A possible reformulation of polymorphism: the simply typed lambda calculus $\mathcal{N}$

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A Possible Reformulation of Polymorphism

1. This talk is about representing the constructive content of **second order deduction** (as expressed in system $\mathcal{F}$) through **operations on tree structures**.

2. We investigate the possibility of representing polymorphic maps on data types definable in system $\mathcal{F}$ in a simply typed λ-calculus, system $\mathcal{N}$.

3. Reduction of $\mathcal{N}$ are:
   - **algebraic** reductions
   - reductions for **primitive recursion** on trees
   - reductions for **uniform recursion**

   Uniform recursion represents polymorphism through **uniform application**, a concrete operation on trees.

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The result we already have about system $\mathcal{N}$ are:

1. **normalization** (with an intuitionistic proof)
2. all trees denoted by some term t in some data type D of system $\mathcal{N}$ are **well-founded**.

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1. system $\mathcal{N}$ defines a **realization model** of second order arithmetic
2. **equivalence between system $\mathcal{N}$ and polymorphism**: for any data types D, E, system $\mathcal{N}$ and system $\mathcal{F}$ define the same maps $f : D \to E$

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Plan of the Talk

- §1. Data types and Types of system $\mathcal{N}$
- §2. Primitive Recursion on Trees
- §3. Recursion in system $\mathcal{N}$: Uniform Recursion
- §4. Definitions and Results for System $\mathcal{N}$
- §5. Conclusions

§1. Data types and Types of system $\mathcal{N}$

We define nested data types as in Martin Lof ([1]).

1. The set **Data** of data types for trees in $\mathcal{N}$ is inductively defined by $D ::= (D_0, \ldots, D_{n-1})$.
2. D denotes the set of well-founded trees whose node all have index set for children the set D_i for some $i < n$.
3. The set **Tp** of types of $\mathcal{N}$ is inductively defined by $T ::= D | T \times T | T \to T$ for any data type $D \in$ **Data**.

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Tree constructors

1. Tree constructors for $D = (D_1, \ldots, D_n)$ are constants $\mathbf{c_i} : (D_i \to D) \to D$, one for each $i = 1, \ldots, n$.
2. We call D_i the index set of $\mathbf{c_i}$.
3. $\mathbf{c_i}$ takes in argument a family of trees in D indexed on D_i, and represented by a term $f : D_i \to D$.
4. $\mathbf{c_i}$ returns the tree $\mathbf{c_i}(f) : D$, whose root has:
   - label $\mathbf{c_i}$
   - children indexed on D_i (for each $e : D_i$, one child $f(e) : D$).
5. The index set D_i of a node (and therefore the arity of the node) may be empty, a singleton, a finite set of any cardinality, or may be infinite.

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Some Examples of Data Types

All finite sets and **Nat** may be represented in **Data** as follows.

1. **Empty set**. Assume $D = () \in \mathbf{Data}$. Then D has no constructors and denotes the empty set $\emptyset$.
2. **Finite sets**. Assume $D = (\emptyset, \ldots, \emptyset) \in \mathbf{Data}$ has n constructors $\mathbf{c_0}, \ldots, \mathbf{c_{n-1}} : (\emptyset \to D) \to D$ with index set empty. (n constant constructors).
3. Given a dummy map $\mathbf{dummy} : \emptyset \to D$, if we abbreviate $i \equiv \mathbf{c_i}(\mathbf{dummy})$ for all $i < n$, then $D = \{0, \ldots, n-1\}$.
4. **Natural Numbers**. Assume $D = (\emptyset, \{0\}) \in \mathbf{Data}$. Then D has one constructor $\mathbf{c_0}$ with empty index set (a constant constructor) and one constructor $\mathbf{c_1}$ with singleton index set (a unary constructor).
5. If we abbreviate $0 \equiv \mathbf{c_0}(\mathbf{dummy})$ and $x + 1 = \mathbf{c_1}(f)$ with $f(0) = x$, we may prove that D is isomorphic to **Nat**.

