Taming Modal Impredicativity: Superlazy Reduction

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Outline

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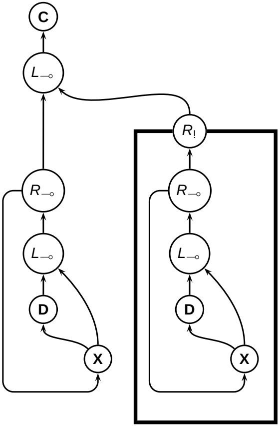
Implicit Computational Complexity.

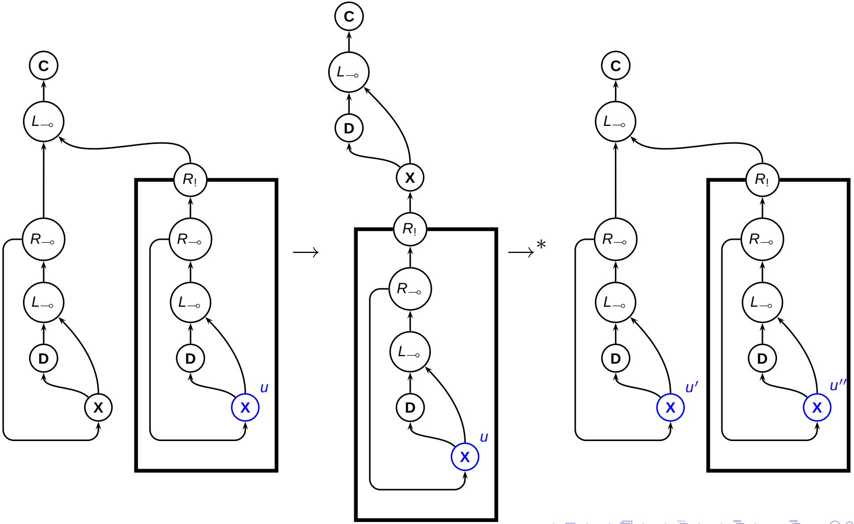
- **ICC** studies Computational Complexity in a more abstract, machine-independent perspective
- Our point of view: *Proofs (of **LL**) as Programs*
- Execution of programs is reduction of **LL** proofs (and proofnets)
- One of the targets of **ICC** is the identification of sub-logics of **LL** characterizing Complexity Classes of programs: **P**, **PSPACE**,. . .
- In this talk we will see that **LL** endowed with a certain reduction strategy will characterize Primitive Recursive Functions.

Formalism: Type-free proofnets



This corresponds to the λ-term $\Delta\Delta$.

A Motivating Example: ΔΔ.



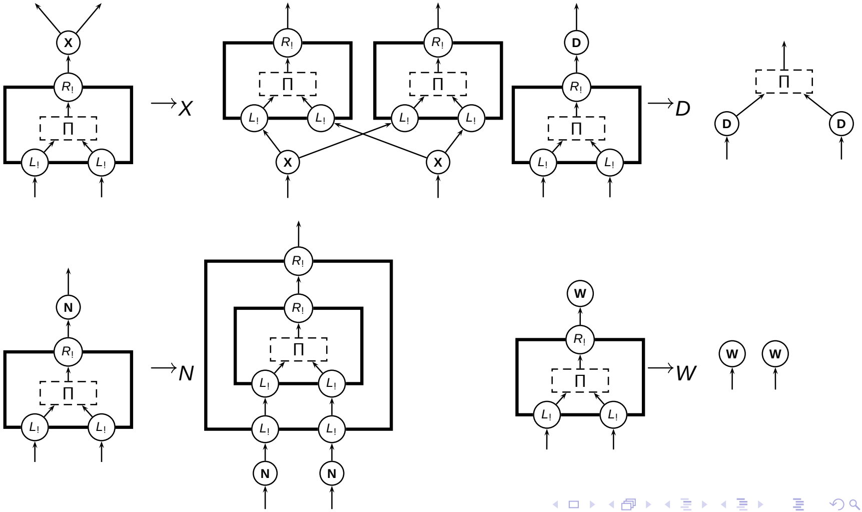
Modal Impredicativity as Source of Complexity.

- Modal impredicativity occurs when *a copy of* a node u inside a box B is allowed to interact with a *copy of* B. For example, in $\Delta\Delta$.
- We have identified modal impredicativity as one of the sources of complexity for **LL** proofs.

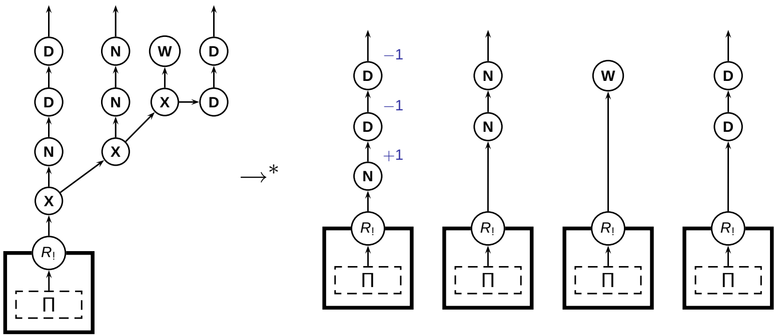
Outline

Usual Reduction Steps **LL**.

