and we apply C1 and C2 **in parallel** to try and make  $c1, c2$  true. C1, C2, may be applied in parallel, because they produce atoms associated to different existential formulas  $\exists y. f_1(n) > f_1(y), \exists y. f_2(n) > f_2(y)$ , hence always consistent each other.

### The learning loop associated to C1, C2



### A sample computation tree for a parallel non-deterministic realizer (2)

- A node labelled "**c1**" represent a situation in which  $c1$  is true and we cannot apply C1.
- A node labelled "**c2**" represent a situation in which  $c2$  is true and we cannot apply C2.
- A node labelled "**c12**" represent a situation in which either  $c1$  or  $c2$  is true and cannot apply C1, C2 in parallel.
- A pair of leaves of the tree labeled with two integers, say, 733, 299, represent a situation in which the current values  $n=733, m=299$  for the minimum of  $f_1, f_2$  solve the original problem (w.r.t. some  $f_1, f_2, g_1, g_2$  fixed at random).



## Appendix. The interpretation of proofs in the Realizability Semantics

- We define a mapping sending any proof with  $EM_1$  of  $A$  into an interactive realizer of  $A$ .
- With a minimum of changes the same procedure works for BHK Realizability for Intuitionistic Arithmetic.

70

## Interactive Realizability and BHK Realizability

- We define a map taking an arithmetical proof in natural deduction form of some formula  $\varphi$ , using  $EM_1$  and returns some interactive realizer  $r \Vdash \varphi$  in Goedel's system  $T_S$  extended with states. For a full account we refer to [As], [As3].
- For a description of arithmetic in natural deduction form we refer to [Re].
- The definition of the realizer is by induction over the proof.
- If we change the clauses for atomic formula and we drop the realizer for  $EM_1$  we obtain a procedure which maps an intuitionistic arithmetical proof of  $\varphi$  into a BHK Realizer  $r \Vdash \varphi$  in Goedel's system  $T$ .

71

## Extending Realizability to more Data Types

- All what we will say applies not just to a language having types:
  - $T = \text{Unit, Bool, N (Natural Numbers), L (Lists of Natural Numbers)}$
- but also to a language having types
  - $T = \text{any Bohm-Berarducci Data Types}$
- We refer Boerio Ph.d [Bo] for a procedure transforming any intuitionistic proof of this extended language into a BHK realizer.

72

### Dummy constants.

- For each simple type  $T$  of  $\mathcal{T}_s$ , we will need some dummy element  $\text{dummy}^T:T$  (just  $d^T$  for short), to be used as default value for such type.
- We define  $d^T:T$  by induction over  $T$ .
  - $\text{dummy}^{\text{Prin}(\text{Atom})} = \emptyset$
  - $\text{dummy}^{\text{Unit}} = \text{unit}$
  - $\text{dummy}^{\text{Bool}} = \text{False}$
  - $\text{dummy}^{\text{N}} = 0$
  - $\text{dummy}^{\text{L}} = \text{nil}$
  - $\text{dummy}^{T \rightarrow U} = \lambda x. \text{dummy}^U$
  - $\text{dummy}^{T \times U} = \langle \text{dummy}^T, \text{dummy}^U \rangle$

73

### The Interactive Realizability Interpretation of proofs

- If  $\underline{x}=x_1, \dots, x_n$  is a vector of variables of types  $T_1, \dots, T_n$ , then  $|\varphi(\underline{x})| = T_1 \times \dots \times T_n \rightarrow |\varphi|$  is the type of all  $r \mid \mid - \varphi(\underline{x})$ .
- Let  $\Gamma = \{\varphi_1, \dots, \varphi_n\}$  be a set of assumptions and  $\underline{x}=x_1, \dots, x_k$ . We write  $r \mid \mid - (\Gamma \mid - \varphi(\underline{x}))$  for:  $r$  is an interactive realizer of  $\varphi(\underline{x})$ , depending on free variables in  $\underline{x}$ , and on the realizer variables  $\xi_1 \mid \mid - \varphi_1(\underline{x}), \dots, \xi_k \mid \mid - \varphi_k(\underline{x})$ .
- We may turn every proof of  $\varphi(\underline{x})$ , with free assumptions in  $\Gamma$ , possibly using  $\text{EM}_1$ , into some  $r \mid \mid - (\Gamma \mid - \varphi(\underline{x}))$ . Definition is by induction on  $p$ , with one clause for each possible rule at the conclusion of  $p$ .
- If the proof of  $\varphi(\underline{x})$  is purely intuitionistic, we may define a BHK realizer  $r \mid \mid - (\Gamma \mid - \varphi(\underline{x}))$  by changing the case of atomic formulas and removing the definition of a realizer of  $\text{EM}_1$ .