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More examples of Data Types

1. **Binary trees**. Assume $D = (\emptyset, \{0,1\}) \in$ **Data**. D has 2 constructors $\mathbf{c_0}, \mathbf{c_2}$ of index sets $\emptyset$ and $\{0,1\}$ (arity 0 and 2). We claim that D is isomorphic to the set of finite binary trees.
2. Indeed, if $f : \emptyset \to D$ and $g : \{0,1\} \to D$ with $g(0) = t_0, g(1) = t_1$, then we may set $\mathbf{leaf} = \mathbf{c_0}(f)$ and $\mathbf{mktree}(t_1, t_2) = \mathbf{c_2}(g)$.
3. **Well-founded trees**. Assume $D = (\emptyset, \mathbf{Nat}) \in$ **Data**. We claim that D denotes the set Ω of well-founded trees whose node are either leaves or have ω children. D has 2 constructors $\mathbf{c_0}, \mathbf{c_2}$ of index sets $\emptyset$ and **Nat** (arity 0 and ω).
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The general pattern for a Tree

Assume we have a data type $D = (D', D'', D''')$ and

1. constructors c', c'', c''' of index sets D', D'', D'''
2. index maps f, g, h of type $D' \to D, D'' \to D, D''' \to D$
3. indexes $y, z : D', \quad t, u : D'', \quad v, w : D'''$

Assume that $f(y)$, $f(y)$ have values $c''(g), c'''(h)$. Then $x = c(f) : D$ denotes the tree:

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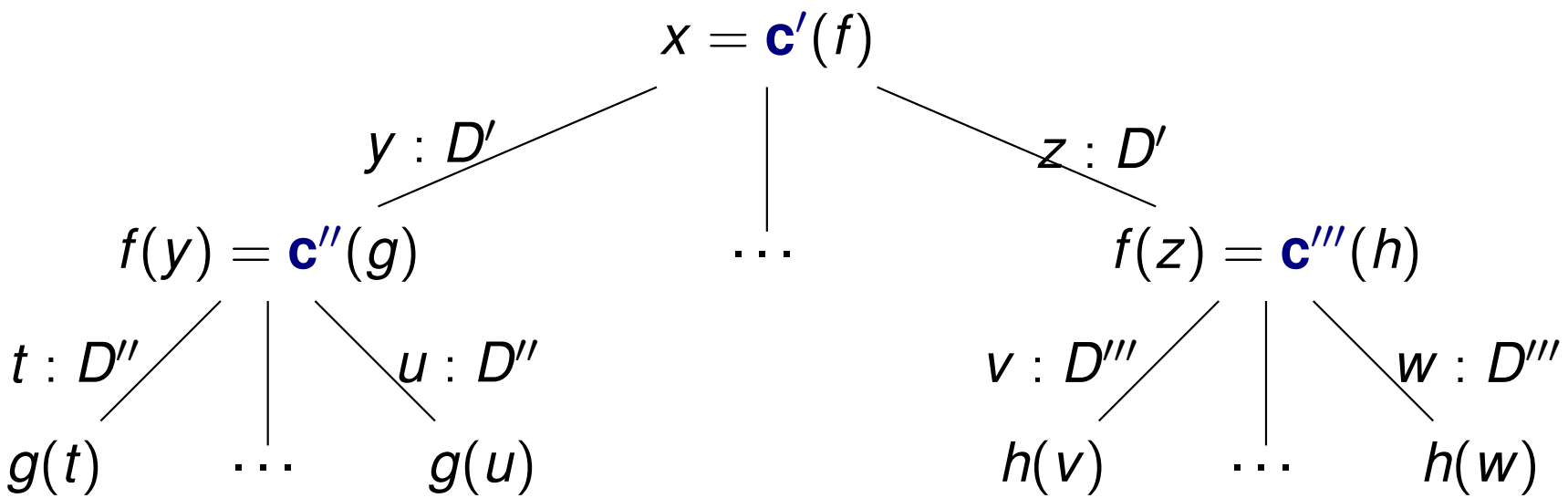

Road Map

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- **§2. Primitive Recursion on Trees**
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§2. Primitive Recursion on Trees

Let $D \in \mathbf{Data}$, $A \in \mathbf{Tp}$. We recall primitive recursive definitions for maps $h : D \to A$, then we explain how to express them in **Data**.

1. Assume $D = \emptyset$. Then recursion defines a single map **dummy** : $\emptyset \to A$.
2. Assume $D = \{0, \ldots, n-1\}$. Given $r_1, \ldots, r_n : A$ we may define $h(i) = r_i$ for all $i < n$.
3. Assume $D = \mathbf{Nat}$. Given $r_0 : A$ and $r_1 : A \to A$ we may define $h(0) = r_0$ and $h(x+1) = r_1(h(x))$ for all $x \in \mathbf{Nat}$.
4. Assume $D =$ the set of binary trees, with constructors $\mathbf{c_0}, \mathbf{c_2}(.,.)$. Given $r_0 : A$ and $r_2 : A, A \to A$ we may define $h(\mathbf{c_0}) = r_0$ and $h(\mathbf{c_2}(t_1, t_2)) = r_2(h(t_1), h(t_2))$.