Some *modal* rewriting rules:



A generic **LL** reduction



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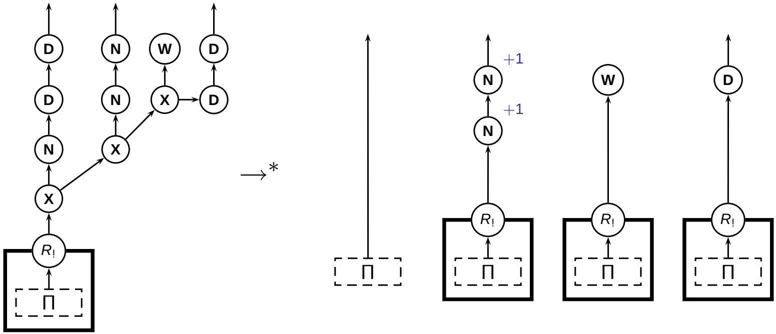

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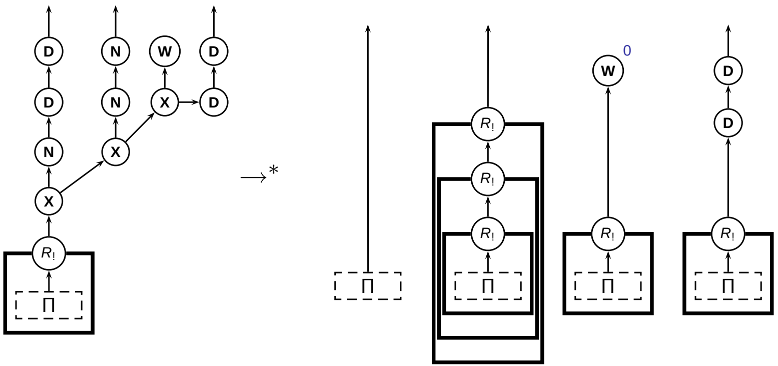
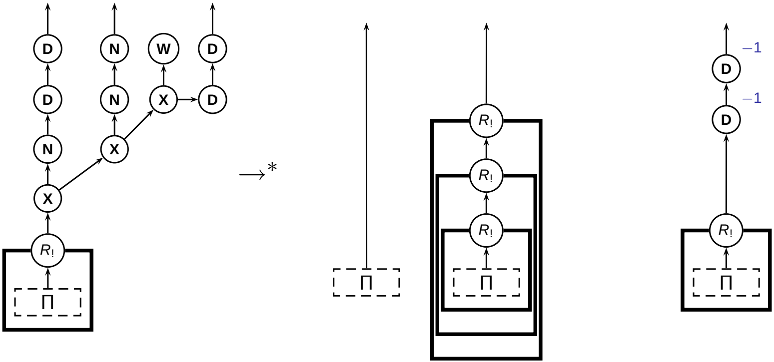
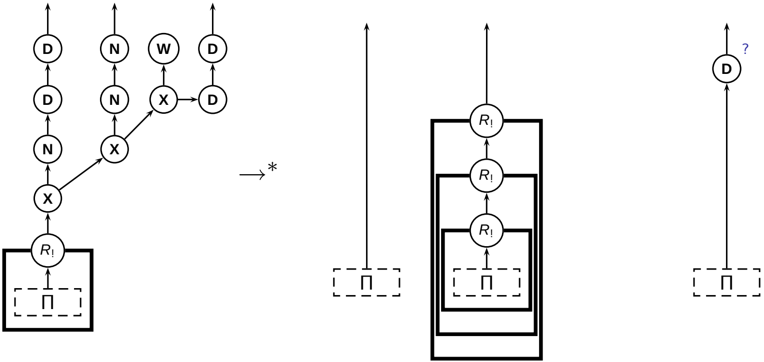


A generic **LL** reduction



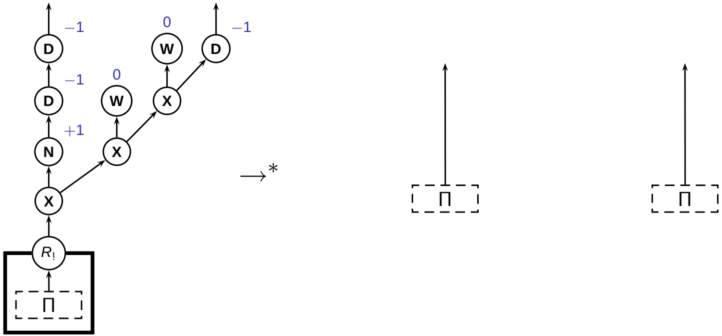
A generic **LL** reduction



Modal impredicativity can occur

Derelicting Trees.