75

### Realizability Interpretation of Formulas.

- Let  $\varphi$  be any closed formula and  $T = \{\text{Unit}, \text{Bool}, \text{N}, \text{L}\}$ . We recall the definition of the simple type  $|\varphi|$  for all interactive realizer  $r$  of  $\varphi$ . The definition of  $|\varphi|$  is by induction over  $\varphi$ .
  - $|\text{P}(t_1, \dots, t_m)| = \text{Prin}(\text{Atom})$
  - $|\varphi_1 \wedge \varphi_2| = |\varphi_1| \times |\varphi_2|$
  - $|\varphi_1 \vee \varphi_2| = \text{Bool} \times |\varphi_1| \times |\varphi_2|$
  - $|\varphi_1 \rightarrow \varphi_2| = |\varphi_1| \rightarrow |\varphi_2|$
  - $|\forall x \in T. \varphi| = T \rightarrow |\varphi|$
  - $|\exists x \in T. \varphi| = T \times |\varphi|$

We obtain the definition  $|\varphi|$  of the type of a BHK realizer of  $\varphi$  if we write  $|\text{P}(t_1, \dots, t_m)| = \text{Unit}$  in the atomic case

74

### The Interactive Realizer for an Atomic rule

**Atomic rules.** If the proofs ends by some Atomic rule, then the realizers of the assumptions are state-extending operators, and we take their union to realize the conclusion.

$$\begin{array}{c} \dots \qquad \dots \\ r_1[s] \mid \mid - \text{P}_1(\underline{t}_1) \quad \dots \quad r_m[s] \mid \mid - \text{P}_m(\underline{t}_m) \\ \hline r_1[s] \text{U} \dots \text{U} r_m[s] \mid \mid - \text{P}(\underline{t}) \end{array}$$

If  $r_1[s] \mid \mid - \Gamma \mid - \text{P}_1(\underline{t}_1), \dots, r_m[s] \mid \mid - \Gamma \mid - \text{P}_m(\underline{t}_m)$ , then  $r_1[s] \text{U} \dots \text{U} r_m[s] \mid \mid - \Gamma \mid - \text{P}(\underline{t})$

76

### The BHK Realizer for an Atomic rule

- **Atomic rules.** If the proofs ends by some Atomic rule, then  $r(x)=unit$ .

$$\frac{\dots \quad \dots}{unit \mid - P(t)}$$

- **If  $unit \mid - \Gamma \mid - P_1(t_1), \dots, unit \mid - \Gamma \mid - P_m(t_m)$ , then  $unit \mid - \Gamma \mid - P(t)$**

77

### Interactive/BHK realizers for Conjunction

- **Rules for  $\wedge$**
- **Introduction rules:**

$$\frac{s_1 \mid - \varphi \quad s_2 \mid - \psi}{\langle s_1, s_2 \rangle \mid - \varphi \wedge \psi}$$

- **If  $s_1 \mid - \Gamma \mid - \varphi$  and  $s_2 \mid - \Gamma \mid - \psi$  then  $\langle s_1, s_2 \rangle \mid - \Gamma \mid - \varphi \wedge \psi$**

78

### Interactive/BHK realizers for Conjunction

- **Elimination rules:**

$$\frac{s \mid - \varphi \wedge \psi}{\pi_1(s) \mid - \varphi} \quad \frac{s \mid - \varphi \wedge \psi}{\pi_2(s) \mid - \psi}$$

- **If  $s \mid - \Gamma \mid - \varphi \wedge \psi$ , then  $\pi_1(s) \mid - \Gamma \mid - \varphi$  and  $\pi_2(s) \mid - \Gamma \mid - \psi$**

79

### Interactive/BHK realizers for Disjunction

- **Rules for  $\vee$ .** Let T=True, F=False, and  $\_ , \_'$  be the dummy elements of type  $\mid - \varphi \mid , \mid - \psi \mid$  (of type  $\mid - \varphi \mid , \mid - \psi \mid$  in the case of a BHK-realizer)

- **Introduction rules:**

$$\frac{r \mid - \varphi}{\langle True, r, \_ \rangle \mid - \varphi \vee \psi} \quad \frac{s \mid - \psi}{\langle False, \_ , s \rangle \mid - \varphi \vee \psi}$$