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How to express Primitive Recursion in Data

All primitive recursions for some $D \in$ **Data** are instances of a single notion of recursion.

1. The general patter for primitive recursion on D is: assume that D has a constructor $\mathbf{c_i}$ with argument list $d_1, \ldots, d_n, \ldots$. We first apply h to each $d_1, \ldots, d_n, \ldots$, obtaining $h(d_1), \ldots, h(d_n), \ldots : A$, then we define

$$\mathbf{h}(\mathbf{c_i}(\mathbf{d_1}, \ldots, \mathbf{d_n}, \ldots)) = \mathbf{r_i}(\mathbf{h}(\mathbf{d_1}), \ldots, \mathbf{h}(\mathbf{d_n}), \ldots)$$

2. In any $D \in$ **Data**, the list $d_1, \ldots, d_n, \ldots$ of arguments of $\mathbf{c_i}$ is replaced by an index map $f : D_i \to D$ such that $f(e_1) = d_1, \ldots, f(e_n) = d_n, \ldots$ for some $e_1, \ldots, e_n, \ldots : D_i$.
3. Thus, instead of forming $h(d_1), \ldots, h(d_n), \ldots : A$, we form $h \circ f : D_i \to A$, an index map for $h(d_1) = h(f(e_1)), \ldots, h(d_n) = h(f(e_n)), \ldots : A$. Then we define

$$\mathbf{h}(\mathbf{c_i}(\mathbf{f})) = \mathbf{r_i}(\mathbf{h} \circ \mathbf{f})$$

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$$\mathbf{h(c_i(f)) = r_i(h \circ f)}$$

A General Pattern for Primitive Recursion

Summing up:

1. Primitive recursion on $D \in \mathbf{Data}$ defines a map $h : D \to A$ using one clause r_i for each D_i.
2. The clause r_i has type $(D_i \to A) \to A$ and it is used on trees $t : D$ of the form $t \equiv \mathbf{c_i}(f)$, for some $f : D_i \to D$.
3. We assume that h is already defined on $f(e)$ for all $e : D_i$, we apply h to the result of f forming $h \circ f : D_i \to A$, then we apply r_i to the result, obtaining $h(t) = r_i(h \circ f)$.

System $\mathcal{N}$ has a stronger version of recursion which we call **uniform recursion**. Uniform recursion includes all clauses of primitive recursion, plus one extra recursive clause, used when we want to extend h to a larger domain.

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§3. Recursion in system $\mathcal{N}$: Uniform Recursion

In order to define uniform recursion (which is recursion for system $\mathcal{N}$), our **first step** is to introduce an operation extending a data type.

1. We define an operation $(.)@E$ adding one index set $E \in$ **Data** to a data structure $D = (D_0, \ldots, D_{n-1})$: $D@E = (D_0, \ldots, D_{n-1}, E) \in$ **Data**.
2. $D@E$ has one tree constructor $\mathbf{c_n} : (E \to D@E) \to D@E$ more than D.
3. $(.)@D$ is extended pointwise and componentwise to all types in **Tp** by:
   - $(A \times B)@E \equiv A@E \times B@E \in$ **Tp**
   - $(A \to B)@E \equiv A \to B@E \in$ **Tp**.
4. We call the type $A@E$ an *extension* of the type A.

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Future constructors

In order to define uniform recursion, our **second step** is to introduce constants $\mathbf{future}_{m,E,D} : (E \to D) \to D$ we call *future constructors*. E is a **term constant** but acts as a type variable.

1. The future constructor **future** represents (an approximation of) the new constructor $\mathbf{c_n} : (E \to D@E) \to D@E$ we will add in last position to $D = (D_0, \ldots, D_{n-1})$ when forming $D@E = (D_0, \ldots, D_{n-1}, E)$.
2. We call E a pending extension.
3. We add a unary **term operator** $\mathbf{Forth}_{m,E}$, executing a pending extension, and acting like a type substitution.
4. $\mathbf{Forth}_{m,E}$ replaces $\mathbf{future}_{m,E,D}$ with $\mathbf{c_n}$:

$$\mathbf{Forth}_{m,E}(\mathbf{future}_{m,E,D}(f)) = \mathbf{c_n}(\mathbf{Forth}_{m,E}(f))$$

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Future trees and Uniform application

In order to define uniform recursion, our **third step** is to introduce is to introduce an operation we call uniform application.