A tree of **X,D,N,W** nodes is a derelicting tree if it opens or erases all the copies of Π that it creates:



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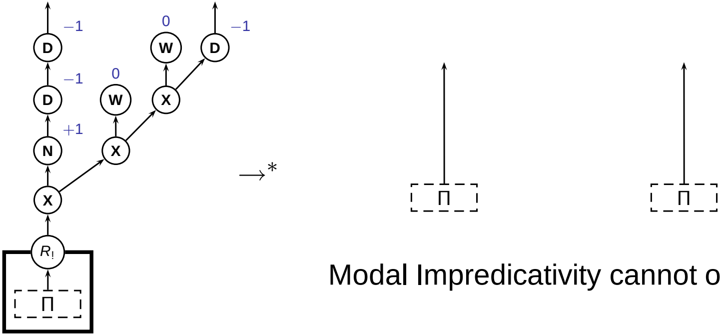
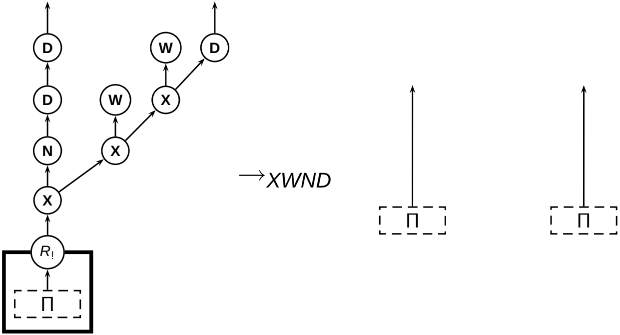


Modal Impredicativity cannot occur

The Superlazy Reduction Strategy.

1. The modal rewriting rules are applied only in presence of a derelicting tree. We consider this rewriting as a single step:



2. Only reductions at depth 0 can be performed.

Primitive Recursion Completeness.

Lemma

*A Church numeral n can be represented as a type-free proofnet $\overline{n}$ of **LL**. $\langle \overline{n}_1, \ldots, \overline{n}_k \rangle$ can be represented too.*

Theorem (Primitive Recursion Completness)

*Let $f(x_1, \ldots, x_k)$ be a Primitive Recursive function. There exists a type-free proofnet G_f of **LL** with 1 premise and 1 conclusion, such that whenever it is plugged to $\langle \overline{n}_1, \ldots, \overline{n}_k \rangle$, it superlazy reduces to $\overline{f(n_1, \ldots, n_k)}$.*

Keypoints: we are able to freely duplicate arguments, and to iterate the application of a function.

Primitive Recursion Soundness.

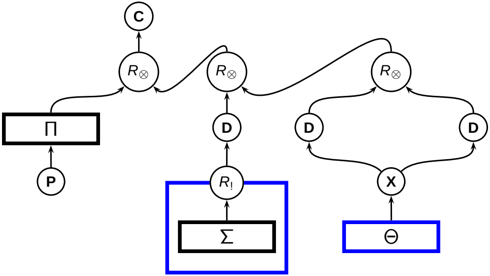
Theorem (Primitive Recursion Soundness)

There exist a family of Primitive Recursive functions $\{f_d(x) \mid d \in \mathbb{N}\}$ such that: if G is a type-free proofnet of ***LL****, of depth d and size $|G|$, and $G \to^k H$, then $f_d(|G|)$ is a bound for both k and $|H|$.*

Primitive Recursion Soundness.

An Idea of the Proof.

- Let's consider a generic reduction $G \rightarrow^* H$.
- Let $b_0, \ldots, b_N$ be the boxes at depth 0 in G that will be opened
- Let δ be the greatest depth of the nodes inside $b_0, \ldots, b_N$.



Primitive Recursion Soundness.

An Idea of the Proof.

- We want to build $h_D(N,\delta)$ as bound for time and size
- The reduction $G \rightarrow^* H$ can be split:

$$\underbrace{G \rightarrow^*}_{b_0,\ldots,b_{N-1}} \underbrace{F \rightarrow_{XWND} J}_{b_N} \rightarrow^* H$$

- Double induction, on δ and on N

$$\begin{array}{rcl} |F| & \leq & h_D(N-1,|G|) \\ |J| & \leq & 2|G|\cdot|F|+|F| \\ |H| & \leq & h_{D-1}(|J|,|J|) \\ \text{time} & \leq & h_D(N-1,|G|)+1+h_{D-1}(|J|,|J|) \end{array}$$

$$h_D(N,|G|) = h_D(N-1,|G|)+1+h_{D-1}(|J|,|J|)$$

Primitive Recursion Soundness.

An Idea of the Proof.

- So, $h_D(N, \delta)$ is a PR bound for time and size
- Then, notice that $h_D(N, \delta) \leq h_D(|G|, |G|)$

Summary

The problem: Modal Impredicativity as source of complexity.

- We keep all the proofnets of **LL**
- But we restrict the reductions to Superlazy reductions.
- Derelicting trees are the technical tool.

Further work

- We would like to control modal impredicativity with more classical methods: keeping the standard cut-elimination, but restricting **LL** proofnets.
- A possibility: to consider different sorted modalities $!_n$. $\dfrac{\vdash \Gamma, A[B/\alpha]}{\vdash \Gamma, \exists \alpha A}$ only if the modalities in B have sort less than the modalities in A.

Summary

Thank you.