- **If  $r \mid - \Gamma \mid - \varphi$  then  $\langle True, r, \_ \rangle \mid - \Gamma \mid - \varphi \vee \psi$**
- **If  $s \mid - \Gamma \mid - \psi$  then  $\langle False, \_ , s \rangle \mid - \Gamma \mid - \varphi \vee \psi$**

80

### Interactive/BHK realizers for **Disjunction**

- **Elimination rules for  $\vee$ .** Let  $u = \text{if } (i=\text{True}) \text{ then } s(a) \text{ else } t(b)$

Then

$$\frac{\begin{array}{c} \xi || - \phi \quad \eta || - \psi \\ \dots \quad \dots \\ \langle i, a, b \rangle || - \phi \vee \psi \quad s(\xi) || - \theta \quad t(\eta) || - \theta \end{array}}{u || - \theta}$$

- If  $r || - \Gamma | - \phi \vee \psi$  and  $s(\xi) || - \Gamma, \xi: \phi | - \theta$  and  $t(\eta) || - \Gamma, \eta: \psi | - \theta$ , then  $u || - \Gamma | - \theta$

81

### Interactive/BHK realizers for **Implication**

- **Elimination rule:**

$$\frac{r || - \phi \rightarrow \psi \quad s || - \phi}{r(s) || - \psi}$$

- If  $r || - \Gamma | - \phi \rightarrow \psi$  and  $s || - \Gamma | - \phi$ , then  $r(s) || - \Gamma | - \psi$ .

83

### Interactive/BHK realizers for **Implication**

- **Rules for  $\rightarrow$ . Introduction rule:**

$$\frac{\begin{array}{c} \xi || - \phi \\ \dots \\ s(\xi) || - \psi \end{array}}{\lambda \xi. s(\xi) || - \phi \rightarrow \psi}$$

- If  $s(\xi) || - \Gamma, \xi: \phi | - \psi$ , then  $\lambda \xi. s(\xi) || - \Gamma | - \phi \rightarrow \psi$

82

### Interactive/BHK realizers for **Existential**

- **Rules for  $\exists$ : Introduction rule.**

$$\frac{\begin{array}{c} \Gamma \\ \dots \\ r || - \phi[t/x] \end{array}}{\langle t, r \rangle || - \exists x \in T. \phi}$$

- If  $r || - \Gamma | - \phi[t/x]$  for some  $t$ , then  $\langle t, r \rangle || - \Gamma | - \exists x \in T. \phi$

84

### Interactive/BHK realizers for **Existential**

- Rules for  $\exists$ : **Elimination** rule.

$$\frac{\begin{array}{c} \Gamma \qquad \Gamma, \xi \Vdash \varphi \\ \dots \\ \langle i, a \rangle \Vdash \exists x \in T. \varphi \quad t(x, \xi) \Vdash \psi \end{array}}{\dots} \quad \dots \\ \hline t(i, a) \Vdash \psi$$

- Provided  $x \notin FV(\Gamma, \psi)$ .
- If  $\langle i, a \rangle \Vdash \Gamma \Vdash \exists x \in T. \varphi$ ,  $t(x, \xi) \Vdash \Gamma, \xi \Vdash \varphi \Vdash \psi$ , and  $x \notin FV(\Gamma, \psi)$ , then  $t(i, a) \Vdash \Gamma \Vdash \psi$

85

### Interactive/BHK realizers for **Universal**

- Rules for  $\forall$ : **Introduction** rule.

$$\frac{\begin{array}{c} \Gamma \\ \dots \\ r \Vdash \varphi \end{array}}{\dots} \quad \dots \\ \hline \lambda x. r \Vdash \forall x \in T. \varphi$$

- Provided  $x \notin FV(\Gamma)$
- If  $r \Vdash \Gamma \Vdash \varphi$  and  $x \notin FV(\Gamma)$ , then  $\lambda x. r \Vdash \Gamma \Vdash \forall x \in T. \varphi$

86

### Interactive/BHK realizers for **Universal**

- Rules for  $\forall$ : **Elimination** rule.

$$\frac{\begin{array}{c} \Gamma \\ \dots \\ f \Vdash \forall x \in T. \varphi \end{array}}{\dots} \quad \dots \\ \hline f(t) \Vdash \varphi[t/x]$$

- If  $f \Vdash \Gamma \Vdash \forall x \in T. \varphi$ , then  $f(t) \Vdash \Gamma \Vdash \varphi[t/x]$  for all  $t$

87

### Interactive/BHK realizers for Induction on **Natural Numbers**

- The Induction Axiom for the type  $N$ =Natural Numbers:

$$\mathbf{Ind}: \forall x \in N. (\varphi[0/x] \rightarrow \forall x \in N. (\varphi \rightarrow \varphi[x+1/x]) \rightarrow \varphi)$$

- The realizer  $Rec$  has type:

$$N \rightarrow ||\varphi|| \rightarrow (N \rightarrow ||\varphi|| \rightarrow ||\varphi||) \rightarrow ||\varphi||$$

BHK realizers have  $|\varphi|$  in the place of  $||\varphi||$ .