1. If $f : E \to D$ denotes a family of trees of D indexed on $e : E$, we call **future**(f) a future tree.
2. **future**(f) represents a tree whose root has an "unknown" label **future**, an "unknown" index set E, and children all $f(e)$ for any $e : E$.
3. We allow a unique way of acting on the tree **future**(f): to apply the same map g on all children $f(e)$ of **future**(f).
4. To this aim, we introduce a constant **u** we call "uniform application", defined by **u**(**future**$(f), g) =$ **future**$(g \circ f)$.
5. We extend recursive definition to future trees using *uniform application*.

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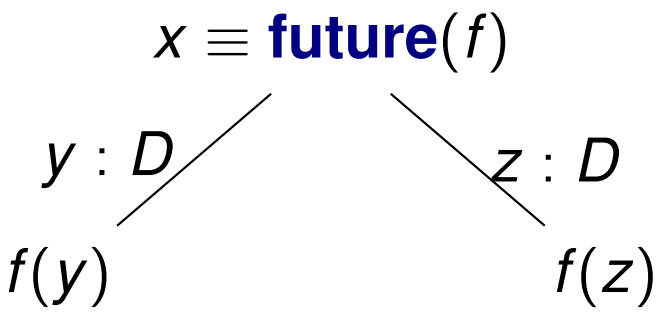
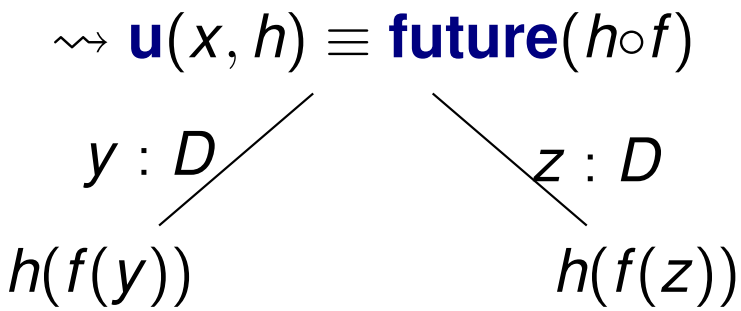
2. **future**(f) represents a tree whose root has an "unknown" label **future**, an "unknown" index set E, and children all $f(e)$ for any $e : E$.

3. We allow a unique way of acting on the tree **future**(f): to apply the same map g on all children $f(e)$ of **future**(f).
4. To this aim, we introduce a constant **u** we call "uniform application", defined by **u**(**future**$(f), g)$ = **future**$(g \circ f)$.
5. We extend recursive definition to future trees using *uniform application*.

Uniform application to a future tree

Uniform application applies a map $h : D \to D$ to all children of a future tree **future**(f).





Definition of Uniform Recursion

We have now all ingredients we need to define uniform recursion.

1. Uniform recursion on D in system $\mathcal{N}$ defines a map $h : D \to A$ using one clause r_i for each D_i, and a single extra clause $r_n : A \to A$, dealing with all pending extensions $A@E$ of A through a uniform application.

2. If $\mathbf{t} \equiv \mathbf{c_i(f)}$, as usual we set
$$\mathbf{h(t) = r_i(h \circ f)}$$

3. The clause r_n is used on trees $\mathbf{t \equiv future(f) : D}$. We assume that $h : D \to A$ is already defined on $f(e) : D$ for all $e : D_i$, we *uniformly* apply h to the tree t forming $\mathbf{future}(h \circ f) : A$, then we apply r_n obtaining
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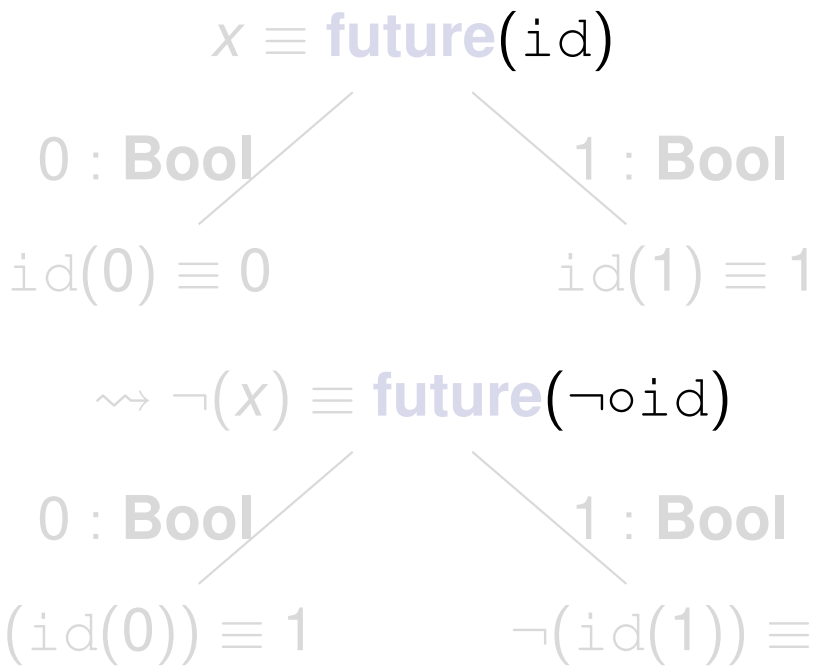
An example: one uniform extension of negation

1. Let **Bool** $= \{0, 1\}$. **Bool** with future constructors is a set of trees whose leaves are booleans.
2. Let $\neg(0) = 1$, $\neg(1) = 0$. We uniformly extend $\neg$ to any future constructor of **Bool** with the clause $\neg(\textbf{future}(f)) = \textbf{future}(\neg \circ f)$.
3. The result is a map negating all leaves of a tree.



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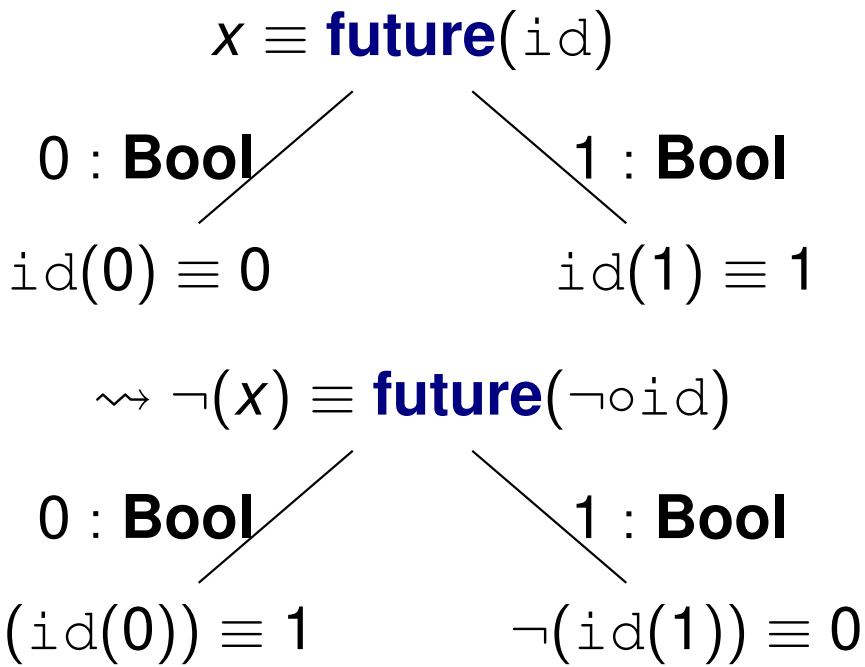

Road Map

- §1. *Data types and Types of system $\mathcal{N}$*
- §2. *Primitive Recursion on Trees*
- §3. *Recursion in system $\mathcal{N}$: Uniform Recursion*
- **§4. Definitions and Results for System $\mathcal{N}$**
- §5. *Conclusions*

§4. Definitions and Results for System $\mathcal{N}$

We mark the new parts in **red**.