- Let  $n: N$ ,  $r \Vdash \varphi[0/x]$  and  $s \Vdash \forall x \in N. (\varphi \rightarrow \varphi[x+1/x])$ .
- We define  $Rec(n, r, s) \Vdash \varphi[n/x]$  by primitive recursion:
  - $Rec(0, r, s) = r$
  - $Rec(n+1, r, s) = s(n, Rec(n, r, s))$

88

## Interactive/BHK realizers for Induction for Induction on Lists

- **Induction Axiom for the type  $L=Lists$  is:**

$$\mathbf{Ind}_L: \forall l \in L. (\varphi[\text{nil}/l] \rightarrow$$

$$\forall l \in L, x \in N. (\varphi \rightarrow \varphi[\text{cons}(x,l)/l]) \rightarrow \varphi)$$

- **We abbreviate  $A \rightarrow B \rightarrow \dots \rightarrow C$  by  $A, B, \dots \rightarrow C$ .**

- **The realizer  $\text{Rec}_L$  has type:**

$$L, ||\varphi||, (L, N, ||\varphi|| \rightarrow ||\varphi||) \rightarrow ||\varphi||$$

BHK realizers have  $|\varphi|$  in the place of  $||\varphi||$ .

- Let  $m: L, r ||\varphi|| - \varphi[\text{nil}/l], s ||\varphi|| - \forall l \in L, x \in N. (\varphi \rightarrow \varphi[\text{cons}(x,l)/l])$

- We define  $\text{Rec}_L(m, r, s) ||\varphi|| - \varphi[m/l]$  by primitive recursion:

1.  $\text{Rec}_L(\text{nil}, r, s) = r$
2.  $\text{Rec}_L(\text{cons}(n, l), r, s) = s(l, n, \text{Rec}_L(l, r, s))$  89

## The Interactive realizer of $\text{EM}_1$

- An interactive realizer  $E_p ||\varphi|| - \forall x. (\exists y. P(x, y) \vee \forall y. P^\perp(x, y))$  of an instance of  $\text{EM}_1$  may be defined as in § 3, by

$$E[s](x) = \langle \chi_p(s, x), \langle \phi_p(s, x), \emptyset \rangle, \lambda y. \text{Add}_p(s, x, y) \rangle ||\varphi|| - \forall x. (\exists y. P(x, y) \vee \forall y. P^\perp(x, y))$$

There is no BHK realizer for  $\text{EM}_1$ , as we said.

## Interactive/BHK realizers for **Well-founded Induction**

- Assume  $R$  is an atomic arithmetical formula defining some well-founded relation (i.e., there is no infinite  $R$ -chain). **The Well-founded Induction Axiom for  $R$  is:**

$$\mathbf{WInd}: (\forall y \in N. (\forall z \in N. R(y, z) \rightarrow \varphi[z/x]) \rightarrow \varphi[y/x]) \rightarrow \forall x \in N. \varphi$$

- **The realizer  $W$  has type (with  $|\varphi|$  or  $||\varphi||$ ):**

$$(N \rightarrow (N \rightarrow P_{\text{fin}}(\text{Atom}) \rightarrow ||\varphi||) \rightarrow ||\varphi||) \rightarrow N \rightarrow ||\varphi||$$

- Let  $r ||\varphi|| - \forall y \in N. (\forall z \in N. R(z, y) \rightarrow \varphi[z/x]) \rightarrow \varphi[y/x]$  and  $n: N$ .

- We define  $W(r, n) ||\varphi|| - \varphi[n/x]$  by fixed point:

$$W(r, n) = r(n, \lambda m: N. \lambda s: P_{\text{fin}}(\text{Atom}). W(r, m)) : ||\varphi||$$

The realizer belongs to  **$T_S$  + fixed point operators**. Terms of this system are convergent if we reduce only closed terms which are not in the minor branch of an "if". <sup>90</sup>

## Talk given at Savoie Technolac