Definition (Terms of $\mathcal{N}$)

Let $n, m \in$ **Nat**, $i < n, j < m$, $D = (D_0, \ldots, D_{n-1})$, $E \in$ **Data**, $A, B \in$ **Tp**, and $\Gamma = E_0, \ldots, E_{m-1}$ any context :

1. $\Gamma \vdash C : A$: if $C(\vec{x}) = \alpha[\vec{x}]$ is a combinator of type A
2. $\Gamma \vdash <\_,\_> : A, B \to A_1 \times A_2$ and $\Gamma \vdash \pi_i : A_1 \times A_2 \to A_i$
3. (constructors) $\Gamma \vdash \mathbf{cons}_{i,D} : (D_i \to D) \to D$
4. If $\Gamma \vdash t : A \to B$ and $\Gamma \vdash u : A$, then $\Gamma \vdash t(u) : B$.
5. **(future constructors)** $\Gamma \vdash \mathbf{future}_{j,E_j,A} : (E_j \to A) \to A$
6. **(uniform application)** $\Gamma \vdash \mathbf{u}_D : D, (D \to D) \to D$
7. **(recursion)** $\Gamma \vdash \mathbf{r}_{D,A} : \vec{R}, D \to A$, with $R_i = (D_i \to A) \to A$ for all $i < n$, and $R_n = (A \to A)$
8. **(Forth)** If $\Gamma, E, \Delta \vdash t : A$, then $\Gamma, \Delta \vdash \mathbf{Forth}_{m,E}.t : A@E$

Lifting application

We explained how to apply a map $h : D \to A$ to a larger domain $D@E$: we call this operation "Lifting". We may internalize lifting in $\mathcal{N}$ through a a term **Lift** : $(D \to A) \to (D@E \to A@E)$

1. We first define a recursive map **Back** : $D@E \to D$, turning the last constructor **c** : $(E \to D@E) \to D@E$ of $D@E$ into a future constructor $\textbf{future}_{m,E} : (E \to D) \to D$ of D
2. Then we apply h to the result **Back**(t), obtaining some $h(\textbf{Back}(t)) : A$
3. Eventually we use the **Forth** operator, replacing **future** with c: the result has type $A@E$.
4. We define **Lift** $\equiv$ **Forth**$(C(\textbf{Back}))$ for $C(x, y, z) = y(x(z))$

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The role of Lifting in system $\mathcal{N}$

1. The only limitation we have for lifting an application from D to $D@E$ is that h may act on any tree $c(f)$ defined by the constructor $c : (E \to D@E) \to D@E$ only using *uniform application*.
2. h cannot select a child $f(e)$ of $c(f)$ for some $e : E$, because **we assume that h does not know the type E**.
3. We believe that this limitation is essential in order insure termination and to match exactly the expressive power of polymorphism of system $\mathcal{F}$.

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The normalization and the well-foundedness results

1. The main feature of $\mathcal{N}$ is that we combine the fact that the domain of a map is extendable with the fact that all maps are total.

2. Indeed, using the notion of approximation *(see Appendix)* and Tait's notion of reducibility we may intuitionistically prove:

Theorem (Normalization and Well-foundedness)

1. *All terms of $\mathcal{N}$ normalize*
2. *all trees denoted by some term $t : D \in$ **Data** of system $\mathcal{N}$ are **well-founded**.*

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§5: Conclusions

1. **We defined a simply typed λ-calculus $\mathcal{N}$** in which primitive recursive definitions on trees may be extended to a larger domain using uniform application
2. System $\mathcal{N}$ is defined in term of **concrete tree operations** and aims to be **equivalent to polymorphism**.
3. **What we proved**: System $\mathcal{N}$ has the usual properties of Subject Reduction, Confluence and Normalization
4. **What we are checking**: whether system $\mathcal{N}$ defines a Realization Model for Second Order Arithmetic, and whether the definable maps on data types are the same in system $\mathcal{N}$ and system $\mathcal{F}$.

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References

1. P. Martin-Lof, Intuitionistic Type Theory, June 1980, Bibliopolis.
2. H. Barendregt, Lambda Calculus with Types. Cambridge University Press, 2013.
3. William W. Tait: Intensional Interpretations of Functionals of Finite Type I. J. Symb. Log. 32(2): 198-212 (1967)

A Century of Constructive Reasoning . . .

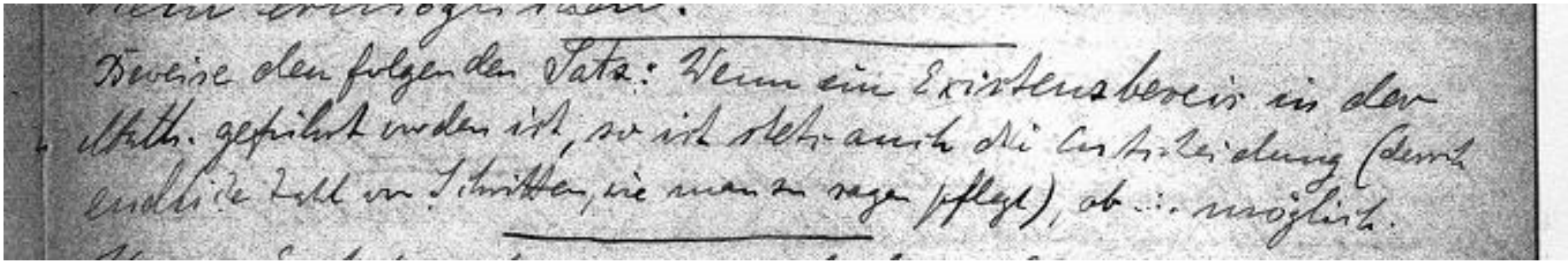

Figure : **Hilbert Constructivization Conjecture** *(Courtesy from Goettingen State and University Library, Germany. Thanks to Benedikt Ahrens for translating)*.

Probably the first version (around 1917) of the following conjecture by Hilbert:

”Prove the following theorem: When a proof of existence has been concluded in mathematics, then also the decision (in a finite number of steps, as one says) is always possible. ”

Appendix: the system $\mathcal{N}$

1. Terms of $\mathcal{N}$ of type A are defined w.r.t. a list $\Gamma \equiv E_0, \ldots, E_{m-1}$ of data structures, denoting *pending extensions* of A.
2. Terms of $\mathcal{N}$ include all algebraic combinators, pairing, projections, tree constructors, future constructors **future** : $(E \to A) \to A$, uniform application **u** : $E, (E \to E) \to E$ and for each $D = (D_0, \ldots, D_{n-1})$ a constant $\mathbf{r} \equiv \mathbf{r}_{D,A}$ denoting recursion on trees of D with result in A.
3. **r** has one recursive clause $r_i : (D_i \to A) \to A$ for each index set D_i, and one extra clause $r_n : A \to A$, dealing with extensions of A.
4. Terms are closed under the unary operator $\mathbf{Forth}_{m,E}$, which removes the type E in position m from a context, and under application.
5. We write $\Gamma \vdash t : A$ for "$t : A$ in the context Γ".

Definition (Algebraic Reductions for $\mathcal{N}$)

1. Let $C(\vec{x}) = \alpha[\vec{t}]$ be any combinator.
   1. $C(\vec{t}) \rightsquigarrow \alpha[\vec{t}/\vec{x}]$.
   2. $\pi_i(< a_1, a_2 >) \rightsquigarrow a_i$ for $i = 1, 2$
   3. If $a \rightsquigarrow b$ then $\pi_i(a) \rightsquigarrow \pi_i(b)$.
   4. If $f \rightsquigarrow g$ then $fa \rightsquigarrow ga$
2. Let $c \equiv \mathbf{future}_{i,E}$.
   1. $c(f)(e) \rightsquigarrow c(Pe \circ f)$
   2. $\pi_i(c(f)) \rightsquigarrow c(\pi_i \circ f)$ for $i = 1, 2$

Definition (Uniform Reductions for $\mathcal{N}$)

1. Let $c \equiv \textbf{future}_{i,E,D}, \textbf{cons}_{i,D}$ and $g : D \to D$ and $c(f) : D$.
   1. $\mathbf{u}(c(f))(g) \rightsquigarrow c(g \circ f) : B$
   2. If $d \rightsquigarrow e : D$ then $\mathbf{u}(d)(g) \rightsquigarrow \mathbf{u}(e)(g)$
2. Assume $D = (D_0, \ldots, D_{n-1})$, $\vec{r} = r_0, \ldots, r_n$.
   1. If $d \equiv c(f)$ and $c \equiv \textbf{cons}_{i,D_i}$ for some $i < n$ then $\mathbf{r}\vec{r}d \rightsquigarrow r_i(\mathbf{r}(\vec{r}) \circ f)$
   2. If $d \equiv c(f)$ and $c \equiv \textbf{future}$ then $\mathbf{r}\vec{r}d \rightsquigarrow r_n(c(\mathbf{r}(\vec{r}) \circ f))$
   3. If $d \rightsquigarrow e$ then $\mathbf{r}\vec{r}d \rightsquigarrow \mathbf{r}\vec{r}e$

If $\Gamma, \Delta \vdash a : A$ then we denote with $a^{i,E}$ the term corresponding of a in the context Γ, E, Δ (*defined in the next slide*).

Definition (Reductions for Forth)

Assume $D = (D_0, \ldots, D_{n-1})$.

1. $\mathbf{Forth}(\mathbf{future}_{i,E,D} f) \rightsquigarrow \mathbf{c_{n,D@E}}(\mathbf{Forth}(f))$
2. $\mathbf{Forth}(\mathbf{future}_{j+1,E,D} f) \rightsquigarrow \mathbf{future}_{j,E,D@E}(\mathbf{Forth}(f))$ for $j \geq i$
3. $\mathbf{Forth}(cf) \rightsquigarrow c(\mathbf{Forth}(f))$ for any other constructor or future constructor.
4. If $d : D \in \mathbf{Data}$ and $d \rightsquigarrow e : D$ then $\mathbf{Forth} d \rightsquigarrow \mathbf{Forth} e$.
5. $\mathbf{Forth}(f)(a) \rightsquigarrow \mathbf{Forth}(f(a^{i,E}))$
6. $\pi_i(\mathbf{Forth}(a)) \rightsquigarrow \mathbf{Forth}(\pi_i(a))$ for $i = 1, 2$.

Context Lifting

Context Lifting is an operation moving a term t in the context Γ, Δ to the corresponding term $t^{i,E}$ in the context Γ, E, Δ. Context lifting adds 1 to the subscripts of future constructors with index in Δ.

Definition (The term $t^{i,E}$)

Assume $\Gamma \vdash t : A$ is a term of $\mathcal{N}$, c is any constant. We define $t^{i,E}$ by induction on t.

1. $(\mathbf{future}_{j,F})^{i,E} \equiv \mathbf{future}_{j+1,F}$ for all $j \geq i$
2. $c^{i,E} \equiv c$ in all other cases.
3. $\mathbf{Forth}_{j,F}(u)^{i,E} \equiv \mathbf{Forth}_{j+1,F}(u^{i,E})$ for all $j \geq i$.
4. $\mathbf{Forth}_{j,F}(u)^{i,E} \equiv \mathbf{Forth}_{j,F}(u^{i+1,E})$ in all other cases.
5. $t(u)^{i,E} \equiv t^{i,E}(u^{i,E})$

Appendix: Models of system $\mathcal{N}$ and a vicious cycle

In order to define a model of system $\mathcal{N}$ we face the following vicious cycle.

1. The definition of a term $E \vdash t : D$ may include a constructor c of index set D.
2. c has type : $(E \to D) \to D$, and has domain all maps $E \to D$.
3. If $E = D$, defining these maps requires to define D **before completing the definition** of any $t : D$.

In order to break this vicious cycle, we have to exploit the fact that a map acts on a tree $c(f)$ by uniform application only, without knowing the type E.

Approximated constructors and a vicious cycle

Let $E \in$ **Data**, and $\mathcal{A}$ be any model of $\mathcal{N}$, and $E_{\mathcal{A}}$ the set of elements of empty context and type E in $\mathcal{A}$ and $X \subseteq E_{\mathcal{A}}$.

1. In the models of $\mathcal{N}$ we add constants $\mathtt{j}_X : (X \to D) \to D$ we call *approximated constructors*.
2. We use $\mathtt{j}_X$ to represent a constructor $c : (E \to D) \to D$.
3. If the constant $\mathtt{j}_X$ is in $\mathcal{A}$, then $X \subset E_{\mathcal{A}}$.
4. Indeed, the terms of empty context and type E defined from $\mathtt{j}_X$ are in $E_{\mathcal{A}}$ and not in X.
5. In every single model $\mathcal{A}$, we found for approximated constructors a vicious cycle similar to the vicious cycle for constructor.
6. This second vicious cycle, however, is easier to break.

Appendix: Breaking the vicious cycle

1. For any model $\mathcal{A}$ there is a model $\mathcal{B} \supset \mathcal{A}$ including the approximated constructor $\mathtt{j}_{E_{\mathcal{A}}}$.
2. Therefore the behavior of the family of terms $c_{\mathcal{A}}$ in context $\Gamma = E$ for all $\mathcal{A}$ may be described from the behavior of the family of terms $\mathtt{j}_{E_{\mathcal{A}}}$ in the empty context for all $\mathcal{A}$.
3. By exploting this idea we may adapt Tait's notion of reducibility ([3]) to system $\mathcal{N}$.
4. We express Tait's reducibility w.r.t. all models of $\mathcal{N}$ and not just w.r.t. a single model of $\mathcal{N}$.